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Integrability of a double bracket system

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Abstract. A group factorization approach is used to show the integrability of a system of infinite equations of Lax type with double bracket.
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Integrabilidad de un sistema con doble conmutador

Resumen. Se utiliza un enfoque algebraico basado en la descomposión de grupos para mostrar la integrabilidad de un sistema de infinitas ecuaciones de Lax con doble corchete.
Palabras claves: Ecuación de Lax, jerarquía Brockett, sistema completamente integrable.

1. Introduction

Mulase [9, 10] introduced a remarkable method to obtain solutions of the KP hierarchy. His results on a feasible extension of concepts such as flat connections, gauge transformations, Frobenius integrability, etc. to the space of pseudo-differential operators (infinite dimensional case) made it possible to consider the hierarchy as only one equation. However, the key point that should be emphasized in Mulase [10] is a theorem of factorization for formal series of the form:

$$\sum_{-\infty}^{\infty} a_k \partial^k, \qquad \partial = \frac{d}{dx}.$$

This factorization theorem has a very similar aspect to the Birkhoff decomposition of loop groups and the Riemann-Hilbert problem for functions of complex variable.

Felipe and Ongay [6] showed that Mulase's ideas can be applied in quite a similar form to the discrete KP hierarchy. In this context, a Borel-Gauss factorization for semi-infinite and bi-infinite matrices plays an important role. We also mention a paper by Schiff [11]

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where the Mulase approach is used to prove the complete integrability of the Camassa-Holm hierarchy (this hierarchy contains the now well known Camassa-Holm equation, see [11] for details).

Bloch, Brockett and Ratiu [2] found integrable gradient flows of ODE's. These equations were defined as Lax type equations with more than one Lie bracket in connection with some least squares matching and sorting problems.

A version of PDE's of the Brockett type equations has been introduced by Felipe [5]. A particular feature of these equations is the existence of an infinite number of conserved quantities, and also that they belong to a hierarchy similar to the well known n-KdV or KP hierarchy. It was proved in [5] that each equation of the Brockett hierarchy is equivalent to a certain gradient flow in the space of pseudo-differential operators. Also, Felipe and Ongay [6] have studied a supersymmetric extension of the Brockett hierarchy.

In this paper we will prove the complete integrability of a double bracket system, the so-called Brockett hierarchy, showing its related group factorization. This is a remarkable fact, because, as it is known, a completely integrable system is always related with some kind of group factorization [13]. We apply the same approach used in [7, 10] for the KP hierarchy, to a more general case where the equations of the hierarchy are defined with double bracket instead of only one. In spite of that, from an algebraic point of view [8], the settings are similar in both the cases; the results developed in this article have important particularities.

2. The Brockett hierarchy

The Brockett hierarchy can be introduced in the following form: let L be a Lax operator, i.e., $L = \partial + \sum_{k=1}^{\infty} a_k \partial^{-k}$. Initially the only requirement on the coefficients a_k of L, that these depend on x and an infinite set of temporal variables $t_1, t_2, \ldots$. We recall that the Brockett hierarchy is defined as

$$\frac{\partial L}{\partial t_n} = \left[L, \left[L, L_+^n\right]\right], \qquad n = 1, 2, \ldots, \tag{1}$$

where we use the notation R_+ to indicate the differential part of a pseudo-differential operator R. We also reserve R_- to denote the integral part of R; it is meant that we can write $R_- = R - R_+$. The important point is that, for a Lax operator L, there exists a dressing operator $S = 1 + s_1\partial^{-1} + s_2\partial^{-2} + \cdots$ such that $L = S\partial S^{-1}$. It is easy to show that the operator S is unique up to right multiplication by those operators $C = 1 + c_1\partial^{-1} + c_2\partial^{-2} + \cdots$ for which $[C, \partial] = 0$.

Note that each equation of (1) can be interpreted as the compatibility condition for the following system of equations:

$$L\phi = \phi\partial, \qquad \frac{\partial \phi}{\partial t_n} = \left[L, L_-^n\right]\phi, \qquad n = 1, 2, \ldots$$

where ϕ is an element of the group of dressing operators. Those pseudo-differential dressing operators S for which

$$\frac{\partial S}{\partial t_n} = \left[L, L_-^n\right] S, \qquad n = 1, 2, \ldots \tag{2}$$

where $L = S\partial S^{-1}$, will be of particular importance to us. Such S will be called Sato-Wilson operators.

From now on, we use $E^{(-1)}$ to denote the subalgebra of pseudo-differential operators R, such that $R_+ = 0$. It means that we are considering the pseudo-differential operators $R \in E = E^{(-1)} \oplus D$, where D denotes the subalgebra of differential operators R such that $R_- = 0$. The "nilpotent" part $E^{(-1)}$ of E has a formal closed Lie group $G = 1 + E^{(-1)}$ which acts on E by adjoint action, preserving the order of elements of E, see [9]. One can easily see that, if $S \in G$ is a Sato-Wilson operator, then $L = S\partial S^{-1}$ is a solution of the Brockett hierarchy.

Note that the system (2) can also be written as

$$dS = B^c\left(S\right)S, \tag{3}$$

where $B^c\left(S\right)$ is the 1-form, $B^c\left(S\right) = \sum_{n=1}^{\infty} B_n^-\left(S\right)dt_n$, $B_n^-\left(S\right) = \left[L, L_-^n\right]$ and d is the usual differentiation on the infinite set of temporal variables $t_1, t_2, \dots$. Obviously, this definition of $B^c\left(S\right)$ also makes sense even if S is not Sato-Wilson operator. In the rest of the paper we will consider 1-form with coefficients in the algebra of pseudo-differential operators. It means we will have formal sums that will be manipulated according to the rules of "exterior algebra."

We can see that system (1) is equivalent to

$$dL = [L, -B^c(S)]. \tag{4}$$

Now, it can be shown that equation (2) implies

$$\frac{\partial B_n^-\left(S\right)}{\partial t_m} - \frac{\partial B_m^-\left(S\right)}{\partial t_n} = \left[B_m^-\left(S\right), B_n^-\left(S\right)\right], \qquad n, m = 1, 2, \dots . \tag{5}$$

An equation of the form (5) is called Zakharov-Shabat or zero-curvature equation. The name "zero-curvature" can be explained as follows in the 1-form: $B^c\left(S\right)$ on $\mathbb{C}$ with coefficients in the Lie algebra $E^{(-1)}$ can be written in virtue of (4) as

$$dB^c(S) = B^c(S) \wedge B^c(S), \tag{6}$$

which is the Maure-Cartan equation. If we interpret $B^c\left(S\right)$ as a connection form on the trivial subbundle $\mathbb{C}\times E^{(-1)}$ and S being a Sato-Wilson operator, then from (6) we can conclude that $B^c\left(S\right)$ is flat. An equivalent expression of (6) is

$$dB^c\left(S\right) = \frac{1}{2}\left[B^c\left(S\right), B^c\left(S\right)\right]. \tag{7}$$

In the same way, if we put $B_n^+(S) = L^n + [L, L_-^n]$, then we can define the 1-form

$$\begin{aligned} B(S) &= \sum_{n=1}^{\infty} B_n^+(S)dt_n \\ &= \sum_{n=1}^{\infty}(L^n + [L, L_-^n])dt_n \\ &= \sum_{n=1}^{\infty} L^n dt_n + B^c(S), \end{aligned} \tag{8}$$

with coefficients in the Lie algebra E. Then, system (1) can be written as

$$dL = [L, B(S)]. \tag{9}$$

Again, if S is Sato-Wilson operator, then the connection $B(S)$ on the trivial subbundle $\mathbb{C} \times E$ is flat and analogously the 1-form $B(S)$ satisfies a zero-curvature equation like (7). That is

$$dB(S) = \frac{1}{2}\left[B(S), B(S)\right]. \tag{10}$$

Lemma 2.1. *Let L be a Lax operator such that L satisfies (1) and let $S \in G$ be a dressing operator for L, i.e., $L = S\partial S^{-1}$. Then*

$$\left[\partial, S^{-1}B_n^-(S)\, S - S^{-1}\frac{\partial S}{\partial t_n}\right] = 0 \tag{11}$$

for $n = 1, 2 \dots$.

Proof. Suppose L satisfies (1) and $L = S\partial S^{-1}$. Calculating the left side of (11), we have

$$\begin{aligned}\left[\partial, S^{-1}B_n^-(S)\, S - S^{-1}\frac{\partial S}{\partial t_n}\right] &= S^{-1}\left[L, B_n^-(S) - \frac{\partial S}{\partial t_n}S^{-1}\right] S \\ &= S^{-1}\left(\left[L, \left[L, L_-^n\right]\right] + \left[\frac{\partial S}{\partial t_n}S^{-1}, L\right]\right) S \\ &= S^{-1}\left(-\left[L, \left[L, L_+^n\right]\right] + \left[\frac{\partial S}{\partial t_n}S^{-1}, L\right]\right) S \\ &= S^{-1}\left(-\frac{\partial L}{\partial t_n} + \frac{\partial L}{\partial t_n}\right) S \\ &= 0,\end{aligned}$$

which is the desired result. ☑

From Lemma 1 it follows that if $L = S\partial S^{-1}$ satisfies (1) then the gauge transformation of $B(S)$, $H^c(S) = S^{-1}B^c(S)S - S^{-1}dS$ has only constant coefficients. It should be remarked that in the suppositions of the Lemma 2.1 the dressing operator S of L is not assumed to be a Sato-Wilson operator. In particular, if S is Sato-Wilson, then $H^c(S) = 0$.

Let Ω be the 1-form, $\Omega = \sum_{n=1}^{\infty} \partial^n dt_n$. Then Ω satisfies the integrability condition $d\Omega = \Omega \wedge \Omega$, because $d\Omega = 0$ and $\Omega \wedge \Omega = 0$.

Proposition 2.2. *Let $S \in G$ be a Sato-Wilson operator and define*

$$H(S) = S^{-1}B(S)S - S^{-1}dS.$$

Then $H(S) = \Omega$.

Proof. Using (8), we obtain

$$\begin{aligned} H(S) &= S^{-1}\left(\sum_{n=1}^{\infty}\left(L^n - [L, L_+^n]\right) dt_n\right) S - S^{-1}dS \\ &= \sum_{n=1}^{\infty} \partial^n dt_n + S^{-1}B^c(S)S - S^{-1}dS \\ &= \Omega + H^c(S). \end{aligned}$$

Now, noting that $H^c(S) = 0$, we have $H(S) = \Omega$. ☑

Proposition 2 indicates that $B(S)$ will be a flat connection on the trivial bundle $\mathbb{C} \times E$, because Ω is clearly a flat connection and

$$B(S) = S\Omega S^{-1} - SdS^{-1}.$$

3. Some auxiliary problems

Let G_0 be the set of all dressing operators corresponding to solutions L of the Brockett hierarchy. Note that if S is a Sato-Wilson operator, then $S \in G_0$. We start with the study of Cauchy problem

$$dY = B(S)Y, \qquad\qquad Y(0) = I, \tag{12}$$

where $S \in G_0$. It is suitable to find the solution of (12) in the form $Y = SZ$. For this we write the corresponding Cauchy problem as

$$dZ = \left(S^{-1}B(S)S - S^{-1}dS\right) Z, \qquad\qquad Z(0) = S^{-1}(0).$$

Lemma 3.1. *If $S \in G_0$, then*

$$\left[\left(S^{-1}B(S)S - S^{-1}dS\right), \partial\right] = 0.$$

Proof. We have

$$\begin{aligned} S^{-1}B(S)S - S^{-1}dS &= S^{-1}\left(\sum_{n=1}^{\infty} L^n - \left[L, L_+^n\right]\right) S - S^{-1}dS \\ &= \sum_{n=1}^{\infty} \partial^n dt_n + \sum_{n=1}^{\infty}\left(S^{-1}\left[L, L_-^n\right] S - S^{-1}\frac{\partial S}{\partial t_n}\right) dt_n. \end{aligned}$$

Now, from Lemma 2.1, we get

$$\left[\left(S^{-1}\left[L, L_-^n\right] S - S^{-1}\frac{\partial S}{\partial t_n}\right), \partial\right] = 0,$$

from which the proof is immediate. ☑

Remark 3.2. Note that if S is Sato-Wilson, then Lemma 3 is evident from Proposition 2.

Let $H(S)$ be the 1-form defined as

$$H(S)=\sum_{n=1}^{\infty} H_n\left(S\right)dt_n.$$

Then, we must resolve the Cauchy problem

$$dZ=H\left(S\right)Z, \qquad\qquad Z(0)=S^{-1}(0) \tag{13}$$

for $S\in G_0$. Notice that if S is Sato-Wilson, then $H(S)=\sum_{n=1}^{\infty}\partial^n dt_n=\Omega$. In this case the coefficient $H\left(S\right)$ in (13) does not depend on x, but it still depends on the temporal variables $t_1,t_2,\ldots$.

Let G_1 be the set of all elements $S\in G_0$ for which there exists $\Theta\left(S,t_1,t_2,\ldots\right)$ such that $d\Theta\left(S,t_1,t_2,\ldots\right)=H\left(S\right)$. It is equivalent to say that $H\left(S\right)$ has a primitive function $\Theta\left(S,t_1,t_2,\ldots\right)=\sum_{n=1}^{\infty}\Theta_n\left(S,t_1,t_2,\ldots\right)$. From Lemma 3.1, we have that if $\Theta\left(S,t_1,t_2,\ldots\right)$ exists, then $\left[\Theta\left(S,t_1,t_2,\ldots\right),\partial\right]=0$ and also $\Theta\left(S,0,0,\ldots\right)=0$. Observe that if S is Sato-Wilson, then $S\in G_1$, which means that G_1 is not empty.

Remark 3.3. Assume that $S\in G_0$ and

$$H\left(S\right)=\sum_{n=1}^{\infty}H_n\left(S\right)dt_n=\sum_{n=1}^{\infty}\frac{\partial\Theta_n\left(S,t_1,t_2,\ldots\right)}{\partial t_n}dt_n.$$

If, for any i and j, we have

$$\frac{H_i\left(S\right)}{\partial t_j}=\frac{H_j\left(S\right)}{\partial t_i},$$

then $S\in G_1$.

On the other hand, $H_n\left(S\right)=\partial^n+C_n\left(S\right)$, where $C_n\left(S\right)\in E^{(-1)}$. Hence if $S\in G_1$, then $\Theta\left(S\right)=\Theta_+ +\Gamma(S)$, with $\left[\Gamma\left(S\right),\partial\right]=0$ and $\Gamma\left(S\right)\in E^{(-1)}$. Moreover, $\Theta_+=\sum_{n=1}^{\infty}t_n\partial^n$. Note that

$$H(S)=d\Theta(S)=d\Theta_+ +d\Gamma(S)=\Omega+d\Gamma(S).$$

In particular, if S is Sato-Wilson then $d\Theta=d\Theta_+=\Omega$ and $d\Gamma=0$.

We say that $\Gamma\left(S\right)\in E^{(-1)}$, $S\in G_1$, is the gauge trace of S if $\Gamma\left(S,t_1,t_2,\ldots\right)$ is the formal series in ∂ such that

$$d\Gamma\left(S\right)=\sum_{n=1}^{\infty}\left(S^{-1}\left[L,L_-^n\right]S-S^{-1}\frac{\partial S}{\partial t_n}\right)dt_n.$$

Theorem 3.4. *Let $S\in G_1$. Then, the general solution of the equation*

$$dZ=H\left(S\right)Z \tag{14}$$

is given by

$$Z_g=e^{\Theta(S)}=e^{\Gamma(S,t_1,t_2,\ldots)}e^{\Theta_+}. \tag{15}$$

Proof. Note that if $S \in G_1$ then there exists $\Theta(S)$ such that $d\Theta(S) = H(S)$ and $\Theta(S) = \Theta_+ + \Gamma(S, t_1, t_2, \ldots)$. For these reasons, by taking $Z = e^{\Theta(S)}$, where $S \in G_1$, we have

$$dZ = e^{\Theta(S)} d\Theta(S) = e^{\Theta(S)} H(S) = H(S) e^{\Theta(S)} = H(S) Z.$$

Hence (15) is a general solution of (14). ☑

Therefore, the corresponding particular solution Z of the Cauchy problem (13) can be written as

$$Z = e^{\Gamma(S,t_1,t_2,\ldots)} e^{\Theta_+} S^{-1}(0). \tag{16}$$

It follows from (16) that formal general solution of (12) is

$$\begin{aligned} Y &= SZ \\ &= S e^{\Gamma(S,t_1,t_2,\ldots)} e^{\Theta_+}. \end{aligned} \tag{17}$$

Note that the right hand side of (17) is formally invertible and $e^{\Gamma(S,t_1,t_2,\ldots)} \in E^{(-1)}$.

Definition 3.5. Define $F(S) = S^{-1} e^{\Gamma(S,t_1,t_2,\ldots)}$, where $S \in G_1$, as the formal Fourier transform of S.

It is clear that $F(S) : G_1 \to G$, and also, if S is Sato-Wilson then $d\Gamma(S, t_1, t_2, \ldots) = 0$, so

$$[\Gamma(S, t_1, t_2, \ldots), \partial] = 0 = [e^{\Gamma(S,t_1,t_2,\ldots)}, \partial].$$

Thus $F(S) = S^{-1} e^{\Gamma(S,t_1,t_2,\ldots)}$ is also Sato-Wilson because of the fact that $e^{\Gamma(S,t_1,t_2,\ldots)}$ is a constant operator.

4. Solutions for the Brockett hierarchy

Denote by $\widehat{E}$ and $\widehat{D}$ the formal Lie groups of the Lie algebras E and D, respectively. We have a group decomposition $\widehat{E} = G \bullet \widehat{D}$ (Lemma 3, [9]) where $G \cap \widehat{D} = \{1\}$.

Let $t = (t_1, t_2, \ldots)$ and let G_B be the set of all S belonging to G for which

(a) there is $\Theta(S, t)$ such that $d\Theta(S, t) = H(S)$,

(b) if $H(S) = S^{-1} B(S) S - S^{-1} dS$, then $[H(S), \partial] = 0$.

Obviously $G_1 \subset G_B$. We can consider the gauge trace $\Gamma(S)$ and the formal Fourier transform $F(S) = S^{-1} e^{\Gamma(S,t)}$ of S.

Theorem 4.1. *The equation* (1) *of the Brockett hierarchy is equivalent to the linear differential equation*

$$dU = \Omega U, \qquad U \in \widehat{E}. \tag{18}$$

Proof. Assume that L is a solution of (1). Then, there exists a Sato-Wilson operator S such that $L = S\partial S^{-1}$. In this case we recall that $H(S) = \Omega$, so $B(S) = S\Omega S^{-1} + dSS^{-1}$ is a flat connection on $\mathbb{C} \times E$. Then, we can find $W \in \widehat{E}$ such that $B(S) = dWW^{-1}$ and define $U = S^{-1}W$ with $W = e^{\Gamma(S,t)}X$, $X \in \widehat{D}$. Thus, $U = S^{-1}e^{\Gamma(S,t)}X$. It is obvious that U satisfies the linear equation (18). In fact,

$$\begin{aligned} dU &= d(S^{-1}e^{\Gamma(S,t)}X) \\ &= dS^{-1} \cdot e^{\Gamma(S,t)}X + S^{-1}d(e^{\Gamma(S,t)}X), \\ &= (-S^{-1}(dS) + B(S)S)S^{-1}e^{\Gamma(S,t)}X = \Omega U. \end{aligned}$$

Conversely, let us assume that $dU = \Omega U$ with $U \in \widehat{E}$. Decompose it into $U = F(S)X$ with $F(S) = S^{-1}e^{\Gamma(S,t)} \in G$ and $X \in \widehat{D}$. Define $L = S\partial S^{-1}$ and $B(S) = d(e^{\Gamma(S,t)}X) \cdot (e^{\Gamma(S,t)}X)^{-1}$. It is clear that

$$\begin{aligned} 0 &= U^{-1}\Omega U - U^{-1}dU = (F(S)X)^{-1}\Omega(F(S)X) - (F(S)X)^{-1}d(F(S)X) \\ &= (S^{-1}e^{\Gamma(S,t)}X)^{-1}\Omega S^{-1}e^{\Gamma(S,t)}X - (S^{-1}e^{\Gamma(S,t)}X)^{-1}d(S^{-1}e^{\Gamma(S,t)}X) \\ &= (e^{\Gamma(S,t)}X)^{-1}(S\Omega S^{-1} - SdS^{-1})e^{\Gamma(S,t)}X - (e^{\Gamma(S,t)}X)^{-1}d(e^{\Gamma(S,t)}X) \\ &= (S\Omega S^{-1} + dS \cdot S^{-1}) - d(e^{\Gamma(S,t)}X) \cdot (e^{\Gamma(S,t)}X)^{-1} \\ &= (S\Omega S^{-1} + dS \cdot S^{-1}) - B(S) \\ &= \Omega + S^{-1}dS - S^{-1}B(S)S, \end{aligned} \tag{19}$$

from which we obtain $S^{-1}B(S)S - S^{-1}dS = H(S) = \Omega$. Consequently, there is a $\Theta(S,t)$ such that $d\Theta(S,t) = \Omega = \sum_{n=1}^{\infty} \partial^n dt_n$. In fact $\Theta = \sum_{n=1}^{\infty} t_n\partial^n$, so $S \in G_B$.

Let us show that $S \in G_1$. If $H(S) = d\Theta\,(S,t) = \Omega$, then $d\Gamma(S,t) = 0$ (see Remark 5). So $S \in G_1$. Moreover S is Sato-Wilson and, as $L = S\partial S^{-1}$, is a Brockett hierarchy solution. ☑

Let us now consider the initial value problem of (18). For $t = 0$,

$$\begin{aligned} U(0) &= S^{-1}(0)e^{\Gamma(S^{-1},0)}X(0) \\ &= S^{-1}(0)X(0). \end{aligned}$$

Since $X(0) = 1$,we have $U(0) = S^{-1}(0)$. Therefore, the solution is given by

$$\begin{aligned} U(t) &= e^{\left(\sum_{n=1}^{\infty} t_n\partial^n\right)}U(0) \\ &= e^{\left(\sum_{n=1}^{\infty} t_n\partial^n\right)}S^{-1}(0). \end{aligned}$$

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