ALGORITHMIC THINKING IN MATHEMATICS FROM THE 16th TO THE 18th CENTURY

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One of the highlights of the so-called scientific revolution was the creation of the infinitesimal calculus by Newton and Leibniz. The importance of this invention is shown in part by the bitterest and fiercest priority dispute the development of science has witnessed so far. The infinitesimal calculus offers one of the rare instances where the inmediate impact of a new tool and its foreseeable further achievements made even contemporary witnesses think and speak of a scientific revolution.

Thus Fontenelle in his Eléments de la géométrie de l'infini, which appeared in the year of Newton's death, 1727, described the several ways in which Johann Bernoulli, the Marquis de l'Hôpital, Varignon, and "all the great mathematicians" carried the infinitesimal calculus forward "with giant steps". The infinitesimal calculus introduced "a previously unhoped for level of simplicity, and thus inaugurated an almost total revolution in mathematics."[1]

No doubt Fontenelle voiced the feelings of his epoch. This holds in part for the priority dispute, too, styled as a war between scientists, which had been lost by Leibniz. Thus Fontenelle attributed to Newton the first discovery of the infinitesimal calculus and to Leibniz only independent co-discovery. Even if the comission of the Royal Society had in 1713 decided in favour of Newton's claims it could not hinder the victory of the Leibnizian form of the calculus in the 18th century, a victory which made English mathematics appear hopelessly backward in the beginning of the 19th century when Charles Babbage, John Peacock, and John Herschel entered Cambridge University. The three translated Lacroix's famous French textbook on Differential and Integral Calculus into English and thereby acknowledged the superiority of Leibniz's differential notation over Newton's dot notation, which was still widely regarded as sacrosanct in England[2].

What finally became clear with Charles Babbage's Reflections on the decline of science in England (London 1830), namely that there had been from the very beginning two different forms of the calculus, one could have already learned from Leibniz, who in letters to Johann Bernoulli had declared Newton's claims to be comparable to the fictive claim that Apollonius had invented the new algebra of Viète and Descartes[3].

One of the key features of Leibniz' form of the calculus was the use of special symbols and the algorithmic rules to which he had condensed it. In contrast, Newton was not very keen on symbols and did not consider them to be essential to his calculus of fluxions[4].

As a philosopher Leibniz had reflected on the value of an appropiate symbolism long before the discovery of the calculus. The development of the Leibnizian calculus by mathematicians like the Bernoullis, Euler, Lagrange, and Laplace confirmed Leibniz' ideas and expectations.

In the following I will be concerned with the different roots of the algorithmic thinking in mathematics that culminated in the calculus and in 18th century analysis.

These roots include the word "algorithm" itself, the teaching methods for elementary arithmetic, the role of symbols after the invention of printing in the 15th century, the necessity to control the practise of craftsmen and artisans by strict directions in the form of a sequence of instructions, Universalism as expressed by the belief in the existence of a method which allows one to solve all the problems in a certain domain, the pedagogical functions of scientific instruments like the proportional circle, and the belief in a mechanical explanation of the Universe including the organic living world. The predominance of symbolism, formalism and algorithms has to do with the role given to intuition, to demonstration and explanation as we will see in the systems of Descartes and Leibniz.

I will try to order these different and partly intertwined ideas. For this I will first concentrate on the origins of "algorithm". Teh word stems from the Latin version of the name Muhamad ibn Mūsā al-Khwārizmī, the author of an arithmetic and an algebra in the first half of the 9th century. Al-Khwārizmī's arithmetic presented comprehensively the system of Indian numerals for the first time. The oldest extant version of this arithmetic is a Latin translation of the 13th century which begins with the words: "dixit algorizmi". In the following centuries the original proper name changed into a term that meant simply "arithmetic".

By the beginning of the 16th century "algorithm" had acquired the connotation of the fundamental operations of arithmetic. Special algorithms had been developed for the domain of natural numbers, for fractions or (positive) rational numbers, and finally for the so-called cossic numbers[5].

Thus one arrived at a higher new arithmetic which was called algebra or

Coss. In this algebra one solved linear and quadratic equations. In order to solve an equation the reader had to learn an algorithm for the different types of equations. These were distinguished because only positive coefficients were allowed.

Theoretically one had to treat 27 different cases according as the given coefficients could be positive, zero, or negative, which in the last case meant that one had to switch sides in order to make the negative coefficients positive. Despite tha fact that some of the 27 cases could be eliminated immediately, it could be understood as a great success to reduce the number of cases to be treated to eight as had been done by Christoff Rudolff. Further progress was made when Michael Stifel in his Arithmetica integra subsumed under one single rule all the eigth cases distinguished by Rudolff. The solution of the - as Stifel could claim now - one quadratic equation could be found by the successive execution of five instructions. These five instructions in some way still reproduce the typical form of acoustic learning during the 16th century. For this reason I will quote it[6]:

> *"First begin with the coefficient of the unknown, halve it; put on its place this half and leave it on its place until you have finished. Secondly square that half. Thirdly add or substract according to its signs the constant term. Fourthly find the square root of the sum respectively the difference. Fifthly add or substract according to the sign or according to your example."*

As an additional crib to keep that rule in the form of a sequence of instructions in mind Stifel called it "AMASIAS", taking the first letters of the separate instructions in their Latin form. The development of an appropiate symbolism in order to represent a sequence of instructions like Stifel's rule AMASIAS had to wait until the method of acoustical learning used for centuries was given up in favour of visually appealing methods which could also be used for autodidactic learning. Stifel and his predecessors had already developed symbols for the unknown and its powers, for plus and minus, and this process had been accelerated by printing. But the purely verbal formulation of the rule AMASIAS still shows the strong influence of the acoustical method of learning by heart. Typical for the transition from an acoustical to a visual mode of perception in the low sciences is the adoption of the "new" Indian method of calculation which replaced the old form using the abacus. Both methods were taught in the form of algorithms, whereby calculating with the abacus can be considered as a kind of mechanical algorithm. The advantage of the abacus is that no intermediate results have to be remembered.

However the higher operations of multiplication and division become relatively complicated in comparison to the written Indian-Arabic method.

The acceptability of the Indian method of calculation has to do with a series of factors out of which printing and the visualization of learning are perhaps the most important.

Despite the many alternative methods of finding the "truth" developed during the Renaissance, there is one thing the different schools all had in common - the hope for a method that would be simple and universally applicable. The idea of an "ars inveniendi" which later dominated the thinking of Leibniz had already been formulated in the Renaissance. The model for this "ars inveniendi" was a geometrical one. Men sought a system that would not hinder cognition and production[7]. People like Petrus Ramus postulated natural laws from which instructions for actions and productions could be derived that took their only justification from practical success. The same attitude, by the way, can still be found in the 17th and 18th centuries when theologians and philosophers like Nieuwentijt (1654-1718) and Berkeley (1685-1753) criticized mathematics because of its inconsistent use of infinitesimals on the basis of the obvious success of analysis. This pragmatic attitude can also be seen in the work of late 16th and early 17th-century mathematicians who had abandoned the rigorous proof methods of the Greeks because these could only be applied to results that were already known. Creative mathematicians were much more interested in constructive methods that would lead to results. The form in which these results were presented had to be chosen according to the capacities of those who were to apply them.

The capacities of the users of the tools offered by mathematicians and especially of the mathematical practitioners can be explored by looking at the use of mathematical instruments. In accordance with the Renaissance search for a simple and universal law, the instrument makers as a part of the group of mathematical practitioners tried to develop mathematical instruments that would cover as comprehensive a domain as possible. Most of the instruments that were designed for multiple purposes acieved the sought-for multiplicity of functions only by adding a large set of accessory parts. As a result the instruments not only became expensive and clumsy, but also very inhomogeneous concerning the different applications. As a consequence of this, simplicity of production and the ability to solve all problems with a minimun of methods ideally with a single method, were seen as criteria of superior quality.

The Italian Fabrizio Mordente invented between 1554 and 1567 an instrument in the form of a circle with which he could practically master all of the first six books of the Euclidian Elements. This instrument was part of an evolution that ended with the proportional circle or sector. The loudest advocate of the proportional circle was Galileo who claimed for his form of it that it could be applied to the entire domain of practical mathematics and this with just one single method. The many descriptions of this instrument that were published afterwards can be seen as algorithms, the steps of wich had to be worked out mechanically in order to solve the problems posed[8].

Another example where mathematical practice was transformed into an algorithm for those who had neither the desire nor the ability to understand the mathematics behind it is offered by tha art of gaugin. Special gaugin rods and instructions for their use were developed in order to find the volume of a wine cask, normally by making only two measurements. The idea behind the design of such a rod was to approximate the volume with a cylinder whose volume was given multiple of the standard measure for fluids used in that city or region. In order to find this number tha gaugers would have had to perform a multiplication. But that was avoided by means of special kinds of scales plotted on the gaugin rods. In this way the gaugers from the 15th century on had only to be led to a single number without any calculation[9].

We have seen in the domains of lower and higher arithmetic, that is to say cossic algebra and practical mathematics, that the algorithmic method of problem solving was justified mainly by the pedagigical aim of offering the user an invariable, reliable procedure for the solution of most or all of the problems in his domain. Because of the limitedness and therefore comprehensibility of the lower sciences it was by no means clear whether it would be possible by means of algorithmic methods to conquer new territory in higher mathematics. The man who was most sceptical about algorithmics and their possibilities in higher mathematics was Descartes, and yet Descartes not only pioneered in framing a mechanical explanation of the world but was also the key figure of the new algebra who considerably improved the algebra of Viète by creating an appropiate symbolism. It should be added that Viète himself distinguished three parts in his algebra or new analysys which he christened zetetical, poristical and exegetical analysis. It is the last part which contains the algorithmic-formal rules for the transformation and solution of the equations found by zetetical analysis[10].

The reason for Descartes' resistance to formal, algorithmic methods can be found in the fact tha only a fraction of human activities can be explained mechanically. Decarted was prepared to give mechanical models for the functions of sensual perception, the transmission of these perceptions to memory, and the control of physical movement by the brain or a part of it; that is to say all these functions can be imitated by mechanical apparatuses. Descartes even gives details of such automata and machines in the second part of Le Monde, which was published porthumously[11].

In the Regulae ad directionem ingenii, especially in rule XII, the domain which can be explained mechanicaly is contrasted with an exclusively intellectual domain wich is responsible for the process of cognition. This domain is dominated by a power that cannot be explained in mechanical terms[12]. This intellectual, spiritual power is described by Descartes as pure intellect, as powers of imagination, as memory, or as sensation. Interestingly, there is a corporeal memory whose function is to store sensual perceptions, which can be explained by a machine, and a memory whose function is to recollect and recognice.

This reservation of a non-mechanical domain for the intellectual capacities of the human being is one of the reasons why Descartes did not think much of algorithms, which can be worked out by an automata.

The second and related reason is that algorithms do not fit into Descartes' idea of the method for attaining cognition. This method consists of a first part called intuition, which reduces the perceived impressions to evident alementary principles, and a second part which deduces all possible phenomena from these principles. For this procedure Descartes offers rules that serve as a help for decisions. This method is realized in what Descartes calls Mathesis universalis, the way to perceive, to gain knowledge. But a human mind that follows al algorithmic program is bound to remain in a restricted, limited domain and lose the possibility of finding new results[13]. Because of this, Descartes objected when Fermat presented his algorithmic methods for finding exteme values of a function an for finding the tangent to a curve given in algebraic form[14].

For Descartes, the only thing that counted was individual intellectual capacity controled by a few evident basic principles. Therefore Descartes refrained from burdening memory with a mass of formal rules that rule out thinking. Instead, Descartes supported the idea of memory by demonstation which connects the demonstrated result with the basic principles evident a priori. This was the way in which Descartes had realized the program to algebraisize geometry. By this method Descartes had solved the most central problem of his Géométrie - to find the normal and the tangent to a point on any given curve in very general terms by clarifying the geometrical facts and by showing how to determine the center of the osculatory circle. When, shortly after the publication of this solution in the Géométrie, Descartes was confronted with Fermat's algorithmic solution of the same problem, doubts immediatly arose about the possibility of giving rules relevant for all problems in an as yet undefined domain of validity. Descartes voiced his doubts in a letter to Fermat in 1638.

> *"In fact it is impossible to master all the cases that could be proposed within the domain of a single rule if one reserves the right to change something at will, as I did in what I wrote, where I did not restruct myself to the domain of any rule but only explained the principles of my procedure and gave some examples so that anyone could apply it afterwards to the various casesthat can occur according to his ability."*[15]

Fermat, however, who felt free from any such considerations, had checked the correctness of his two rules for the determination of extreme values and of tangents. Therefore he thought he was perfectly justified when he claimed universal validity for his two methods. Fermat had formulated them in keeping with the algorithmical tendencies of the 16th century as a sequence of operations to be applied to the function describing the problem. What was new with Fermat was not the claim for the universal validity of his methods, but the fact that this claim was applied to an undefined domain. The single steps of Fermat's rule for the determination of extreme values amounts to equating the first derivative of a polynomial with zero. After the general formulation of his rule, Fermat applied it to the problem of finding the maximum rectangle for which the sum of the length and the breadth is a given line segment. After finding the result that the maximum is achieved by halving the line segment to obtain a square, Fermat stated: "We can hardly expect a more general method"[16].

Fermat expressed himself even less modestly at the end of his second rule for finding tangents. Here the amazed Descartes could read:

> *"This method never fails and could be extended to a number of beautiful problems; with its aid, we have found the centers of gravity of figures bounded by straight lines and curves, as well as those of solids, and a number of other results which we may treat elsewhere if we have time to do so."*[16]

It is true that Descartes' reaction to Fermat's claims was determined as much by personal feelings as by relevant objections. It is also true that Descartes managed to find a counterexample for which Fermat's rule failed; but neither Fermat nor Descartes had the means at their disposal to explain why this was so. For Descartes, of course, it sufficed to give a counterexample in order to justify his dislike of universal claims for an algorithmic rule. For Fermat, on the other hand, it sufficed to circumvent the difficulty posed by Descartes by reformulating the problem in a way that would allow the application of his rule[14]. However, the majority of those who in reading Descartes' Géométrie took an interest in the determination of tangents on curves found Fermat's rules, particularly his tangent rule, much more appealing than Descartes' comparatively vague description of how to procede. Descartes presupposed an understanding of what and why whereas with Fermat it was only necessary to follow his precise instructions in order to find the required result. Moreover, from a technical point of view Fermat's tangent rule was simpler and more practical because there were fewer calculations and transformations. No wonder then that in 17th century mathematics the algorithmic style of presenting a method as exemplified by Fermat prevailed. Exactly these virtues of simplicity and the dispensability of mathematical knowledge were stressed in teh development following Fermat. Both of Fermat's rules were extended to the domain of implicit functions of the form where an integer (?) function of two variables

$$f(x,y) = \sum_{i,h} a_{i,h}\, x^i y \qquad \text{with} \qquad \left.\begin{matrix} i \\ h \end{matrix}\right| = 0,\ldots,n$$

is equated to zero. This was done by Ian Hudde for the method of extreme values in van Schooten's famous Latin edition of Descartes' geometry in 1659 and by de Sluse for the tangent rule in an article in the Philosophical Transactions in 1673. De Sluse explicitely boasted that a boy without any mathematical education, puer ἀγεωμετρός, could learn and successfully apply his method without any effort of calculation[17].

The different algorithmic rules for the determination of extreme values and tangents developed after Fermat constitute one part of the prehistory of the infinitesimal calculus. Another root is the method of indivisibilia created by Cavalieri. Cavalieri's method is based on the methods of determining area and volume developed by Archimedes, Stevin and Kepler. Based on the principle called after him, Cavalieri found one of the most fundamental results for the future calculus, which we now state in the following form:

$$\int x^n\, dx = \frac{1}{n+1}\, x^{n+1}.$$

The impact of Cavalieri's work can best be seen in England in the person of John Wallis, who integrated the method of indivisibles into his Aritmetica infinitorum[18].

Before the invention of his calculus, Leibniz had first read all of the Geometria indivisibilibus of Cavalieri[19]; and Newton owed the decisive methods for his theory of series in large part to the inspirations he got from the reading of John Wallis.

The most distinctive criteria of the calculus in Newton's and Leibniz' time which were stated explicitly by Leibniz and less clearly in Newton's various drafts are the following two[20]:

1. Diferentiation and integration are inverse operations.
2. Both methods for differentiation and integration have to be expressed in an appropiate algorithmic form.

Newton had found the basic principles of his calculus of fluxions in the winter of 1665/66. However, the formation of Leibniz' calculus, which took place during his stay in Paris from 1672 to 1676, is more interesting from a formal point of view. Newton appears comparatively much more conservative in following up as a mathematician the paths opened by Fermat and Cavalieri.

Leibniz, on the other hand, had thought of an epistemological program for a new mathesis universalis, an ars inveniendi, long before he began to be interested more profoundly in mathematics. He reports[21] that he first hit on this idea at the age of 18.

A draft of how to realize this program can already be found in his Dissertatio de arte combinatoria[22] from the year 1666.

Here Leibniz proposes to break down all imaginable concepts into a small number of simple, consistent elements and to find for these elements typical aymbols which he calls "characters". He is convinced that it is possible to express all results known to be true immediatly in a generally intelligible way and moreover to find new results simply by combining these "characters". This Leibnizian characteristica universalis is of special relevance for his calculus. In a letter to Oldembourg[23] in 1675 Leibniz tries to explain this symbolic language for his dreamed of ars inveniendi. It should secure a solid and visible truth in a so to speak mechanical way.

It is distinguished by preventing us from error even against our own will. In a later letter to Oldembourg[24] Leibniz claims that this symbolic language goes on inside the human mind during the speaking process and thus enables even the lesser intelligent to utter extraordinary sentences.

In saying this, Leibniz clarifies the contrast which distinguishes him from Descartes. The Cartesian distinction between a mechanically explicable domain and one that cannot be explained in this way is given up in the program of Leibniz' ars inveniendi. With this in mind it seems very natural for

Leibniz to be involved in the construction of the most effective four calculating machines of his time. Of course, this does not create a contradiction with Descartes. The contradiction arises only with the attempt to construct an essentially more complex machine with the aim of reproducing and replacing all the functions of the human mind. This is exactly the idea behind Leibniz' program, the realization of which, as has been said, would have made unnecessary Leibniz' own mind, in which for the last time all human knowledge of an epoch was concentrated.

Leibniz' descriptions of his symbolic language in letters to Henry Oldenbourg coincide with the creation of his infinitesimal calculus. In some way the development of Leibniz' calculus can be understood as a test for his program of a characteristica universalis.

In his investigations which culminated in the creation of the calculus, Leibniz tried according to the lines of his ars inveniendi to summarize the available results concerning the determination of areas, centres of gravity, etc. This is to say that he tried to clarify for himself as a student the results of his predecessors by creating a new symbolism which summed up their achievements in a form better adapted to his way of thinking and which left room for greater generalizations. I can only touch on the formation of this special symbolism[25].

In the middle of a manuscript written in the fall of 1675 which was decisive for the first version of his calculus, Leibniz changes from a terminology stemming from Cavalieri to the modern integral sign, which he had derived from the word summa, and to the little "d" for the differential operator, which was meant as an abbreviation of differentia[26].

In this manuscript Leibniz had already stated the simplest rules of the integral calculus which he described by the word calculus summatorius in contrast to his calculus differentialis. Here one can observe the degree to which Leibniz trusted in the viability (?) of his symbolism. Leibniz wanted to know if the operator d applied to the product xy would result in the product $dx \cdot dy$ as he had assumed. The assumption shows impressively that Leibniz had rid himself completely from any geometrical idea in order to rely on purely formal calculations. But one can also see what Leibniz meant when he claimed that his characteristica universalis leads the mind by means of the symbolism, that is to say, a way to right understanding that is perceptible to the senses even if we make mistakes. In a second attempt Leibniz used the reversibility of the integral and differential operators in order to show that by assuming

$$d(xy) = d(x) \cdot d(y)$$

then

$$\int dxdy = \int d(xy) = xy = \int dx \int dy$$

which does not hold generally for sums. So Leibniz was forced to give up his assumption that

$$d(xy) = dx\, dy.$$

The analogous assumption for the quotient $\frac{x}{y}$ had to be given up too, although he was still not able to say what the meaning of $d(xy)$ or $d(\frac{x}{y})$ was. By the end of the year 1675 Leibniz had completed the main rules for the calculus differentialis and with them to some degree the process of adopting the results of his predecessors to this new symbolism. The calculus so far created had to be elaborated to include infinitesimals as an example for the realization of Leibniz' ars inveniendi.

He also had to find out how to apply his differential calculus to the determination of tangents and he had to learn how to operate with series. It took several more years before he achieved all this. In 1684 Leibniz published the most important rules of his differential calculus in a very short paper in the Acta Eruditorum. Here one can see that Leibniz was fully aware of the algorithmic character if this calculus as well as of the consequences of it[27].

> *"Knowing thus the Algorithm (as I may say) of this calculus, which I call differential calculus, all other differential equations can be solved by a common method. We can find maxima and minima as well as tangents without the necessity of removing fractions, irrationals, and other restrictions, as had to be done according to the methods that have been published hitherto. . . It is clear that our method also covers transcendental curves - those that cannot be reduced by algebraic computation, or have no particular degree - and thus holds in a most general way without any particular and not always satisfactory assumptions... And this is only the beginning of a much more sublime Geometry, pertaining to even the most difficult and most beautiful problems of applied mathematics, which without our differential calculus or something similar no one could attack with any such ease".*

Leibniz espressed similar views for his integral calculus two years later[28]

The success of Leibniz' form of the calculus resulted in a purely formal elaboration of infinitesimal methods. Thus Leibniz derived in analogy to the binomial theorem for natural exponents a rule for higher differentials of products of several variables. This rule implies in the case of two variables:

$$d^n(xy) = (dx + dy)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} d^\nu x \ d^{n-\nu} y \ .$$

Leibniz had comunicated this rule to Johan I Bernoulli in a letter[29] of 1695.

Obviously this induced Johan I Bernoulli to continue this formal procedure in an even more uninhibited fashion. Where Leibniz had introduced the symbol $\overset{0}{d}x$, which leaves the function invariant because the operator works zero times so to speak, so Bernoulli extended now the notion to negative exponents, that is to say:

$$d^{-n} = \int^{+n} .$$

In this way Bernoulli, by using an algorithm for division by a binomial, hit on a special form of the Taylor series[30].

Despite an increasing number of critical voices, these impressive results encouraged the mathematicians who followed Leibniz to stick fo formalism. No wonder that Basel had become already, before Newton died, the centre of further mathematical research in analysis. This was mainly due to the activities of Johan I Bernoulli, who passed on his insights and methods to the next generation of the Bernoullis and to the young Leonhard Euler. Euler, whose books on analysis became classics and were used as textbooks well into the middle of the 19th century, transmitted this formal tradition of his teacher Johan I Bernoulli. Euler's calculus of Zeros became famous and even notorious. Joseph Louis Lagrange, who followed Euler in the second half of the 18th century as the leading mathematician in Europe, tried to free Analysis from the metaphysics of infinitesimal quantities as used by Euler. He did this by reverting to formal algebraic methods but in vàin. Even a growing awareness of the necessity to lay better foundations for analysis did not prevent him from joining the club of Leibnizian formalists. This becomes crystal clear in the following comment[31]:

> *"Though the principle of this analogy between positive powers and differentials, negative powers and integrals is not obvious by itself, yet I will use it in this papers since the conclusions one draws from it are nevertheless exact, as we can convince ourselves afterwards in order to discover several general theorems concerning the differentiation and integration of functions of several variables. These are mostly new theorems which can be found by means of other methods only with great difficulty".*

As Leibniz had used the analogy between the natural powers of a binomial and higher differentials of the same order, Lagrange so introduced Euler's exponential function as a formal analogy of the Taylor series. This means that the exponential function serves as an algorithm for the application of Leibniz' differential operator in order to get the Taylor expansion of a function. Ia a similar way, Lagrange, using the Bernoulli analogy between differentials of a negative order and integrals of the corresponding positive order, derived what has become known as the Euler-MacLaurin summation formula[32].

With the aid of this formula it is possible to approximate the finite sum of the discrete values of a function with the integral of the function. But the Euler-MacLaurin formula leads to a kind of series that was later caller "asymptotic". The means of handling asymptotic series could be developed only when mathematicians ceased to rely blindly on formal algorithmic methods. One pioneer of this new mathematics was Cauchy who postulated new standards of mathematical rogour and developed the method of counter examples which had already seen advocated by Descartes in the 17th century. These postulates can be understood in part as reactions against the universalism of the formal style of 18th-century mathematics and especially against its last prominent representative, Laplace. He took Lagrange's results as a starting point for the creation of the theory of generating functions and made this the analytical tool for probability theory.

The decline of the very fertile formalistic era in the 18th century was not entirely a result of a growing number of mathematical problems with divergent o semidivergent series. I rate two other factors as more influential: The formal methods had come to an end; the formal period had exhausted its possibilities. This was clearly seen by Lagrange himself who compared the mathe matics of his time with a nine that had been abandoned because it had become unprofitable[33].

The second factor has to do with the social justification for the development of algorithmic rules. Algorithms had been created forst of all for those who did not need or were not able to understand the principles underlying mathematical methods and results. The number of those who depend as mathematical illiterated on a sequence of unexplained instructions decreased drasticly as a consequence of the reforms of the educational system beginning with teh French revolution. All of this contributed to a profound change in the character of mathematics in the first decades of the 19th century.

As one of the key figures of this change and so as a witness to it, the <u>princeps mathematicorum</u> himself, Gauss, could join Descartes' party (even though he was unaware of it), when in 1843 he stated in a letter to his friend

Schumacher[34]:

> "*In general it holds for all these new calculuses that we can achieve nothing with their help that could not be achieved without them*".

In contrast to Descartes, Gauss was ready to acknowledge the practical advantages of a calculus if its domain of validity is taken to be limited.

NOTES

1. Compare I, Bernard Cohen, The Newtonian Revolution, Cambridge University Press, 1980, p. 43.

2. See Norman T. Gridgeman's article on Charles Babbage in the Dictionary of Scientific Biografy (ed. by Charles C.Gillispie in 16 volc.), vol.I, N.Y. 1970, p.354-356.

3. See letter from Leibniz to Johan Bernoulli, June 28, 1713. G.W.Leibniz Mathematische Schriften (ed.C.I.Gerhardt), vol.III/2, Halle 1856,p.912-915, esp.913."... sed apparet non magis, eum cognovisse Calculum nostrum, quam Apollonius cognovit calculum Vietae et Cartesii speciosum. Fluxiones cognovit non Calculum fluxionum..."

4.See Annotatio 247 to the Commercium epistolicum D.Johannis Collins et aliorun de analysi promota, London 1722 (2nd edition), which reads: "Methodus fluxionum utique non consistit in forma symbolorum".

5. See Heinrich Schreiber alias Grammateus, Ein new künstlich behend vnd gewiss Rechenbüchlin vff alle Kauffmanschafft, ca. 1519 f. 38v. and Christopf Rudolff, Behend und hübsch Rechnung durch die künstreichen Regeln Algebre so gemeincklich die Coss genent werden, Wien (?) 1525, f.AIII v.

6. Michael Stifel, Arithmetica integra, Nürnberg 1544, f.240v.

7. Compare Stephan Otto (ed.), Renaissance und frühe Neuzeit, Stuttgart (Reclam) 1984 (= Geschichte der Philosophie in Text und Darstellung (ed.Rüdiger Bubner), vol.III).

8. See Ivo Schneider, "Der Proportionalzirkel - ein universelles Analogrecheninstrument der Vergangenheit", Abhandlungen und Berichte des Deutschen Museums Jg.38, 1970, Heft 2, p.1-72.

9. Compare Menso Folkerts, "Die Entwicklung und Bedeutung der Visierkunst als Beispiel der praktischen Mathematik der frühen Neuzeit", Humanismus und Neuzeit 18, Heft 1, 1974, p.1-41.

10. See Ivo Schneider, "François Viète", in Exempla historica Epochen der Weltgeschichte in Biographien, vol.27, Die Konstituierung der neuzeitlichen Welt. Naturwissenschafter und Mathematiker, Frankfurt/M. 1984, p.57-84, esp. p.65.

11. See Oeuvres de Descartes (ed. Ch.Adam and P. Tannery), vol.XI, p.119f. and vol.I, p.254f. and 263.

12. Oeuvres de Descartes, vol.X, p.410-430, esp.415.

13. See Ivo Schneider, "Die Rolle des Formalen und des Individuums in der Mathematik, bei Descartes und Leibniz", Sudhoffs Archiv 58, 1974, p.225-234.

14. See Ivo Schneider, "Descartes' Diskussion der Fermatschen Extremwertmethode - ein Stück Ideengeschichte der Mathematik", Archive for History of Exact Sciences 7, 1971, p.354-374.

15. Letter to Fermat from July 27, 1638 in Oeuvres de Fermat, vol.II, p. 163f.

16. Oeuvres de Fermat, vol.I, p.133. 1891-1922. Tannery, Paris

17. R.F.de Sluse, "A method of drawing tangents af all geometrical curves" Philosophical Transactions of the Royal Society, 7, 1672, p.5143-5147 and vol. 8, 1673, p.6059.

18. John Wallis, Arithmetica infinitorum, Oxford, 1656.

19. Bonaventura Cavalieri, Geometria indivisibilibus continuorum nova quadam ratione promota, Bologna 1635 (2nd ed.1653).

20. Derek Thomas Whiteside, "Patterns of mathematical thought in the later seventeenth century", Archive for history of exact sciences, 1, 1961, p.179-388.

21. See G.W.Leibniz, Opuscules et fragments inédits (ed L.Couturat), Paris 1903, reprint Hildesheim 1966, p.157.

22. Leibniz' philosophische Schriften (ed. C.J.Gerhardt), vol.IV, p.72f.

23. See Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern (ed. C.J.Gerhardt), vol.1, Berlin 1899, p.143-147; letter to Oldenbourg from 28.December 1675.

24. Letter from 18/29.October 1676 in Leibniz' philosophische Schriften (ed.C.J.Gerhardt), vol.VII, p.11-15.

25. For details see Joseph E.Hofmann, Leibniz in Paris 1672-1676, Cambridge University Press 1974.

26. See manuscript "Analysis Tetraganista ex Centrobarycis" part 2 from 29.October 1675 in Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern (ed. C.J.Gerhardt) volI, Berlin 1899, p.151-156.

27. "Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus", Acta Eruditorum 3, 1684, p.467-473; quoted from D.J.Struik, A source book in mathematics, 1200-1800, Harvard University Press, 1969, p.276, and 279.

28. G.W. Leibniz, "De geometria recondita et analysi indivisibilium atque infinitorum", Acta eruditorum, 5, 1686.

29. Leibniz to Johan Bernoulli, 16. May 1695, in Leibniz mathematische Schriften (ed. C.J.Gerhardt), vol.III/1, Halle 1855, p.174-179.

30. See Johan I Bernoulli to Leibniz, 18 June 1695, ibidem p.179-190. Bernoulli starts from the formal relationship

$\frac{d^0 x}{d^1 x} = \frac{d^{-1} x}{d^0 x} = \frac{x}{d^0 x}$ and applies it to $y \cdot dx$ which lead to

$$\frac{\int y\, dx}{d^0 y\, dx} = \frac{d^0 y\, dx}{d(ydx)} = \frac{d^0 y\, dx}{d^0 d^2 x + dydx} \quad \text{or}$$

$$\int ydx = \frac{d^0 y\, d^2 x}{d^0 y\, d^2 x + dydx} = \frac{d^0 y\, dx}{d^0 dx + dyd^0 x}$$

Now he divides formally and obtains

$$\int ydx = d^0 y\, d^0 x - dyd^{-1} x + d^2 y\, d^{-2} x \pm \ldots = yx - \frac{x^2}{2} dy + \frac{x^3}{3} d^2 y \mp \ldots$$

31. J.L.Lagrange, "Sur une nouvelle espèce de calcul relatif à la différentiation et à l'integration des quantités variables", in Nouvelles Mémoires de l'Académie Royale de Berlin 1772 (=Oeuvres de Lagrange vol.3, Paris 1869, p.441 f.

32. Lagrange deduced formally

$$\Sigma f = \frac{1}{e^{\frac{df}{dx} h} - 1}$$

33. Letter to d'Alembert from 21 September 1781 in Oeuvres de Lagrange, vol.XIII, Paris 1882, p.368.

34. See letter from 15. May 1843 in Gauss' Werke, vol:VIII, p.297.