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Analysis of perturbations of moments associated with orthogonality linear functionals through the Szegő transformation

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Abstract. In this paper, we consider perturbations to a sequence of moments associated with an orthogonality linear functional that is represented by a positive measure supported in $[-1, 1]$. In particular, given a perturbation to such a measure on the real line, we analyze the perturbation obtained on the corresponding measure on the unit circle, when both measures are related through the Szegő transformation. A similar perturbation is analyzed through the inverse Szegő transformation. In both cases, we show that the applied perturbation can be expressed in terms of the singular part of the measures, and also in terms of the corresponding sequences of moments.

Keywords: Orthogonal polynomials, Stieltjes and Carathéodory functions, Hankel and Toeplitz matrices, Szegő transformation.
MSC2010: 42C05, 33C45, 33D45, 33C47.

Análisis de perturbaciones de momentos asociados a funcionales de ortogonalidad a través de la transformación de Szegő

Resumen. En el presente trabajo, analizamos las perturbaciones a una sucesión de momentos asociada a un funcional lineal de ortogonalidad que se representa por una medida positiva con soporte en $[-1, 1]$. En particular, dada una cierta perturbación a dicha medida en la recta real, analizamos la perturbación obtenida en la correspondiente medida en la circunferencia unidad, cuando dichas medidas están relacionadas por la transformación de Szegő. También se analiza una perturbación similar a través de la transformación inversa de

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Szegő. En ambos casos, se muestra que la perturbación aplicada puede ser expresada en términos de la parte singular de las medidas, y también a través de las correspondientes sucesiones de momentos.
Palabras clave: Polinomios ortogonales, funciones de Stieltjes y Carathéodory, matrices de Hankel y Toeplitz, transformación de Szegő.

1. Introduction

The theory of orthogonal polynomials with respect to measures supported on the real line has numerous applications, including numerical integration, integrable systems, spectral theory, approximation, and moment problem, among others. In the last decades, spectral properties of Jacobi matrices (the matrix representation of the multiplication operator in terms of the orthonormal basis) have been studied (see, for instance, [11], [14]) in connection with some spectral transformations of the orthogonality measures, which can be expressed as LU factorizations of the corresponding Jacobi matrices, and also in terms of the corresponding Stieltjes functions (the Cauchy transform of the orthogonality measure). Such transformations have been studied in the context of Darboux transformations, which are related with the bispectral problem: to find all situations in which a pair of differential operators have a common eigenfunction.

On the other hand, orthogonal polynomials with respect to nontrivial probability measures supported on the unit circle have been studied more recently. The basic references on this topic are the classical Szegő book [12] and the recent monograph [10]. Some of the more studied applications of orthogonal polynomials on the unit circle are numerical integration, prediction theory and control theory. Much of the research efforts in this area are conducted using analogies with the real line case: If orthogonal polynomials on the real line satisfy some property, the corresponding property for orthogonal polynomials on the unit circle is studied. In particular, the analysis of spectral transformations on the unit circle has been developed in [7, 8], among others.

Furthermore, the analysis of perturbations on the moments associated with an orthogonality measure on the real line has been developed in [1], and on the unit circle in [2]. In both cases, the authors show that such perturbations constitute particular cases of linear spectral transformations, and obtain some properties of the perturbed sequence of orthogonal polynomials. In this paper, we analyze perturbations of moments from the perspective of the Szegő transformation: a well known relation between measures in $[-1, 1]$ and measures in the unit circle. The structure of the manuscript is as follows: in Section 2, we present some basic results regarding orthogonal polynomials on the real line, as well as some results about moment perturbations for measures supported on the real line. The analogous results for measures supported on the unit circle are described in Section 3. Finally, the relation of these perturbations through the Szegő transformation and the inverse Szegő transformation is studied in Sections 4 and 5, respectively. This constitutes the original contribution of this paper. Some illustrative examples are provided.

2. Perturbation of a Hankel matrix

2.1. Orthogonal polynomials and linear spectral transformations on the real line

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of complex numbers and let $\mathcal{L}$ be a linear functional defined in the linear space $\mathbb{P}$ of polynomials with complex coefficients such that

$$\langle \mathcal{L}, x^n \rangle = \mu_n, \quad n \geq 0. \tag{1}$$

$\mathcal{L}$ is called a moment linear functional, and the complex numbers $\{\mu_n\}_{n\geq 0}$ are the moments associated with $\mathcal{L}$ (see [3, 12]).

A sequence of polynomials $\{p_n(x)\}_{n\geq 0}$, where

$$p_n(x) = \gamma_n x^n + \delta_n x^{n-1} + \cdots, \quad \gamma_n \neq 0, \ n \geq 0,$$

is called an orthogonal polynomials sequence (**OPS**) with respect to $\mathcal{L}$, if for every non negative integers n and m, the conditions

1. $p_n(x)$ is a polynomial of degree n,
2. $\langle \mathcal{L}, p_m(x)p_n(x) \rangle = 0$, for $m \neq n$,
3. $\langle \mathcal{L}, p_n(x)p_n(x) \rangle = \langle \mathcal{L}, p_n^2(x) \rangle \neq 0, \ n \geq 0$,

hold. The corresponding monic sequence (**MOPS**), denoted by $\{P_n(x)\}_{n\geq 0}$, whose leading coefficient is 1, is defined by $P_n(x) = \frac{p_n(x)}{\gamma_n}$.

The Gram matrix associated with the bilinear form $\mathcal{B}$ associated with $\mathcal{L}$, defined by $\mathcal{B}(p,q) = \langle \mathcal{L}, pq \rangle$, $p, q \in \mathbb{P}$, with respect to the monomial basis $\{x^n\}_{n\geq 0}$ of $\mathbb{P}$, is

$$\mathbf{H} = [\langle \mathcal{L}, x^{i+j} \rangle]_{i,j=0,1,\ldots} = [\mu_{i+j}]_{i,j=0,1,\ldots} = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}.$$

These matrices with constant values along the anti diagonals are known in the literature as **Hankel matrices**.

Not every linear functional has an associated OPS. A necessary and sufficient condition for the existence of an OPS associated with $\mathcal{L}$ is

$$\Delta_n = \det \mathbf{H}_n = \det(\mu_{i+j})_{i,j=}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \neq 0, \quad n \geq 0,$$

where $\mathbf{H}_n$ is the $(n+1) \times (n+1)$ principal leading submatrix of $\mathbf{H}$. $\mathcal{L}$ is called quasi-definite if $\Delta_n \neq 0$ for every $n \geq 0$, and positive definite if and only if all the moments are real and $\Delta_n > 0$ for every $n \geq 0$. If $\mathcal{L}$ is positive definite, then there exists a unique sequence of polynomials

$$p_n(x) = \gamma_n x^n + \delta_n x^{n-1} + \cdots, \ \gamma_n > 0, \ n \geq 0$$

satisfying

$$\langle \mathcal{L}, p_n(x) p_m(x) \rangle = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker's delta. $\{p_n(x)\}_{n\geq 0}$ is called the sequence of orthonormal polynomials associated with $\mathcal{L}$. From the Riesz's representation theorem, there exists an integral representation (not necessarily unique)

$$\langle \mathcal{L}, x^n \rangle = \int_E x^n d\alpha(x),$$

where α is a positive non trivial Borel measure, whose support E is an infinite subset of the real line. Some of the most remarkable properties of orthonormal polynomials are the following.

1. The sequence $\{p_n(x)\}_{n\geq 0}$ satisfies the three term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \ n \geq 0,$$

with initial conditions $p_{-1}(x) = 0$, $p_0(x) = \mu_0^{-1/2}$, and recurrence coefficients given by

$$a_n = \int_E x p_{n-1}(x) p_n(x) d\alpha(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0, \ b_n = \int_E x p_n^2 d\alpha(x) = \frac{\delta_n}{\gamma_n} - \frac{\delta_{n+1}}{\gamma_{n+1}}.$$

2. The n-th orthonormal polynomial, $p_n(x)$, can be expressed in terms of Hankel determinants as follows:

$$p_n(x) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}, \ n \geq 0,$$

with $\Delta_{-1} = 1$. This is called the Heine's formula. The leading coefficient is given by the ratio of two Hankel determinants

$$\gamma_n = \sqrt{\frac{\Delta_{n-1}}{\Delta_n}}.$$

3. The zeros of $p_n(x)$ are real, simple, and located in the interior of the convex hull of the support E of the measure.

4. Let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ be the zeros $p_n(x)$. The zeros of $p_n(x)$ and $p_{n+1}(x)$ satisfy the interlacing property

$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \ i = 1, 2, ..., n.$$

The Stieltjes function associated with α is defined as

$$S(x) = \int_E \frac{d\alpha(t)}{x-t}.$$

If $\mathcal{L}$ is quasi-definite, $S(x)$ admits the formal series expansion at infinity

$$S(x) = \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}},$$

where μ_k are the moments associated with α, given by (1). In what follows, we will assume that $\mu_0 = 1$. A rational spectral transformation of a Stieltjes function $S(x)$ is a transformation of the form

$$\tilde{S}(x) = \frac{A(x)S(x) + B(x)}{C(x)S(x) + D(x)},$$

where $A(x), B(x), C(x)$ y $D(x)$ are polynomials in x with $AD - BC \neq 0$ and such that $\tilde{S}(x)$ has a formal series expansion around infinity. The transformation is said to be linear if $C(x) \equiv 0$.

Given a linear functional $\mathcal{L}$, some canonical perturbations analyzed in the literature are:

1. The Christoffel transformation $d\tilde{\alpha}_c = (x-\beta)d\alpha$, $\beta \notin supp(\alpha)$.
2. The Uvarov transformation $d\tilde{\alpha}_u = d\alpha + M_r\delta(x-\beta)$, $\beta \notin supp(\alpha)$, $M_r \in \mathbb{R}$.
3. The Geronimus transformation $d\tilde{\alpha}_g = \frac{d\alpha}{x-\beta} + M_r\delta(x-\beta)$, $\beta \notin supp(\alpha)$, $M_r \in \mathbb{R}$,

where $\delta(x-\beta)$ is the Dirac's delta functional, defined by

$$\langle \delta(x-\beta), q\rangle = q(\beta), \quad q \in \mathbb{P}, \quad \beta \in \mathbb{R}.$$

The three canonical perturbations defined above correspond to linear spectral transformations of the corresponding Stieltjes functions (see [7, 8, 14]).

2.2. Perturbation of the $j-$th moment of the Hankel matrix

Let $\mathcal{L}$ be a positive definite linear functional associated with a Borel measure α supported in $E \in \mathbb{R}$. Instead of considering the monomial basis of $\mathbb{P}$, we deal with the basis $\{1, (x-a), (x-a)^2, ...\}$, where $a \in \mathbb{R}$. The reason for this will be evident in what follows. The sequence of moments associated with this basis, $\{\nu_n\}_{n\geq 0}$, is given by

$$\nu_n = \mu_n + \sum_{j=0}^{n-1} \binom{n}{j} (-1)^{n+j} a^{n-j} \mu_j, \tag{2}$$

where $\mu_0 = \nu_0$ (see [1]).

We will be interested in analyzing a perturbation of the j-th moment of a given sequence of moments. Before that, the following definition is needed. Given a linear functional $\mathcal{L}$, its distributional derivative $D\mathcal{L}$ (see [13]) is defined as

$$\langle D\mathcal{L}, q\rangle = -\langle \mathcal{L}, q'\rangle, \quad q \in \mathbb{P}.$$

If j is a non negative integer, the $j-$th distributional derivative is

$$\langle D^j \mathcal{L}, q\rangle = (-1)^j \langle \mathcal{L}, q^{(j)}\rangle, \quad q \in \mathbb{P}.$$

In particular, for the functional $\delta(x-a)$, it is easy to see that the the $j-$th distributional derivative is

$$\langle D^j \delta(x-a), q\rangle = (-1)^j q^{(j)}(a).$$

Definition 2.1 ([1]). Let $\mathcal{L}$ be a quasi-definite linear functional. The linear functional $\mathcal{L}_j$ is defined by

$$\begin{aligned}\langle \mathcal{L}_j, p(x)\rangle &= \langle \mathcal{L}, p(x)\rangle + (-1)^j \frac{m_j}{j!} \langle D^{(j)} \delta(x-a), p(x)\rangle \\ &= \langle \mathcal{L}, p(x)\rangle + \frac{m_j}{j!} p^{(j)}(a),\end{aligned} \tag{3}$$

where m_j and a are real constants, and $p^{(j)}(x)$ denotes $j-$th derivative of $p(x)$.

Necessary and sufficient conditions for $\mathcal{L}_j$ to be quasi-definite can be found in [1], as well as the explicit relation between the corresponding MOPS. If both $\mathcal{L}$ and $\mathcal{L}_j$ are positive definite, then the previous transformation can be expressed in terms of the orthogonality measures as follows:

$$d\tilde{\alpha}_j = d\alpha + (-1)^j \frac{m_j}{j!} D^{(j)} \delta(x-a). \tag{4}$$

On the other hand, from (3) it is easily obtained that

$$\tilde{\nu}_k = \langle \mathcal{L}_j, (x-a)^k\rangle = \begin{cases} \nu_k, & \text{if} \quad k < j, \\ \nu_j + m_j, & \text{if} \quad k = j, \\ \nu_k, & \text{if} \quad k > j. \end{cases}$$

In other words, $\mathcal{L}_j$ only perturbs the generalized $j-$th moment of the linear functional $\mathcal{L}$. Furthermore, if $\mathcal{L}_j$ is quasi-definite and $\tilde{S}(x)$ denotes its corresponding Stieltjes function, $S(x) = \sum_{k=0}^{\infty} \frac{\nu_k}{(x-a)^{k+1}}$ and $\tilde{S}(x)$ are related by

$$\tilde{S}_j(x) = S(x) + \frac{m_j}{(x-a)^{j+1}}. \tag{5}$$

As a consequence, (5) is a linear spectral transformation of $S(x)$, where $A(x) = (x-a)^{j+1}$, $B(x) = m_j$ and $D(x) = (x-a)^{j+1}$.

3. Perturbation of a Toeplitz matrix

3.1. Orthogonal polynomials and spectral transformations on the unit circle

The following definitions and theorems can be found in [4, 8, 10, 12]. Notice the analogy with the concepts presented in the previous section. Let $\mathcal{L}$ be a linear functional in the linear space of Laurent polynomials ($\Lambda = span\{z^k\}_{k\in\mathbb{Z}}$) such that $\mathcal{L}$ is Hermitian, i.e.

$$c_n = \langle \mathcal{L}, z^n\rangle = \overline{\langle \mathcal{L}, z^{-n}\rangle} = \overline{c}_{-n}, \quad n \in \mathbb{Z}.$$

The complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ are said to be the moments associated with $\mathcal{L}$. Under this conditions, a bilinear functional can be defined in the linear space $\mathbb{P} = span\{z^k\}_{k\in\mathbb{N}}$ of polynomials with complex coefficients by

$$\langle p(z), q(z)\rangle_{\mathcal{L}} = \langle \mathcal{L}, p(z)\bar{q}(z^{-1})\rangle, \quad p, q \in \mathbb{P}.$$

The Gram matrix associated with the canonical basis $\{z^n\}_{n\geq 0}$ of $\mathbb{P}$ is

$$\mathbf{T} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{6}$$

known in the literature as **Toeplitz** matrix. As in the real line case, $\mathcal{L}$ is said to be quasi-definite if the $(n+1)\times(n+1)$ principal leading submatrices $(\mathbf{T}_n)_{n\geq 0}$ are non singular, i.e.,

$$\det \mathbf{T}_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & \cdots & c_0 \end{vmatrix} \neq 0, \quad n \geq 0.$$

If $\mathcal{L}$ is quasi-definite, there exists a sequence of monic polynomials $\{\Phi_n\}_{n\geq 0}$ satisfying:

1. $\deg \Phi_n = n$, $n \geq 0$,
2. $\langle \Phi_n, \Phi_m\rangle_{\mathcal{L}} = 0$, for $m \neq n$, $m, n \geq 0$,
3. $\langle \Phi_n, \Phi_n\rangle_{\mathcal{L}} = \mathbf{K}_n$, $n \geq 0$,

where $\mathbf{K}_n \neq 0$ for every $n \geqslant 0$. $\{\Phi_n\}_{n\geq 0}$ is called the sequence of monic orthogonal polynomials sequence associated with the Hermitian linear functional $\mathcal{L}$.

If $\mathbf{T}_n$, $n \geq 0$, has positive determinant, then $\mathcal{L}$ is said to be positive definite. In such a case we can define an inner product, and we have

$$\langle \mathcal{L}, \Phi_n(z)\bar{\Phi}_n(z^{-1})\rangle = \langle \Phi_n, \Phi_n\rangle = \|\Phi_n\|^2 = \mathbf{K}_n,\ \mathbf{K}_n > 0.$$

There also exists an integral representation

$$\langle \mathcal{L}, p(z)\rangle = \int_{\mathbb{T}} p(z)d\sigma,$$

where σ is a nontrivial probability measure supported in the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

$\{\Phi_n\}_{n\geq 0}$ satisfies the forward and backward recurrence relations

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z),\ n \geq 0,$$

$$\Phi_{n+1}(z) = (1 - |\Phi_{n+1}(0)|^2) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z),\ n \geq 0,$$

where $\Phi_n^*(z) = z^n \overline{\Phi}_n(z^{-1})$. In the literature (see [10]), the polynomial $\Phi_n^*(z)$ is known as reversed polynomial. The complex numbers $\{\Phi_n(0)\}_{n\geq 1}$ are called Verblunsky (reflection, Schur) parameters and play a central role in the study of orthogonal polynomials on the unit circle. If $\mathcal{L}$ is positive definite, we have $|\Phi_n(0)| < 1$, for every $n \geq 1$.

Assume $\mathcal{L}$ to be positive definite, and let σ be its corresponding measure on the unit circle. Then, there exists (see [5]) a sequence of polynomials $\{\varphi_n\}_{n\geq 0}$,

$$\varphi_n(z) = \kappa_n z^n + \cdots,\quad \kappa_n \geq 0,$$

such that

$$\int_{-\pi}^{\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})} d\sigma(\theta) = \delta_{m,n},\quad z = e^{i\theta},\quad m, n \geq 0. \tag{7}$$

Clearly, we have $\Phi_n(z) = \frac{\varphi_n(z)}{\kappa_n}$. κ_n can be expressed in terms of two Toeplitz determinants,

$$\kappa_n = \sqrt{\frac{\det \mathbf{T}_{n-1}}{\det \mathbf{T}_n}},\ n \geq 1,$$

and from the above equation we have

$$\mathbf{K}_n = \frac{1}{\kappa_n^2} = \frac{\det \mathbf{T}_n}{\det \mathbf{T}_{n-1}},\quad n \geq 1.$$

On the other hand, the k-th moment c_k associated with σ is defined by

$$c_k = \int_{-\pi}^{\pi} e^{ik\theta} d\sigma(\theta),\quad k \in \mathbb{Z},$$

and the n-th orthonormal polynomial can be computed by

$$\varphi_n(z) = \frac{1}{\sqrt{\det \mathbf{T}_n \det \mathbf{T}_{n-1}}} \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \dots & \ddots & \vdots \\ c_{-(n-1)} & c_{-(n-2)} & c_{-(n-3)} & \cdots & c_1 \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix},\ n \geq 0,$$

with the convention $\det \mathbf{T}_{-1} = 1$.

Given an analytic function $F : \mathbb{D} \to \mathbb{C}$, we say F is a Carathéodory function if and only if $F(0) \in \mathbb{R}$ and $\mathfrak{Re}(F(z)) > 0$ in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If $\mathcal{L}$ is positive definite, the Taylor series

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k. \tag{8}$$

is analytic in $\mathbb{D}$ and $\mathfrak{Re}(F(z)) > 0$ in $\mathbb{D}$. (8) is thus called the Carathéodory function associated with $\mathcal{L}$. $F(z)$ can be represented as the Riesz-Herglotz transform of σ,

$$F(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta).$$

Furthermore, σ can be decomposed into a part that is absolutely continuous with respect to the normalized Lebesgue measure $\frac{d\theta}{2\pi}$ and a singular part (see [10]). If we denote by $\omega(\theta) = \sigma'$ the Radon-Nikodyn derivative of σ and by σ_s singular measure, then

$$d\sigma(\theta) = \omega(\theta)\frac{d\theta}{2\pi} + d\sigma_s(\theta). \tag{9}$$

Also, there exists a relation between the Carathéodory function and $\omega(\theta)$ (see [9]). Namely, if $\theta \in \partial\mathbb{D}$, then

$$F(e^{i\theta}) \equiv \lim_{r\to 1} F(re^{i\theta}),$$

and therefore,

$$\omega(\theta) = \mathfrak{Re} F(e^{i\theta}). \tag{10}$$

The singular part σ_s is supported in $\{\theta | \lim_{r\uparrow 1} \mathfrak{Re}(re^{i\theta}) = \infty\}$.

Given a linear functional $\mathcal{L}$, some perturbations that have been studied in the literature are (see [2]):

1. The Christoffel transformation $d\tilde{\sigma}_C = |z-\xi|^2 d\sigma$, $|z| = 1, \xi \in \mathbb{C}$,
2. The Uvarov transformation $d\tilde{\sigma}_U = d\sigma + M_c\delta(z-\xi) + \overline{M}_c\delta(z-\overline{\xi}^{-1}), \xi \in \mathbb{C} - \{0\}$, $M_c \in \mathbb{C}$,
3. The Geronimus transformation $d\tilde{\sigma}_G = \frac{d\sigma}{|z-\xi|^2} + M_c\delta(z-\xi) + \overline{M}_c\delta(z-\overline{\xi}^{-1})$, $\xi \in \mathbb{C} - \{0\}, M_c \in \mathbb{C}, |\xi| \neq 1$.

Again, notice the analogy with the transformations defined in the previous section. A rational spectral transformation of a Carathéodory function $F(z)$ is a transformation of the form

$$\tilde{F}(z) = \frac{A(z)F(z) + B(z)}{C(z)F(z) + D(z)},$$

where $A(z), B(z), C(z)$ and $D(z)$ are polynomials in z with $AD - BC \neq 0$, and such that $\tilde{F}(z)$ is analytic in $\mathbb{D}$ and has positive real part therein. Again, if $C(z) \equiv 0$, the transformation is said to be linear. The three transformations defined above correspond to linear spectral transformations, when they are expressed in terms of the corresponding Carathéodory functions (see [8]).

3.2. Perturbation of the $j-$th moment of a Toeplitz matrix

Perturbations in the subdiagonal of a Toeplitz matrix were studied in [2].

Definition 3.1. Let $\mathcal{L}$ be an Hermitian linear functional and define a linear functional $\mathcal{L}_j$ such that the associated bilinear functional satisfies

$$\langle p(z), q(z)\rangle_{\mathcal{L}_j} = \langle p(z), q(z)\rangle_{\mathcal{L}} + M_j\langle z^j p(z), q(z)\rangle_{\mathcal{L}_\theta} + \overline{M}_j\langle p(z), z^j q(z)\rangle_{\mathcal{L}_\theta}, \tag{11}$$

where $M_j \in \mathbb{C}$, $p, q \in \mathbb{P}$, $j \in \mathbb{N}$ is fixed, and $\langle\cdot,\cdot\rangle_{\mathcal{L}_\theta}$ is the bilinear functional associated with the normalized Lebesgue measure in the unit circle.

If $\mathcal{L}$ is quasi-definite, necessary and sufficient conditions in order for $\mathcal{L}_j$ to be quasi-definite, as well as the connection formula between the corresponding families of monic orthogonal polynomials, were obtained in [2]. If $\mathcal{L}$ is a positive definite linear functional, then the above transformation can be expressed in terms of the corresponding measures as

$$d\tilde{\sigma}_j = d\sigma + M_j z^j \frac{d\theta}{2\pi} + \overline{M}_j z^{-j} \frac{d\theta}{2\pi}. \tag{12}$$

From (11), one easily sees that

$$\tilde{c}_k = \langle \mathcal{L}_j, z^k \rangle = \langle z^k, 1 \rangle_{\mathcal{L}_j} = \begin{cases} c_k, & \text{si} \quad k \notin \{j, -j\}, \\ c_{-j} + M_j, & \text{si} \quad k = -j, \\ c_j + \overline{M}_j, & \text{si} \quad k = j. \end{cases} \tag{13}$$

In other words, $\mathcal{L}_j$ perturbs the moments c_j and c_{-j} from the sequence of moments associated with $\mathcal{L}$. The rest of the moments remain unchanged. This is, the Toeplitz matrix associated with $\mathcal{L}_j$ is equal to the Toeplitz matrix associated with $\mathcal{L}$, except for the j-th and $-j$-th moments, which are equal to $c_j + \overline{M}_j$ and $c_{-j} + M_j$, respectively. In matrix form, we have

$$\widetilde{\mathbf{T}}_n = \begin{bmatrix} c_0 & \cdots & c_j + \overline{M}_j & c_{j+1} & \cdots & c_n \\ c_{-1} & \cdots & c_{j-1} & c_j + \overline{M}_j & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{-j} + M_j & c_{-j+1} & \cdots & c_0 & \cdots & c_{n-j} \\ c_{-j-1} & c_{-j} + M_j & \cdots & c_{-1} & \cdots & c_{n-j-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & \cdots & c_{-n+j} & \cdots & c_0 \end{bmatrix}, \quad n \geq 0.$$

Clearly, the Caratheódory function associated with $\mathcal{L}_j$ is

$$\widetilde{F}_j(z) = F(z) + 2M_j z^j, \tag{14}$$

which is a linear spectral transformation of $F(z)$ with $A(z) = D(z) = 1$ and $B(z) = 2M_j z^j$.

4. *Analysis of moment perturbations through the Szegő transformation*

From a positive, nontrivial Borel measure α supported in $[-1, 1]$, we can define a positive, nontrivial Borel measure σ supported in $[-\pi, \pi]$ by

$$d\sigma(\theta) = \frac{1}{2} |d\alpha(\cos\theta)|, \tag{15}$$

in such a way that if $d\alpha(x) = \omega(x)dx$, then

$$d\sigma(\theta) = \frac{1}{2} \omega(\cos\theta) |\sin\theta| d\theta. \tag{16}$$

There exists a relation between the corresponding families of orthogonal polynomials (see [5]). On the other hand, since the moments $\{c_n\}_{n\geq 0}$ are real (see [5])), $F(z)$, the Carathéodory function associated with σ has real coefficients. Therefore, we have

$$\mathfrak{Re} F(e^{i\theta}) = \mathfrak{Re} F(e^{i(2\pi-\theta)}),$$

and then $d\sigma(\theta) + d\sigma(2\pi - \theta) = 0$. Thus, there exists a simple relation between the Stieltjes and Carathéodory functions associated with α and σ, respectively, given by

$$F(z) = \frac{1-z^2}{2z} \int_{-1}^{1} \frac{d\alpha(t)}{x-t} = \frac{1-z^2}{2z} S(x), \tag{17}$$

where $x = \frac{z+z^{-1}}{2}$ and $z = x + \sqrt{x^2-1}$ (see [9]). In the literature, this relation is known as the Szegő transformation. Conversely, if σ is a positive, nontrivial Borel measure with support in the unit circle such that its moments are real, then there exists a positive, nontrivial Borel measure α, supported in $[-1,1]$, such that (15) holds. This is called the inverse Szegő transformation.

In this Section, we consider some perturbations of sequences of moments associated with measures supported in $[-1,1]$ and then analyze the type of perturbation obtained in the corresponding measure supported on the unit circle, when both measures are related through the Szegő transformation. The main tool is the relation (17).

4.1. Perturbation of the j-th moment. Case $a = 0$

Let α be a nontrivial probability measure supported in $[-1,1]$ and let $\{\nu_n\}_{n\geq 0}$ be its corresponding sequence of moments, associated with the basis $\{1, x-a, (x-a)^2, \ldots\}$. If we apply the perturbation (4), then we obtain a measure whose corresponding sequence of moments coincide with $\{\nu_n\}_{n\geq 0}$, except for the j-th moment, which is perturbed by adding to it a mass m_j. Notice that, in order to apply the Szegő transformation (15), the perturbed linear functional $\mathcal{L}_j$ has to be positive definite (i.e. it has an integral representation in terms of a positive measure). More generally, we can define the Szegő transformation for quasi-definite linear functionals using (17).

Assume $a = 0$,[1] so that the Stieltjes function is

$$\tilde{S}_j(x) = S(x) + \frac{m_j}{x^{j+1}}, \tag{18}$$

where $S(x) = \sum_{k=0}^{\infty} \frac{\nu_k}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}}$, since in this case the two bases coincide. Applying the Szegő transformation (17) to (18), and using $x = \frac{z+z^{-1}}{2}$, we obtain the Carathéodory function

$$\begin{aligned} \tilde{F}(z) &= \frac{1-z^2}{2z} \tilde{S}_j(x) \\ &= \frac{1-z^2}{2z} \left(\frac{2z}{1-z^2} F(z) + \frac{m_j}{x^{j+1}} \right) \\ &= F(z) + \frac{1-z^2}{2z} \frac{m_j}{(\frac{z+z^{-1}}{2})^{j+1}}, \end{aligned}$$

[1] This means that the mass associated with the Dirac's delta is added at the origin. Furthermore, if $j = 0$, then $\mathcal{L}_0$ will be positive definite if $m_0 > 0$.

which becomes

$$\tilde{F}(z) = F(z) + \frac{2^j m_j z^j (1-z^2)}{(z^2+1)^{j+1}}. \tag{19}$$

Notice that, from the relation $z = x + \sqrt{x^2-1}$, the mass point at $x = 0$ gives two mass points $z = \pm i$, which are the poles of $\tilde{F}(z)$. Furthermore, since we have poles at $z = \pm i$ with multiplicity $j+1$, the corresponding measure on the unit circle can be obtained directly using the general Riesz-Herglotz transform. Naturally, the perturbation only affects the singular part of the measure. From (19), we can deduce the moments associated with the perturbed Carathéodory function or, in other words, the effect on the moments associated with σ caused by the perturbation.

Comparing (19) with (14), we obtain

$$M_j = \frac{2^{j-1} m_j (1-z^2)}{(z^2+1)^{j+1}}. \tag{20}$$

From this, we conclude that a perturbation of the j-th moment associated with a measure supported in $[-1,1]$ does not translate to a perturbation on the j-th moment associated with the corresponding measure supported in the unit circle, when both measures are related through the Szegő transformation. In other words, a perturbation on the j-th anti diagonal of a Hankel matrix does not result in a perturbation of the j-th sub diagonal of the corresponding Toeplitz matrix.

In order to determine the type of perturbation obtained, set

$$\tilde{F}(z) = F(z) + 2z^j R(z), \tag{21}$$

where

$$R(z) = \frac{2^{j-1} m_j (1-z^2)}{(z^2+1)^{j+1}} = 2^{j-1} m_j (1-z^2) \left(\frac{1}{(z^2+1)^{j+1}} \right). \tag{22}$$

To deduce the perturbed moments, we will write (22) as a power series. Notice that

$$\frac{1}{(z^2+1)^{j+1}} = \sum_{n=j}^{\infty} (-1)^{n-j} \binom{n}{j} z^{2(n-j)}, \tag{23}$$

and replacing (23) in (22), we obtain

$$\begin{aligned} R(z) &= 2^{j-1} m_j (1-z^2) \sum_{n=j}^{\infty} (-1)^{n-j} \binom{n}{j} z^{2(n-j)} \\ &= 2^{j-1} m_j (1-z^2) \sum_{n=0}^{\infty} (-1)^n \binom{n+j}{j} z^{2n}. \end{aligned} \tag{24}$$

Substituting (24) in (21), we get

$$\begin{aligned} \tilde{F}(z) &= F(z) + 2z^j \left(2^{j-1} m_j (1-z^2) \sum_{n=0}^{\infty} (-1)^n \binom{n+j}{j} z^{2n} \right) \\ &= F(z) + 2^j m_j \sum_{n=0}^{\infty} (-1)^n \binom{n+j}{j} z^{2n+j} - 2^j m_j \sum_{n=0}^{\infty} (-1)^n \binom{n+j}{j} z^{2n+2+j}. \end{aligned} \tag{25}$$

From (25) we conclude that, if the $j-$th moment associated with α is perturbed and we apply the Szegő transformation, then the obtained perturbation in $F(z)$ corresponds to a perturbation of the moments associated with σ in the following way:

- If j is even, all even moments starting from c_j are perturbed.
- If j is odd, all odd moments starting from c_j are perturbed.

Also, the explicit perturbation is

$$\tilde{c}_{-n} = \begin{cases} c_{-n}, & \text{if} \quad n < j \text{ or } n = j+2k+1, \\ & \qquad \text{for } k \in \mathbb{N}, \\ c_{-n} + i^{n-j} 2^{j-1} m_j \left(\binom{(n+j)/2}{j} + \binom{(n+j-2)/2}{j} \right), & \text{if} \quad n = j+2k, \\ & \qquad \text{for } k \in \mathbb{N}, \end{cases} \tag{26}$$

with $\binom{j-1}{j} := 0$. We summarize our findings in the following Proposition.

Proposition 4.1. *Let α be a nontrivial positive Borel measure with support in $[-1,1]$ and let σ be its associated measure with support in the unit circle, defined through the Szegő transformation. Let $\{\nu_n\}_{n\geq 0}$ and $\{c_n\}_{n\in\mathbb{Z}}$ be their corresponding sequences of moments. If we apply the perturbation (4) with parameter $a = 0$ to α, then the perturbed measure $\tilde{\sigma}$ obtained in the unit circle is a linear spectral transformation of σ, with moments given by*

$$\tilde{c}_{-n} = c_{-n} + i^{n-j} 2^{j-1} m_j \left(\binom{(n+j)/2}{j} + \binom{(n+j-2)/2}{j} \right), \quad n = j+2k, \quad k \geq 0,$$

with the convention $\binom{j-1}{j} := 0$. The remaining moments remain unchanged. Furthermore, the perturbation affects only the singular part of σ.

Example 4.2. *If $j = 0$ in (25), then the perturbed Carathéodory function is*

$$\tilde{F}_0(z) = F(z) + m_0 + 2m_0 \sum_{n=1}^{\infty} (-1)^n z^{2n},$$

and, as a consequence, the perturbed moments associated with the measure σ are

$$\tilde{c}_{-n} = \begin{cases} c_{-n}, & \text{if} \quad n \text{ is odd}, \\ c_{-n} + i^n m_0, & \text{if} \quad n \text{ is even}, \end{cases} \tag{27}$$

or, equivalently,

$$\tilde{c}_{-n} = c_{-n} + 2i^n \left(\left\lceil \frac{n+1}{2} \right\rceil - \frac{n+1}{2} \right) m_0, \tag{28}$$

where $\lceil \cdot \rceil$ is the ceiling function. This means that all even moments associated with σ are perturbed. In matrix form, we have

$$\widetilde{\boldsymbol{H}} = \begin{bmatrix} \mu_0 + m_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_2 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_1 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\Downarrow$$

Szegő transformation

$$\Downarrow$$

$$\widetilde{\boldsymbol{T}} = \begin{bmatrix} c_0 + m_0 & c_1 & c_2 - m_0 & c_3 & \cdots \\ c_{-1} & c_0 + m_0 & c_1 & c_2 - m_0 & \cdots \\ c_{-2} - m_0 & c_{-1} & c_0 + m_0 & c_1 & \cdots \\ c_{-3} & c_{-2} - m_0 & c_{-1} & c_0 + m_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For instance, for the third kind Chebyshev polynomials, the orthogonality measure is

$$d\alpha(x) = 2\sqrt{\frac{1-x}{1+x}}\frac{dx}{\pi},$$

supported in $[-1,1]$. *Applying the Szegő transformation, the corresponding measure on the unit circle is*

$$\begin{aligned} d\sigma &= (1-\cos\theta)\frac{d\theta}{2\pi} \\ &= |z-1|^2\frac{d\theta}{2\pi}, \end{aligned}$$

i.e., a Christoffel transformation of the normalized Lebesgue measure, with parameter 1. *The corresponding moments are*

$$c_{-n} = \begin{cases} 1, & \textit{if} \quad n = 0, \\ -\frac{1}{2}, & \textit{if} \quad n = 1 \ \textit{or}\ n = -1, \\ 0, & \textit{otherwise.} \end{cases}$$

Thus, if the first moment μ_0 *of the Hankel matrix is perturbed by adding it a real mass* m_0, *the perturbed Carathéodory function is*

$$\tilde{F}_0(z) = 1 - z + m_0 + 2m_0\sum_{n=1}^{\infty} z^{2n},$$

which means that the perturbed moments are

$$c_{-n} = \begin{cases} 1 + m_0, & \textit{if} \quad n = 0, \\ -\frac{1}{2}, & \textit{if} \quad n = 1 \ \textit{or}\ n = -1, \\ i^n m_0, & \textit{if} \quad n \ \textit{is even.} \\ 0, & \textit{if} \quad n \ \textit{is odd, different from} \ \pm 1. \end{cases}$$

The perturbed Toeplitz matrix is

$$\widetilde{\boldsymbol{T}} = \begin{bmatrix} 1+m_0 & -1/2 & -m_0 & & \cdots \\ -1/2 & 1+m_0 & -1/2 & -m_0 & \cdots \\ -m_0 & -1/2 & 1+m_0 & -1/2 & \cdots \\ 0 & -m_0 & -1/2 & 1+m_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and the perturbed measure $\tilde{\sigma}$ on the unit circle is

$$\begin{aligned} d\tilde{\sigma} &= (1-\cos\theta)\frac{d\theta}{2\pi} + d\tilde{\sigma}_s, \\ &= |z-1|^2 \frac{d\theta}{2\pi} + d\tilde{\sigma}_s. \end{aligned}$$

4.2. Perturbation of the j-th moment. Case $a \neq 0$

Let α be a nontrivial probability Borel measure supported in $[-1,1]$, and let $\{\upsilon_n\}_{n\geq 0}$ be its corresponding sequence of moments, associated with the monomial basis $\{1, x, x^2, \cdots\}$, i.e.,

$$\upsilon_k = \int_{-1}^{1} x^k d\alpha(x), \quad k \geq 0. \tag{29}$$

The use of the monomial basis is required because the Szegő transformation (17) is defined for the Stieltjes function that uses this basis. Applying the perturbation (4) to α, it is not difficult to see that the perturbed sequence of moments, $\{\tilde{\upsilon}_n\}_{n\geq 0}$, is given by

$$\tilde{\upsilon}_k = \int_{-1}^{1} x^k d\alpha(x) = \begin{cases} \upsilon_k, & \text{if} \quad k < j, \\ \upsilon_j + m_j, & \text{if} \quad k = j, \\ \upsilon_k + m_j \binom{k}{j} a^{k-j}, & \text{if} \quad k > j, \end{cases} \tag{30}$$

i.e., all moments υ_k, $k \geq j$, are perturbed. Again, we will assume that the perturbed linear functional $\mathcal{L}_j$ is positive definite.

As a consequence, the perturbed Stieltjes function is

$$\tilde{S}_j(x) = S(x) + \sum_{k=j}^{\infty} \binom{k}{j} \frac{m_j}{x^{k+1}} a^{k-j}. \tag{31}$$

Now, we apply the Szegő transformation to (31), in order to determine the perturbation obtained in the corresponding measure on the unit circle. We have

$$\begin{aligned} \tilde{F}_j(z) &= \frac{1-z^2}{2z}\left(S(x) + \sum_{k=j}^{\infty} \binom{k}{j} \frac{m_j}{x^{k+1}} a^{k-j}\right) \\ &= F(z) + \frac{1-z^2}{2z} \sum_{k=j}^{\infty} \binom{k}{j} \frac{m_j a^{k-j} (2z)^{k+1}}{(z^2+1)^{k+1}} \\ &= F(z) + (1-z^2) m_j \sum_{k=j}^{\infty} \binom{k}{j} a^{k-j} (2z)^k \left(\frac{1}{(z^2+1)^{k+1}}\right), \end{aligned} \tag{32}$$

and replacing (23) in (32), we obtain

$$\tilde{F}_j(z) = F(z) + (1-z^2)m_j \sum_{k=j}^{\infty}\left(\binom{k}{j}a^{k-j}(2z)^k\left(\sum_{n=0}^{\infty}(-1)^n\binom{n+k}{k}z^{2n}\right)\right). \tag{33}$$

Now (33) can be rewritten as

$$\begin{aligned}
&\sum_{k=j}^{\infty}\left(\binom{k}{j}a^{k-j}(2z)^k\left(\sum_{n=0}^{\infty}(-1)^n\binom{n+k}{k}z^{2n}\right)\right)\\
&=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}(-1)^{n+k}a^{2n}2^{j+2n}\binom{j+2n}{j}\binom{j+k+n}{j+2n}\right)z^{j+2k}\\
&+\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}(-1)^{n+k}a^{2n+1}2^{j+1+2n}\binom{j+1+2n}{j}\binom{j+1+k+n}{j+1+2n}\right)z^{j+1+2k},
\end{aligned} \tag{34}$$

and, therefore, using (34) and (33), the perturbed Carathédory function is

$$\begin{aligned}
\tilde{F}_j(z) = F(z) &+ 2m_j\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}(-1)^{n+k}a^{2n}2^{j-1+2n}\binom{j+2n}{j}\binom{j+k+n}{j+2n}\right)z^{j+2k}\\
&+2m_j\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}(-1)^{n+k}a^{2n+1}2^{j+2n}\binom{j+1+2n}{j}\binom{j+1+k+n}{j+1+2n}\right)z^{j+1+2k}\\
&-2m_j\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}(-1)^{n+k}a^{2n}2^{j-1+2n}\binom{j+2n}{j}\binom{j+k+n}{j+2n}\right)z^{j+2k+2}\\
&-2m_j\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}(-1)^{n+k}a^{2n+1}2^{j+2n}\binom{j+1+2n}{j}\binom{j+1+k+n}{j+1+2n}\right)z^{j+3+2k}.
\end{aligned} \tag{35}$$

From the previous expression, we conclude that if the perturbation (4) is applied to a measure on $[-1,1]$ in such a way that all moments starting from υ_j are perturbed, then all moments associated with the measure on the unit circle, starting from c_j, are perturbed. Namely, we have

$$\tilde{c}_{-(j+k)} = \begin{cases} c_0 + 2A_1(0), & \text{if } \; j=k=0,\\ c_{-(j+k)} + A_1(k), & \text{if } \; k \text{ is even},\\ c_{-(j+k)} + A_2(k), & \text{if } \; k \text{ is odd}, \end{cases} \tag{36}$$

where

$$\begin{aligned}
A_1(k) = m_j &\sum_{n=0}^{k/2}(-1)^{n+k/2}a^{2n}2^{j-1+2n}\binom{j+2n}{j}\binom{j+k/2+n}{j+2n}\\
&- m_j\sum_{n=0}^{k/2-1}(-1)^{n+k/2-1}a^{2n}2^{j-1+2n}\binom{j+2n}{j}\binom{j+(k/2-1)+n}{j+2n},
\end{aligned}$$

and

$$A_2(k) = m_j \sum_{n=0}^{(k-1)/2} (-1)^{n+(k-1)/2} a^{2n+1} 2^{j+2n} \binom{j+1+2n}{j} \binom{j+1+(k-1)/2+n}{j+1+2n}$$
$$- m_j \sum_{n=0}^{(k-3)/2} (-1)^{n+(k-3)/2} a^{2n+1} 2^{j+2n} \binom{j+1+2n}{j} \binom{j+1+(k-3)/2+n}{j+1+2n}.$$

In order to compute the absolutely continuous part of the perturbed measure in the unit circle, notice that

$$\begin{aligned} \tilde{F}_j(e^{i\theta}) &= \lim_{r\to 1} \left(F(re^{i\theta}) + (1-(re^{i\theta})^2) m_j \sum_{k=j}^{\infty} \binom{k}{j} a^{k-j} (2re^{i\theta})^k \left(\frac{1}{((re^{i\theta})^2+1)^{k+1}} \right) \right) \\ &= F(e^{i\theta}) + (1-e^{2i\theta}) m_j \sum_{k=j}^{\infty} \binom{k}{j} a^{k-j} (2e^{i\theta})^k \left(\frac{1}{(e^{2i\theta}+1)^{k+1}} \right), \end{aligned} \tag{37}$$

and thus

$$\begin{aligned} \overline{\tilde{F}_j(z)} &= \overline{F(z)} + (1-e^{-2i\theta}) m_j \sum_{k=j}^{\infty} \binom{k}{j} a^{k-j} (2e^{-i\theta})^k \left(\frac{1}{(e^{-2i\theta}+1)^{k+1}} \right) \\ &= \overline{F(e^{i\theta})} - (1-e^{2i\theta}) m_j \sum_{k=j}^{\infty} \binom{k}{j} a^{k-j} (2e^{i\theta})^k \left(\frac{1}{(e^{2i\theta}+1)^{k+1}} \right). \end{aligned} \tag{38}$$

As a consequence, we obtain

$$\tilde{\sigma}'(\theta) = \mathfrak{Re}(\tilde{F}_j(e^{i\theta})) = \sigma'(\theta), \tag{39}$$

where, again, we assume that $\tilde{F}(z)$ is analytic and has real part in $\mathbb{D}$. Notice that this implies that a is a real number with $|a| < 1$. As before, the absolutely continuous part of the measure remains invariant with respect to the Szegő transformation, and we have

$$\begin{aligned} d\tilde{\sigma} &= \tilde{\sigma}'(\theta) \frac{d\theta}{2\pi} + d\tilde{\sigma}_s(\theta) \\ &= \sigma'(\theta) \frac{d\theta}{2\pi} + d\tilde{\sigma}_s(\theta). \end{aligned} \tag{40}$$

As a conclusion, we have the following result.

Proposition 4.3. *Let α, σ, $\{\nu_n\}_{n\geq 0}$ and $\{c_n\}_{n\in\mathbb{Z}}$ be as in Proposition 4.1. If we apply the perturbation (4) with parameter $a \neq 0$ to α, then the perturbed measure $\tilde{\sigma}$ obtained in the unit circle is a linear spectral transformation of σ, affecting only the singular part of σ, with moments given by $\tilde{c}_i = c_i$ for $0 \leq i \leq j-1$ and*

$$\tilde{c}_{-(j+k)} = \begin{cases} c_0 + 2A_1(0), & \text{if} \quad j = k = 0, \\ c_{-(j+k)} + A_1(k), & \text{if} \quad k > 0 \text{ is even}, \\ c_{-(j+k)} + A_2(k), & \text{if} \quad k > 0 \text{ is odd}, \end{cases}$$

where $A_1(k)$ and $A_2(k)$ are given above.

Example 4.4. *Setting $j=0$ in (30) and (36), we have*

$$\tilde{\upsilon}_k = \begin{cases} \upsilon_0 + m_0, & \text{if} \quad k=0, \\ \upsilon_k + m_0 k a^k, & \text{if} \quad k>0, \end{cases} \tag{41}$$

and

$$\tilde{c}_{-k} = \begin{cases} c_0 + 2A_1(0), & \text{if} \quad k=0, \\ c_{-k} + A_1(k), & \text{if} \quad k \text{ is even}, \\ c_{-k} + A_2(k), & \text{if} \quad k \text{ is odd}, \end{cases} \tag{42}$$

where

$$\begin{aligned} A_1(k) &= m_0 \sum_{n=0}^{k/2} (-1)^{n+k/2} a^{2n} 2^{-1+2n} \binom{2n}{0} \binom{k/2+n}{2n} \\ &- m_0 \sum_{n=0}^{k/2-1} (-1)^{n+k/2-1} a^{2n} 2^{-1+2n} \binom{2n}{0} \binom{(k/2-1)+n}{2n} \\ &= \frac{m_0}{2} \left((2a)^k + \sum_{n=0}^{\frac{k}{2}-1} (-1)^{n+\frac{k}{2}} (2a)^{2n} \left[\binom{\frac{k}{2}+n}{2n} + \binom{\frac{k}{2}-1+n}{2n} \right] \right), \end{aligned}$$

and

$$\begin{aligned} A_2(k) &= m_0 \sum_{n=0}^{(k-1)/2} (-1)^{n+(k-1)/2} a^{2n+1} 2^{2n} \binom{1+2n}{0} \binom{1+(k-1)/2+n}{1+2n} \\ &- m_0 \sum_{n=0}^{(k-3)/2} (-1)^{n+(k-3)/2} a^{2n+1} 2^{2n} \binom{1+2n}{0} \binom{1+(k-3)/2+n}{1+2n} \\ &= \frac{m_0}{2} \left((2a)^k + \sum_{n=0}^{\frac{k-3}{2}} (-1)^{n+\frac{k-1}{2}} (2a)^{2n+1} \left[\binom{\frac{k+1}{2}+n}{1+2n} + \binom{\frac{k-1}{2}+n}{1+2n} \right] \right). \end{aligned}$$

From (42), the first perturbed moments on the unit circle are $\tilde{c}_0 = c_0 + 2A_1(0) = c_0 + m_0$, $\tilde{c}_{-1} = c_{-1} + A_2(1) = c_{-1} + m_0 a$, $\tilde{c}_{-2} = c_{-2} + A_1(2) = c_{-2} - m_0 + 2m_0 a^2$, $\tilde{c}_{-3} = c_{-3} + A_2(3) = c_{-3} - 3m_0 a + 4m_0 a^3$, $\tilde{c}_{-4} = c_{-4} + A_1(4) = c_{-4} + m_0 - 8m_0 a^2 + 8m_0 a^4$, *and in matrix form we have*

$$\widetilde{\boldsymbol{H}} = \begin{bmatrix} v_0+m_0 & v_1+m_0a & v_2+2m_0a^2 & v_3+3m_0a^3 & \cdots \\ v_1+m_0a & v_2+2m_0a^2 & v_3+3m_0a^3 & v_4+4m_0a^4 & \cdots \\ v_2+2m_0a^2 & v_3+3m_0a^3 & v_4+4m_0a^4 & v_5+5m_0a^5 & \cdots \\ v_3+3m_0a^3 & v_4+4m_0a^4 & v_5+5m_0a^5 & v_6+6m_0a^6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\Downarrow$$

$$\textit{Szegő transformation}$$

$$\Downarrow$$

$$\widetilde{\boldsymbol{T}} = \begin{bmatrix} c_0+m_0 & c_1+m_0a & c_2-m_0+2m_0a^2 & c_3-3m_0a+4m_0a^3 & \cdots \\ c_{-1}+m_0a & c_0+m_0 & c_1+m_0a & c_2-m_0+2a^2 & \cdots \\ c_{-2}-m_0+2m_0a^2 & c_{-1}+m_0a & c_0+m_0 & c_1+m_0a & \cdots \\ c_{-3}-3m_0a+4m_0a^3 & c_{-2}-m_0+2a^2 & c_{-1}+m_0a & c_0+m_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Notice that setting $a = 0$, we obtain the same results as in the previous subsection.

Remark 4.5. Notice that the previous results can be easily generalized to the case when the perturbation is applied to a finite number of moments, since each perturbed moment adds an additional term to the power series on the Stieltjes and Carathéodory functions.

5. *Analysis of moment perturbations through the inverse Szegő transformation*

Let σ be a positive measure supported in the unit circle such that its corresponding moments $\{c_n\}_{n\in\mathbb{Z}}$ are real. Assume also that the perturbed measure $\tilde{\sigma}_j$, defined by (12), is positive and that M_j is real, so the moments associated with $\tilde{\sigma}_j$ are also real. Our goal in this Section is to determine the relation between the positive Borel measures α and $\tilde{\alpha}_j$, supported in $[-1, 1]$, which are associated with σ and $\tilde{\sigma}_j$, respectively, via the inverse Szegő transformation. As in the previous Section, this relation will be stated in terms of the corresponding sequences of moments.

Notice that, setting $z = e^{i\theta}$, $x = \cos\theta$, and taking into account that the inverse Szegő transformation applied to the normalized Lebesgue measure $d\theta/2\pi$ yields the first kind Chebyshev measure $\frac{dx}{\pi\sqrt{1-x^2}}$, the measure $\tilde{\alpha}_j$ obtained by applying the inverse Szegő transformation to $\tilde{\sigma}_j$ is given by

$$\begin{aligned} d\tilde{\alpha}_j &= d\alpha + M_j(x+i\sqrt{1-x^2})^j \frac{dx}{\pi\sqrt{1-x^2}} + M_j(x+i\sqrt{1-x^2})^{-j}\frac{dx}{\pi\sqrt{1-x^2}} \\ &= d\alpha + M_j(\cos(j\theta)+i\sin(j\theta))\frac{dx}{\pi\sqrt{1-x^2}} + M_j(\cos(j\theta)-i\sin(j\theta))\frac{dx}{\pi\sqrt{1-x^2}} \\ &= d\alpha + 2M_j\frac{\cos(j\theta)dx}{\pi\sqrt{1-x^2}} \\ &= d\alpha + 2M_j\frac{T_j(x)}{\pi}\frac{dx}{\sqrt{1-x^2}}, \end{aligned}$$

where $T_j(x) := \cos(j\theta)$ are the Chebyshev polynomials of the first kind. Notice that a measure that changes its sign in the interval $[-1,1]$ is added to $d\alpha$. Then, the moments associated with $\tilde{\alpha}_j$ are given by

$$\tilde{\mu}_n = \int_{-1}^{1} x^n d\tilde{\alpha}(x) = \mu_n + \frac{2M_j}{\pi}\int_{-1}^{1} x^n T_j(x)\frac{dx}{\sqrt{1-x^2}}.$$

As a consequence, by the orthogonality of $T_j(x)$, we obtain

$$\tilde{\mu}_n = \begin{cases} \mu_n, & \text{if} \quad 0 \leq n < j, \\ \mu_n + \frac{2M_j}{\pi}\int_{-1}^{1} x^n T_j(x)\frac{dx}{\sqrt{1-x^2}}, & \text{if} \quad j \leq n. \end{cases} \tag{43}$$

Notice that, because of the symmetry, the integral above is different from zero if n and j are both odd or both even, and vanish otherwise. This means that

$$\tilde{\mu}_n = \begin{cases} \mu_n + \frac{2M_j}{\pi}\int_{-1}^{1} x^n T_j(x)\frac{dx}{\sqrt{1-x^2}}, & \text{if} \quad j \geq n,\ n+j \text{ is even}, \\ \mu_n, & \text{otherwise}. \end{cases} \tag{44}$$

In other words, if j is even (odd), all even (odd) moments starting from μ_j are perturbed. Furthermore (see [6]), we have

$$T_j(x) = \frac{j}{2}\sum_{k=0}^{[j/2]} \frac{(-1)^k (j-k-1)!(2x)^{j-2k}}{k!(j-2k)!}, \quad j = 1,2,3,...,$$

where $[j/2] = j/2$ if j is even and $[j/2] = (j-1)/2$ if j is odd. Therefore, if $j \geq n$ and $n+j$ is even, we have

$$\begin{aligned}\int_{-1}^{1} x^n T_j(x)\frac{dx}{\sqrt{1-x^2}} &= \frac{j}{2}\int_{-1}^{1} x^n \sum_{k=0}^{[j/2]} \frac{(-1)^k (j-k-1)!(2x)^{j-2k}}{k!(j-2k)!}\frac{dx}{\sqrt{1-x^2}} \\ &= \frac{j}{2}\sum_{k=0}^{[j/2]} \frac{(-1)^k (j-k-1)!(2)^{j-2k}}{k!(j-2k)!}\int_{-1}^{1} x^{j+n-2k}\frac{dx}{\sqrt{1-x^2}};\end{aligned}$$

and, since

$$\int_{-1}^{1} x^k \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & \text{if} \quad k=0, \\ 0, & \text{if} \quad \text{k is odd}, \\ \left(\prod_{i=1}^{k/2}\frac{k-(2i-1)}{k-2(i-1)}\right)\pi, & \text{if} \quad \text{k is even}, \end{cases}$$

and $j+n-2k$ is even, we get

$$\int_{-1}^{1} x^{j+n-2k}\frac{dx}{\sqrt{1-x^2}} = \left(\prod_{i=1}^{(j+n-2k)/2}\frac{j+n-2k-(2i-1)}{j+n-2k-2(i-1)}\right)\pi.$$

The previous integral can also be computed using the Residue's Theorem, taking into account that the change of variables $x = (z + z^{-1})/2$ yields an integral along the unit circle $\mathbb{T}$. Indeed, the integral is zero when $n + j$ is an odd integer number. Otherwise, one gets

$$\frac{1}{2i}\int_{\mathbb{T}}\left(\frac{z+z^{-1}}{2}\right)^n\left(\frac{z^j+z^{-j}}{2}\right)\frac{dz}{z},$$

and one only needs to find the coefficient of $1/z$ inside the integral. As a consequence, (44) becomes

$$\tilde{\mu}_n = \begin{cases} \mu_n + M_j B(n,j), & \text{if} \quad j \geq n,\ n+j \text{ is even}, \\ \mu_n, & \text{otherwise}, \end{cases} \tag{45}$$

where

$$B(n,j) = j\sum_{k=0}^{[j/2]}\left(\frac{(-1)^k(j-k-1)!(2)^{j-2k}}{k!(j-2k)!}\prod_{i=1}^{(j+n-2k)/2}\frac{j+n-2k-(2i-1)}{j+n-2k-2(i-1)}\right).$$

This means that the perturbation of the moments c_j and c_{-j} of a measure σ supported in the unit circle results in a perturbation, defined by (45), of the moments μ_n associated with a measure α supported in $[-1,1]$, when both measures are related through the inverse Szegő transformation. We summarize our findings in the following Proposition.

Proposition 5.1. *Let σ be a positive nontrivial Borel measure with real moments supported in the unit circle, and let α be its corresponding measure in $[-1,1]$, obtained through the inverse Szegő transformation. Let $\{c_n\}_{n\in\mathbb{Z}}$ and $\{\mu_n\}_{n\geq 0}$ be their corresponding sequences of moments. Assume that $\tilde{\sigma}_j$, defined by (12) with $M_j \in \mathbb{R}$, is positive. Then, the measure $\tilde{\alpha}_j$, obtained by applying the inverse Szegő transformation to $\tilde{\sigma}_j$, is given by*

$$d\tilde{\alpha}_j = d\alpha + 2M_j\frac{T_j(x)}{\pi}\frac{dx}{\sqrt{1-x^2}},$$

and its corresponding sequence of moments is given by (45).

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