Analítika

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Sobre la Perpetuidad Trasladada de Dufresne y Algunas Anualidades de Black-Scholes

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Abstract

Let $(\mathcal{E}_t, t \geq 0)$ be a geometric Brownian motion. In this paper, we compute the law of a generalization of Dufresne's translated perpetuity (following the terminology of Salminen-Yor) :

$$\int_0^{+\infty} \frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2 + 2a\mathcal{E}_s + b)^2} ds,$$

and show that, in some cases, this perpetuity is identical in law with the first hitting time of a three-dimensional Bessel process with drift. We also study the law of the following pair of annuities

$$\left(\int_0^t (\mathcal{E}_s - 1)^+ \, ds, \quad \int_0^t (\mathcal{E}_s - 1)^- \, ds \right)$$

via a Feynman-Kac approach, and discuss some particular cases for which we are able to recover the associated perpetuities.

Keywords: Geometric Brownian motion; Bessel processes ; Feynman-Kac formula.

Resumen

Sea $(\mathcal{E}_t, t \geq 0)$ un movimiento geométrico Browniano. En este artículo, calculamos la ley de una generalización de la perpetuidad trasladada de Dufresne (con la terminología de Salminen-Yor) :

$$\int_0^{+\infty} \frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2 + 2a\mathcal{E}_s + b)^2} ds,$$

y mostramos que en algunos casos, esta perpetuidad tiene la misma ley que el primer tiempo en el que un proceso de Bessel de dimensión tres con deriva alcanza una cierta barrera. Estudiamos también la ley del par de anualidades siguientes

$$\left(\int_0^t (\mathcal{E}_s - 1)^+ \, ds, \quad \int_0^t (\mathcal{E}_s - 1)^- \, ds \right)$$

con un teorema de Feynman-Kac, y discutimos algunos casos en los cuales podemos recuperar la ley de la perpetuidad asociada.

Palabras clave: Movimiento geométrico Browniano ; Proceso de Bessel ; Fórmula de Feynman-Kac.

Código MSC2000 : 60J65, 60J60, 60J55.

1 Introduction

One of the most important concept in finance theory is the *time value of money* which states that a given amount of money at the present time is worth more that the same amount in the future. This is due to the fact that you may invest the money you hold today and earn interest. Assuming that the rate of interest is constant and equal to $r > 0$, the value of the money $V(k)$ at time $k \in \mathbb{N}$ satisfies:

$$V(k) = V(0) \times (1+r)^k.$$

This formula may be used to compute the present value of an annuity, that is, of a series of equal payments P that occur at evenly spaced intervals. For a period of n payments, the present value of an annuity $A_n(0)$ equals:

$$A_n(0) = \sum_{k=1}^{n} \frac{P}{(1+r)^k} = \frac{P}{r}\left(1 - \frac{1}{(1+r)^n}\right). \qquad (1)$$

Classic examples of annuities are for instance lease payments, insurance payments or regular deposits to a savings account. Letting n tend towards $+\infty$ in Formula (1), we obtain the present value of a perpetuity $A_\infty(0)$, that is, when the payments are not limited in time:

$$A_\infty(0) = \frac{P}{r}.$$

All these computations may be extended to a continuous-time framework. In the continuous case, the value of the money $V(s)$ at time $s \geq 0$ satisfies :

$$V(s) = V(0)\exp(rs).$$

A natural generalization may be obtained by considering varying interest rates, in which case r is no longer a positive constant but a function which fluctuates over time. The value of the money $V(s)$ at time $s \geq 0$ then satisfies :

$$V(s) = V(0)\exp\left(\int_0^s r(u)du\right).$$

By analogy with (1), the present value of an annuity paid continuously up to time $t > 0$ is thus given by :

$$A_t(0) = P\int_0^t \exp\left(-\int_0^s r(u)du\right) ds$$

and, letting t tend to $+\infty$, the analogous perpetuity equals :

$$A_\infty(0) = P\int_0^{+\infty} \exp\left(-\int_0^s r(u)du\right) ds.$$

Unfortunately, the future evolution of interest rates (i.e. the function r) is unknown, and is therefore often modeled by a stochastic process. In other words, when dealing with long term guaranteed payments, one is often led to study the law of random variables of the form

$$\int_0^{+\infty} X_s\, ds$$

where $(X_s, s \geq 0)$ is a given (positive) stochastic process. One of the most famous example is certainly Dufresne's perpetuity. In his study of the value of a pension fund, Dufresne [5] proved the equality

$$\int_0^{+\infty} e^{aB_s - \nu s} ds \overset{(\text{law})}{=} \frac{2}{a^2 \gamma_{2\nu/a^2}}$$

where $(B_t, t \geq 0)$ is a Brownian motion started from 0 and γ_μ denotes a Gamma random variable with parameter μ:

$$\mathbb{P}(\gamma_\mu \in dz) = \frac{1}{\Gamma(\mu)} e^{-z} z^{\mu-1} 1_{\{z>0\}} dz.$$

Since then, many perpetuities involving geometric Brownian motion $\left(\exp\left(\nu B_t - \frac{\nu^2 t}{2}\right),\ t \geq 0\right)$ have been considered, which is of no surprise due to its prominent role in the celebrated Black-Scholes model. Observe that, in the set-up of perpetuities, the parameter ν may be removed by scaling since:

$$\int_0^{+\infty} f\left(e^{\nu B_s \pm \frac{\nu^2 s}{2}}\right) ds \overset{(\text{law})}{=} \int_0^{+\infty} f\left(e^{B_{\nu^2 s} \pm \frac{\nu^2 s}{2}}\right) ds$$
$$= \frac{1}{\nu^2}\int_0^{+\infty} f\left(e^{B_u \pm \frac{u}{2}}\right) du.$$

Therefore, in the following, we shall emphasize our study on the processes :

$$\mathcal{E}_t = \exp\left(B_t + \frac{t}{2}\right) \qquad \text{and} \qquad \mathcal{M}_t = \exp\left(B_t - \frac{t}{2}\right).$$

In [16], Salminen and Yor introduced a translated version of Dufresne's perpetuity, to circumvent the fact that the original one does not have all his moments finite. They prove in particular the equality :

$$\int_0^{+\infty} \frac{ds}{(2a+\mathcal{E}_s)^2} \overset{(\text{law})}{=} \inf\left\{s \geq 0,\ B_s + as = \frac{1}{2a}\ln(1+2a)\right\}$$

which we shall recover in Section 2. In fact, many perpetuities involving Brownian motion with drift are seen to be identical in law with the first hitting time of some associated diffusions, see [17] for a discussion and many examples on this subject.

In this paper, we study a generalization of Dufresne's translated perpetuity, that is,

$$\int_0^{+\infty} \frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2 + 2a\mathcal{E}_s + b)^2} ds,$$

and show, in particular, the following equalities in law:

Theorem 1. *Assume that $b > 0$ and $a^2 - b \geq 0$. Let $(R_t^{(3,\sqrt{a^2-b})}, t \geq 0)$ be a three-dimensional Bessel process with drift $\sqrt{a^2-b}$. Then, if $\mathcal{E}_0 = \mathcal{M}_0 = x > 0$,*

$$\int_0^{+\infty} \frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2 + 2a\mathcal{E}_s + b)^2} ds$$
$$\overset{(\text{law})}{=} \inf\left\{t \geq 0,\ R_t^{(3,\sqrt{a^2-b})} = \eta(\infty)\right\},$$

with $R_0^{(3,\sqrt{a^2-b})} = \eta(x)$, *and*

$$\int_0^{+\infty} \frac{\mathcal{M}_s^2}{(\mathcal{M}_s^2+2a\mathcal{M}_s+b)^2}ds \overset{\text{(law)}}{=} \inf\left\{t\geq 0,\ R_t^{(3,\sqrt{a^2-b})}=\eta(\infty)\right\},$$
$$\text{with } R_0^{(3,\sqrt{a^2-b})}=\eta(b/x).$$

where

$$\eta(x)=\int_0^x \frac{dz}{z^2+2az+b}.$$

We give two proofs of this result:

i) a direct proof relying on a martingale approach and on the weak absolute continuity formula between Brownian motion and the three-dimensional Bessel process,

ii) and a slightly more probabilistic proof relying on the third Ray-Knight theorem and on a decomposition of the three-dimensional Bessel paths at its last passage times.

In the remainder of the paper, we study via a Feynman-Kac approach the law of the pair :

$$\left(\int_0^t \left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds,\quad \int_0^t \left(e^{2b(B_s+\nu s)}-1\right)^- ds\right)$$

where $x^+ = \text{máx}(0,x)$ and $x^- = \text{mín}(0,x)$. This study answers a problem raised in the monograph [14, Chapter 4], where the authors compute the Laplace transform of the Black-Scholes call perpetuity

$$\int_0^{+\infty} (e^{x+B_u+\nu u}-K)^+ du$$

and leave as an open question the study of the analogous annuity. We then discuss several special cases (among which the Black-Scholes call annuity, the positive sojourn time of Brownian motion with drift and Yor's functional), and recover the Laplace transform of the associated perpetuities.

Note that annuities also appear in the computation of Asian options, where the payoff is determined by the average price of the underlying asset $(S_t, t\geq 0)$ on the considered period, see [7]. For instance, the price of an Asian Call option with exercise price K and maturity T is given by

$$\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t\,dt-K\right)^+\right].$$

This is somewhat different from the price of a classic European Call option, where only the final value of the underlying asset at time T is considered (see [3]) :

$$\mathbb{E}\left[(S_T-K)^+\right].$$

Of course, Asian options are harder to compute in practice as they depend on the entire past history of the underlying asset, but they make it possible to reduce the risk of price manipulation near the maturity date.

The paper is finally concluded by a short appendix on Bessel functions and Bessel processes (with drift).

2 A generalization of Dufresne's translated perpetuity

In this section, we compute the law of the perpetuities :

$$\int_0^{+\infty} \frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2+2a\mathcal{E}_s+b)^2}ds \quad \text{and} \quad \int_0^{+\infty} \frac{\mathcal{M}_s^2}{(\mathcal{M}_s^2+2a\mathcal{M}_s+b)^2}ds.$$

THEOREM 2. *Assume that the polynomial $z^2+2az+b$ does not have positive roots. For $2\lambda+a^2-b\geq 0$, we have:*

$$\mathbb{E}_x\left[\exp\left(-\lambda\int_0^{+\infty}\frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2+2a\mathcal{E}_s+b)^2}ds\right)\right]= \frac{\sqrt{x^2+2ax+b}}{x}\,\frac{\sinh\left(\sqrt{2\lambda+a^2-b}\int_0^x\frac{dz}{z^2+2az+b}\right)}{\sinh\left(\sqrt{2\lambda+a^2-b}\int_0^{+\infty}\frac{dz}{z^2+2az+b}\right)}$$

and

$$\mathbb{E}_x\left[\exp\left(-\lambda\int_0^{+\infty}\frac{\mathcal{M}_s^2}{(\mathcal{M}_s^2+2a\mathcal{M}_s+b)^2}ds\right)\right]= \frac{\sqrt{b+2ax+x^2}}{\sqrt{b}}\,\frac{\sinh\left(\sqrt{2\lambda+a^2-b}\int_0^{b/x}\frac{dz}{z^2+2az+b}\right)}{\sinh\left(\sqrt{2\lambda+a^2-b}\int_0^{+\infty}\frac{dz}{z^2+2az+b}\right)}.$$

The equality in law given in Theorem 1 follows directly from this result, since when $a^2-b\geq 0$, one recognizes in the right-hand side the expression of the Laplace transform of the first passage time of a three-dimensional Bessel process with drift $\sqrt{a^2-b}$. A short review of Bessel processes with drift is given in Section A, where these Laplace transforms are inverted thanks to Jacobi's theta function.

2.1 A martingale approach to Dufresne's translated perpetuity

Let $x>0$ and assume that $\mathcal{E}_0=x$. By Lamperti's relation (see Theorem 7), there exists a three-dimensional Bessel process $(R_t, t\geq 0)$ started from x such that:

$$\mathcal{E}_t = R_{A_t} \qquad \text{with } A_t=\langle\mathcal{E}\rangle_t=\int_0^t(\mathcal{E}_s)^2ds.$$

Using this transform and the change of variable $u = A_s$, we obtain :

$$\int_0^{+\infty} \frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2 + 2a\mathcal{E}_s + b)^2} ds = \int_0^{+\infty} \frac{R_{A_s}^2}{(R_{A_s}^2 + 2aR_{A_s} + b)^2} ds = \int_0^{+\infty} \frac{1}{(R_u^2 + 2aR_u + b)^2} du. \tag{2}$$

To compute the law of this perpetuity, we shall construct the appropriate martingale, thanks to the following lemma:

Lemma 3 ([13]). *Let $\Delta = a^2 - b$ and $2\lambda + \Delta > 0$. Set for $x \geq 0$:*

$$\eta(x) = \int_0^x \frac{dz}{z^2 + 2az + b}.$$

Then, the functions

$$\phi_\pm(x) = \sqrt{x^2 + 2ax + b} \exp\left(\pm\eta(x)\sqrt{2\lambda + \Delta}\right)$$

are two independent solutions of the equation:

$$f'' - \frac{2\lambda}{(x^2 + 2ax + b)^2} f = 0.$$

Let $(W_t, t \geq 0)$ be a Brownian motion. Applying Itô's formula, we deduce that the process

$$M_t = \sqrt{W_t^2 + 2aW_t + b} \sinh\left(\eta(W_t)\sqrt{2\lambda + \Delta}\right) \exp\left(-\lambda \int_0^t \frac{du}{(W_u^2 + 2aW_u + b)^2}\right)$$

is a continuous martingale, and with $T_0 = \text{ínf}\{t \geq 0, W_t = 0\}$, Doob's optional stopping theorem implies:

$$\begin{aligned}
&\sqrt{x^2 + 2ax + b}\ \sinh\left(\sqrt{2\lambda + \Delta}\eta(x)\right) \\
&= \mathbb{E}_x\left[\sqrt{W_{t\wedge T_0}^2 + 2aW_{t\wedge T_0} + b}\ \sinh\left(\eta(W_{t\wedge T_0})\sqrt{2\lambda + \Delta}\right) \exp\left(-\lambda \int_0^{t\wedge T_0} \frac{du}{\left(W_u^2 + 2aW_u + b\right)^2}\right)\right] \\
&= \mathbb{E}_x\left[\sqrt{W_t^2 + 2aW_t + b}\ \sinh\left(\eta(W_t)\sqrt{2\lambda + \Delta}\right) \exp\left(-\lambda \int_0^{t} \frac{du}{\left(W_u^2 + 2aW_u + b\right)^2}\right) 1_{\{t<T_0\}}\right].
\end{aligned}$$

Now, from the absolute continuity formula between Brownian motion and the three-dimensional Bessel process (14), we deduce that:

$$\begin{aligned}
&\sqrt{x^2 + 2ax + b} \sinh\left(\eta(x)\sqrt{2\lambda + \Delta}\right) \\
&= x\mathbb{E}_x^{(3)}\left[\frac{1}{R_t}\sqrt{R_t^2 + 2aR_t + b} \sinh\left(\eta(R_t)\sqrt{2\lambda + \Delta}\right) \exp\left(-\lambda \int_0^{t} \frac{du}{\left(R_u^2 + 2aR_u + b\right)^2}\right)\right]
\end{aligned}$$

and, letting t go towards $+\infty$, we obtain from the dominated convergence theorem :

$$\begin{aligned}
&\mathbb{E}_x^{(3)}\left[\exp\left(-\lambda \int_0^{+\infty} \frac{du}{(R_u^2 + 2aR_u + b)^2}\right)\right] \\
&= \frac{1}{x}\sqrt{x^2 + 2ax + b} \frac{\sinh\left(\eta(x)\sqrt{2\lambda + \Delta}\right)}{\sinh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right)},
\end{aligned} \tag{3}$$

which, thanks to (2), gives the first part of Theorem 2.
Next, the analogous formula for $\mathcal{M}$ follows by symmetry. Indeed, since when $B_0 = 0$,

$$\mathcal{M}_t = \exp\left(B_t - \frac{t}{2}\right) \overset{\text{(law)}}{=} \exp\left(-B_t - \frac{t}{2}\right) = \frac{1}{\mathcal{E}_t}, \tag{4}$$

we obtain :

$$\begin{aligned}
&\int_0^{+\infty} \frac{x^2\mathcal{M}_s^2}{(x^2\mathcal{M}_s^2 + 2ax\mathcal{M}_s + b)^2} ds \\
&= \int_0^{+\infty} \frac{x^2\mathcal{E}_s^2}{(x^2 + 2ax\mathcal{E}_s + b\mathcal{E}_s^2)^2} ds \\
&= x^2 \int_0^{+\infty} \frac{b^2\mathcal{E}_s^2}{(bx^2 + 2axb\mathcal{E}_s + b^2\mathcal{E}_s^2)^2} ds
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left(-\lambda \int_0^{+\infty} \frac{\mathcal{M}_s^2}{(\mathcal{M}_s^2 + 2a\mathcal{M}_s + b)^2} ds\right)\right] \\
&= \frac{\sqrt{b + ax + x^2}}{\sqrt{b}} \frac{\sinh\left(x\sqrt{2\lambda + \Delta}\int_0^b \frac{dz}{z^2 + 2axz + bx^2}\right)}{\sinh\left(x\sqrt{2\lambda + \Delta}\int_0^{+\infty} \frac{dz}{z^2 + 2axz + bx^2}\right)} \\
&= \frac{\sqrt{b + 2ax + x^2}}{\sqrt{b}} \frac{\sinh\left(\sqrt{2\lambda + \Delta}\int_0^{b/x} \frac{dy}{y^2 + 2ay + b}\right)}{\sinh\left(\sqrt{2\lambda + \Delta}\int_0^{+\infty} \frac{dy}{y^2 + 2ay + b}\right)}.
\end{aligned}$$

after the change of variable $y = xz$.

□

Remark 4. Letting x tend towards 0 in (3), we obtain the simple formula:

$$\mathbb{E}_0^{(3)}\left[\exp\left(-\lambda \int_0^{+\infty} \frac{du}{(R_u^2 + 2aR_u + b)^2}\right)\right] = \frac{\sqrt{2\lambda + \Delta}}{\sqrt{b}\sinh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right)}.$$

2.2 A probabilistic approach to Dufresne's translated perpetuity

We now give another proof of Theorem 2, with a slightly more probabilistic approach. Let $x > 0$ and assume that $\mathcal{M}_0 = x$. From the Dambis, Dubins-Schwarz theorem (see [15, Theorem 1.6, p.181]), there exists a Brownian motion $(B_t, t \geq 0)$ started from x such that:

$$\mathcal{M}_t = B_{\langle \mathcal{M} \rangle_t} \qquad \text{with } \langle \mathcal{M} \rangle_t = \int_0^t (\mathcal{M}_s)^2 ds.$$

Therefore:

$$\begin{aligned}
&\int_0^{+\infty} \frac{\mathcal{M}_s^2}{(\mathcal{M}_s^2 + 2a\mathcal{M}_s + b)^2} ds \\
&= \int_0^{+\infty} \frac{B^2_{\langle \mathcal{M} \rangle_s}}{(B^2_{\langle \mathcal{M} \rangle_s} + 2aB_{\langle \mathcal{M} \rangle_s} + b)^2} ds \\
&= \int_0^{T_0} \frac{1}{(B_u^2 + 2aB_u + b)^2} du,
\end{aligned} \tag{5}$$

where $T_0 = \inf\{u \geq 0,\ B_u = 0\}$. By the classic time-reversal result of Brownian motion (see Section A), this last expression is seen to be identical in law with:

$$\begin{aligned}
&\int_0^{T_0} \frac{1}{(B_u^2 + 2aB_u + b)^2} du \\
&= \int_0^{T_0} \frac{1}{(B^2_{T_0 - s} + 2aB_{T_0 - s} + b)^2} ds \\
&\overset{\text{(law)}}{=} \int_0^{G_x} \frac{1}{(R_s^2 + 2aR_s + b)^2} ds
\end{aligned} \tag{6}$$

where $(R_s, s \geq 0)$ is a three-dimensional Bessel process starting from 0 and $G_x = \sup\{s \geq 0,\ R_s = x\}$.
To compute the law of this last expression, we shall first consider the whole perpetuity

$$\int_0^{+\infty} \frac{ds}{(R_s^2 + 2aR_s + b)^2}.$$

From the occupation time formula and Ray-Knight's third theorem (see Theorem 10):

$$\begin{aligned}
\int_0^{+\infty} \frac{ds}{(R_s^2 + 2aR_s + b)^2} &= \int_0^{+\infty} \frac{L_\infty^y(R)}{(y^2 + 2ay + b)^2} dy \\
&\overset{\text{(law)}}{=} \int_0^{+\infty} \frac{Z_y^{(2)}}{(y^2 + 2ay + b)^2} dy
\end{aligned}$$

where $(Z_y^{(2)}, y \geq 0)$ denotes a two-dimensional squared Bessel process starting from 0. Then, from Theorem 8, the Laplace transform of the right-hand side equals :

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^{+\infty} \frac{Z_y^{(2)}}{(y^2 + ay + b)^2} dy\right)\right] = F(\infty)$$

where F is the unique solution on $[0, +\infty[$ of :

$$F'' = \frac{2\lambda}{(x^2 + 2ax + b)^2} F$$

such that F is positive, non increasing, and $F(0) = 1$.

Therefore, from Lemma 3, there exist two constants α and β such that :

$$\begin{aligned}
F(x) = &\sqrt{x^2 + 2ax + b} \\
&\left(\alpha \cosh\left(\eta(x)\sqrt{2\lambda + \Delta}\right) + \beta \sinh\left(\eta(x)\sqrt{2\lambda + \Delta}\right)\right).
\end{aligned}$$

Since $F(0) = 1$, we deduce that:

$$\alpha = \frac{1}{\sqrt{b}}.$$

Next, as F is positive and non increasing, the limit $F(\infty)$ necessarily exists, so we must have :

$$\frac{1}{\sqrt{b}} \cosh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right) + \beta \sinh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right) = 0$$

which yields

$$\beta = -\frac{1}{\sqrt{b}} \frac{\cosh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right)}{\sinh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right)}.$$

Finally, thanks to the additivity formula of sinh:

$$\begin{aligned}
F(x) = &\frac{\sqrt{x^2 + 2ax + b}}{\sqrt{b}} \frac{\sinh\left(\sqrt{2\lambda + \Delta}(\eta(\infty) - \eta(x))\right)}{\sinh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right)} \\
&\xrightarrow[x \to +\infty]{} \frac{\sqrt{2\lambda + \Delta}}{\sqrt{b} \sinh\left(\eta(+\infty)\sqrt{2\lambda + \Delta}\right)}
\end{aligned}$$

and we recover the result of Remark 4. Now, to obtain the result for any $x > 0$ we shall use a decomposition of the paths of the three-dimensional Bessel process at its last passage time $G_x = \sup\{t \geq 0,\ R_t = x\}$. From Theorem 9, we may write :

$$\begin{aligned}
&\int_0^{+\infty} \frac{ds}{(R_s^2 + 2aR_s + b)^2} \\
&= \int_0^{G_x} \frac{ds}{(R_s^2 + 2aR_s + b)^2} + \int_{G_x}^{+\infty} \frac{ds}{(R_s^2 + 2aR_s + b)^2} \\
&= \int_0^{G_x} \frac{ds}{(R_s^2 + 2aR_s + b)^2} + \int_0^{+\infty} \frac{ds}{\left(R^2_{G_x + s} + 2aR_{G_x + s} + b\right)^2} \\
&\overset{\text{(law)}}{=} \int_0^{G_x} \frac{ds}{(R_s^2 + 2aR_s + b)^2} + \\
&\int_0^{+\infty} \frac{ds}{\left((x + \widetilde{R}_s)^2 + 2a(x + \widetilde{R}_s) + b\right)^2}
\end{aligned}$$

where $\left(\widetilde{R}_s, s \geq 0\right)$ is an independent copy of $(R_s, s \geq 0)$. Taking the Laplace transform of both sides, we obtain, from

(5) and (6):

$$\mathbb{E}_0^{(3)}\left[\exp\left(-\lambda\int_0^{+\infty}\frac{ds}{(R_s^2+2aR_s+b)^2}\right)\right]$$
$$=\mathbb{E}_0^{(3)}\left[\exp\left(-\lambda\int_0^{G_x}\frac{ds}{(R_s^2+2aR_s+b)^2}\right)\right]$$
$$\mathbb{E}_0^{(3)}\left[\exp\left(-\lambda\int_0^{+\infty}\frac{ds}{\left((x+\widetilde{R}_s)^2+2a(x+\widetilde{R}_s)+b\right)^2}\right)\right]$$
$$=\mathbb{E}_x\left[\exp\left(-\lambda\int_0^{+\infty}\frac{\mathcal{M}_s^2}{(\mathcal{M}_s^2+2a\mathcal{M}_s+b)^2}ds\right)\right]$$
$$\mathbb{E}_0^{(3)}\left[\exp\left(-\lambda\int_0^{+\infty}\frac{ds}{(R_s^2+(2a+2x)R_s+x^2+2ax+b)^2}\right)\right],$$

which yields the formula:

$$\mathbb{E}_x\left[\exp\left(-\lambda\int_0^{+\infty}\frac{\mathcal{M}_s^2}{(\mathcal{M}_s^2+2a\mathcal{M}_s+b)^2}ds\right)\right]$$
$$=\frac{\sqrt{x^2+2ax+b}}{\sqrt{b}}$$
$$\frac{\sinh\left(\sqrt{2\lambda+a^2-b}\int_0^{+\infty}\frac{dz}{z^2+2(a+x)z+x^2+2ax+b}\right)}{\sinh\left(\sqrt{2\lambda+a^2-b}\int_0^{+\infty}\frac{dz}{z^2+2az+b}\right)}.$$

This new expression is seen to agree with Theorem 2 by applying the change of variable $z=\frac{b}{y}-x$ in the integral on the numerator. The other relation follows as before thanks to (4).

□

2.3 A few particular cases

i) When $a=1$ and $b=1$, we recover a particular case of Hariya's identity :

$$\int_0^{+\infty}\frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2+2\mathcal{E}_s+1)^2}ds$$
$$=\int_0^{+\infty}\frac{ds}{(R_s+1)^4}\overset{\text{(law)}}{=}\inf\{t\geq 0,R_t=1\}.$$

with $\mathcal{E}_0=x$ and $R_0=1-\frac{1}{1+x}$.

ii) More generally, when $a=\sqrt{b}$,

$$\int_0^{+\infty}\frac{\mathcal{E}_s^2}{(\mathcal{E}_s^2+2b\mathcal{E}_s+b^2)^2}ds$$
$$=\int_0^{+\infty}\frac{ds}{(R_s+b)^4}\overset{\text{(law)}}{=}\inf\left\{t\geq 0,R_t=\frac{1}{b}\right\},$$

with $\mathcal{E}_0=x$ and $R_0=\frac{1}{b}-\frac{1}{x+b}$.

iii) We may recover the result of Salminen-Yor [17] by letting $b\to 0$. Indeed, for $0<b<a^2$, we have:

$$\int_0^x\frac{dz}{z^2+2az+b}$$
$$=\frac{1}{2\sqrt{a^2-b}}\left(\ln\left(\frac{x+a-\sqrt{a^2-b}}{x+a+\sqrt{a^2-b}}\right)-\right.$$
$$\left.\ln\left(\frac{a-\sqrt{a^2-b}}{a+\sqrt{a^2-b}}\right)\right)$$

so that, letting b go towards 0, we obtain:

$$\sinh\left(\sqrt{2\lambda+a^2-b}\int_0^x\frac{dz}{z^2+2az+b}\right)$$
$$\underset{b\to 0}{\sim}\frac{1}{2}\exp\left(\frac{\sqrt{2\lambda+a^2}}{2a}\left(\ln\left(\frac{x}{x+2a}\right)-\right.\right.$$
$$\left.\left.\ln\left(\frac{1-\sqrt{1-\frac{b}{a^2}}}{2}\right)\right)\right)$$

and

$$\mathbb{E}_x\left[\exp\left(-\lambda\int_0^{+\infty}\frac{ds}{(\mathcal{E}_s+2a)^2}\right)\right]$$
$$=\frac{\sqrt{x+2a}}{\sqrt{x}}\left(\frac{x}{x+2a}\right)^{\frac{\sqrt{2\lambda+a^2}}{2a}}=\left(\frac{x}{x+2a}\right)^{\frac{\sqrt{2\lambda+a^2}}{2a}-\frac{1}{2}}.$$

This last expression is seen to coincide with the Laplace transform of the first hitting time at level $\frac{1}{2a}\ln\left(\frac{x+2a}{x}\right)$ of a Brownian motion with drift a started from 0, which was the announced result in the introduction, with $x=1$.

We refer to Salminen & Yor [16, 17] and Decamps, De Schepper, Goovaerts & Schoutens [4] for similar articles on this subject.

3 Some Black-Scholes annuities

Let $(B_t+\nu t,\, t\geq 0)$ be a standard Brownian motion with drift ν started from 0. In this section, we study the law of the pair of annuities :

$$\left(\int_0^t\left(e^{2\beta(B_s+\nu s)}-1\right)^+ds,\quad\int_0^t\left(e^{2b(B_s+\nu s)}-1\right)^-ds\right)\quad(7)$$

where $x^+=\max(0,x)$ and $x^-=\min(0,x)$.

Theorem 5. *Let $\alpha,\beta,a,b,\lambda\geq 0$ and $\nu\in\mathbb{R}$. The double Laplace transform of the couple (7) is given by:*

$$\int_0^{+\infty}e^{-(\lambda+a)t}\mathbb{E}_0\left[\exp\left(-\alpha\int_0^t\left(e^{2\beta(B_s+\nu s)}-1\right)^+ds-\right.\right.$$
$$\left.\left.a\int_0^t\left(e^{2b(B_s+\nu s)}-1\right)^-ds\right)\right]dt$$
$$=\frac{2}{\omega_\lambda}\left(K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)\int_{-\infty}^0 e^{\nu y}I_{2c}\left(\frac{\sqrt{2a}}{b}e^{by}\right)dy+\right.$$
$$\left.I_{2c}\left(\frac{\sqrt{2a}}{b}\right)\int_0^{+\infty}e^{\nu y}K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}e^{\beta y}\right)dy\right)$$

where the Wronskien ω_λ equals:

$$\omega_\lambda = \sqrt{2\alpha} K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right) I'_{2c}\left(\frac{\sqrt{2a}}{b}\right) - \sqrt{2\alpha} K'_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right) I_{2c}\left(\frac{\sqrt{2a}}{b}\right)$$

and I_{2c} and $K_{2\gamma}$ denote the third modified Bessel functions with respective indexes

$$c = \frac{1}{2b}\sqrt{2\lambda+\nu^2} \qquad \text{and}$$
$$\gamma = \begin{cases} \frac{1}{2\beta}\sqrt{2(\lambda+a)-2\alpha+\nu^2} & \text{if } 2(\lambda+a)-2\alpha+\nu^2 \geq 0 \\ \frac{i}{2\beta}\sqrt{2\alpha-2(\lambda+a)-\nu^2} & \text{otherwise.} \end{cases}$$

The proof of this result is given in the next Sections 3.1 and 3.2. We will discuss in Section 4 some special cases for which the expression on the right-hand side simplifies.

3.1 A useful version of the Feynman-Kac formula

To prove Theorem 5, we shall apply the following well-known Feynman-Kac formula, see for instance Janson [9, Appendix C] where many other Brownian areas are also studied.

THEOREM 6 (Feynman-Kac)**.** *Let $V(x) \geq 0$ be a positive continuous function on $\mathbb{R}$, $\lambda > 0$, and let ϕ_+ and ϕ_- be two $\mathcal{C}^2$-solutions of the differential equation*

$$\frac{1}{2}\phi''(x) = (V(x)+\lambda)\phi(x) \tag{8}$$

such that, for A large enough :

$$\begin{aligned} &\phi_+ \text{ is positive and bounded on } [A,+\infty[\qquad \text{and} \\ &\phi_- \text{ is positive and bounded on }]-\infty,-A]. \end{aligned} \tag{9}$$

Let $w_\lambda := \phi_+(0)\phi'_-(0) - \phi_-(0)\phi'_+(0)$ and assume that $\omega_\lambda \neq 0$. Then, for any positive and measurable function f on $\mathbb{R}$ and any $x \in \mathbb{R}$:

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[e^{-\int_0^t V(B_s)ds} f(B_t)\right] dt = \frac{2}{\omega_\lambda}\left(\phi_+(x)\int_{-\infty}^x \phi_-(y)f(y)dy + \phi_-(x)\int_x^{+\infty}\phi_+(y)f(y)dy\right). \tag{10}$$

Demostración.
We sketch the proof of this result for the sake of completeness. First, define the Wronskien:

$$W_\lambda(x) = \phi_+(x)\phi'_-(x) - \phi_-(x)\phi'_+(x).$$

Since ϕ_+ and ϕ_- are solutions of (8), we deduce that $W'_\lambda(x) = 0$, hence for any $x \in \mathbb{R}$, $W_\lambda(x) = W_\lambda(0) = \omega_\lambda$.

Assume first that f is continuous and has compact support, and define:

$$\phi(x) = \phi_+(x)\int_{-\infty}^x \phi_-(y)f(y)dy + \phi_-(x)\int_x^{+\infty}\phi_+(y)f(y)dy.$$

ϕ is a function of $\mathcal{C}^1$-class, and differentiation yields:

$$\phi'(x) = \phi'_+(x)\int_{-\infty}^x \phi_-(y)f(y)dy + \phi'_-(x)\int_x^{+\infty}\phi_+(y)f(y)dy.$$

We thus deduce that ϕ is of $\mathcal{C}^2$-class, and from (8):

$$\begin{aligned}\phi''(x) &= 2(V(x)+\lambda)\phi(x) - W_\lambda(x)f(x) \\ &= 2(V(x)+\lambda)\phi(x) - \omega_\lambda f(x).\end{aligned}$$

Observe also that, since f is a function with compact support, the function ϕ is bounded on $\mathbb{R}$. Consider now the process

$$M_t = e^{-\lambda t - \int_0^t V(B_s)ds}\phi(B_t) + \frac{\omega_\lambda}{2}\int_0^t e^{-\lambda u - \int_0^u V(B_s)ds} f(B_u)du.$$

From Itô's formula, this process is a local martingale and we have the estimate:

$$\begin{aligned}|M_t| &\leq \sup_{x\in\mathbb{R}}|\phi(x)| + \frac{\omega_\lambda}{2}\sup_{x\in\mathbb{R}}|f(x)|\int_0^t e^{-\lambda u}du \\ &\leq \sup_{x\in\mathbb{R}}|\phi(x)| + \frac{\omega_\lambda}{2\lambda}\sup_{x\in\mathbb{R}}|f(x)|.\end{aligned}$$

Therefore M is uniformly bounded, i.e. M is a bounded martingale and

$$\begin{aligned}\phi(x) &= \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty] \\ &= \frac{\omega_\lambda}{2}\mathbb{E}_x\left[\int_0^{+\infty} e^{-\lambda u - \int_0^u V(B_s)ds} f(B_u)du\right].\end{aligned}$$

By a monotone class argument, the assumption on the continuity of f may be dropped, so Relation (10) is in fact valid for any positive and measurable function with compact support. Let now f by a positive and measurable function, and consider the sequence of functions

$$f_n(y) = f(y)1_{\{|y|\leq n\}}.$$

Since the f_n have compact support, we may apply Relation (10) and write, for n large enough:

$$\begin{aligned}&\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[e^{-\int_0^t V(B_s)ds} f(B_t)1_{\{|B_t|\leq n\}}\right] dt \\ &= \frac{2}{\omega_\lambda}\left(\phi_+(x)\int_{-n}^x \phi_-(y)f(y)dy + \phi_-(x)\int_x^n \phi_+(y)f(y)dy\right).\end{aligned}$$

We finally end the proof by letting $n \to +\infty$ and applying the monotone convergence theorem thanks to the Condition (9).

□

3.2 Proof of Theorem 5

We restrict our attention to the case $x = 0$, for which the formulae take a simpler form. To prove Theorem 5, we first remove the drift thanks to the Cameron-Martin formula, which states that for every functional F:

$$\begin{aligned}\mathbb{E}_0\left[F(B_s + \nu s, s \le t)\right] &= \mathbb{E}_0\left[e^{\nu B_t - \frac{\nu^2 t}{2}} F(B_s, s \le t)\right] \\ &= e^{-\frac{\nu^2 t}{2}} \mathbb{E}_0\left[e^{\nu B_t} F(B_s, s \le t)\right].\end{aligned}$$

Therefore the double Laplace transform reads:

$$\begin{aligned}&\int_0^{+\infty} e^{-(\lambda+a)t}\mathbb{E}_0\Big[\exp\Big(-\alpha\int_0^t \left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds - \\ &\qquad a\int_0^t \left(e^{2b(B_s+\nu s)}-1\right)^- ds\Big)\Big]dt \\ &= \int_0^{+\infty} e^{-(\lambda+\frac{\nu^2}{2})t}\mathbb{E}_0\Big[e^{\nu B_t}\exp\Big(-\alpha\int_0^t \left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds \\ &\qquad - at - a\int_0^t \left(e^{2b(B_s+\nu s)}-1\right)^- ds\Big)\Big]dt\end{aligned}$$

Consider the ordinary differential equation:

$$\begin{aligned}\frac{1}{2}\phi''(x) =& \Big(\alpha(e^{2\beta x}-1)1_{\{x>0\}} + a(e^{2bx}-1)1_{\{x<0\}} + \\ & a + \frac{\nu^2}{2} + \lambda\Big)\phi(x).\end{aligned}$$

The function V defined by

$$V(x) = \alpha(e^{2\beta x}-1)1_{\{x>0\}} + a(e^{2bx}-1)1_{\{x<0\}} + a + \frac{\nu^2}{2}$$

is positive and continuous, so we may apply the Feynman-Kac Theorem 6. To simplify the notation, we set:

$$c = \frac{1}{2b}\sqrt{2\lambda+\nu^2} \qquad \text{and}$$

$$\gamma = \begin{cases} \frac{1}{2\beta}\sqrt{2(\lambda+a)-2\alpha+\nu^2} & \text{if } 2(\lambda+a)-2\alpha+\nu^2 \ge 0 \\ \frac{i}{2\beta}\sqrt{2\alpha-2(\lambda+a)-\nu^2} & \text{otherwise.}\end{cases}$$

In this framework, the solutions ϕ_+ and ϕ_- which satisfy (9) are given by (see Section A.1):

$$\phi_+(x) = K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}e^{\beta x}\right) \qquad \text{for } x \ge 0,$$

and

$$\phi_-(x) = I_{2c}\left(\frac{\sqrt{2a}}{b}e^{bx}\right) \qquad \text{for } x \le 0.$$

Their Wronskien equals

$$\begin{aligned}\omega_\lambda =& \sqrt{2a}K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)I'_{2c}\left(\frac{\sqrt{2a}}{b}\right) - \\ & \sqrt{2\alpha}K'_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)I_{2c}\left(\frac{\sqrt{2a}}{b}\right),\end{aligned}$$

so the Feynman-Kac formula yields, for $x = 0$:

$$\begin{aligned}&\int_0^{+\infty} e^{-(\lambda+a)t}\mathbb{E}_0\Big[\exp\Big(-\alpha\int_0^t \left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds - \\ &\qquad a\int_0^t \left(e^{2b(B_s+\nu s)}-1\right)^- ds\Big)\Big]dt \\ &= \frac{2}{\omega_\lambda}\left(\phi_-(0)\int_0^{+\infty} e^{\nu y}\phi_+(y)dy + \phi_+(0)\int_{-\infty}^0 e^{\nu y}\phi_-(y)dy\right)\end{aligned}$$

which ends the proof.

□

4 Some special annuities

We now look at some particular cases of Theorem 5, according to the values of α, β, a and b. Most of the formulae we obtain may be found in [2].

4.1 The Black-Scholes call annuity

We first let $a \to 0$ in Theorem 5. Recall the asymptotics (see Section A.1):

$$I_\mu(z) \underset{z\to 0}{\sim} \frac{z^\mu}{2^\mu\Gamma(\mu+1)} \qquad \text{and}$$

$$zI'_\mu(z) = zI_{\mu+1}(z) + \mu I_\mu(z) \underset{z\to 0}{\sim} \frac{\mu z^\mu}{2^\mu\Gamma(\mu+1)}.$$

We thus obtain the double Laplace transform of the Black-Scholes call annuity:

$$\begin{aligned}&\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-\alpha\int_0^t\left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds}\right]dt \\ &= \frac{2}{\omega_\lambda}\left(\int_0^{+\infty} e^{\nu y}K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}e^{\beta y}\right)dy + \frac{K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)}{\nu+\sqrt{2\lambda+\nu^2}}\right)\end{aligned}$$

where the Wronskien ω_λ equals:

$$\omega_\lambda = \sqrt{2\lambda+\nu^2}K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right) - \sqrt{2\alpha}K'_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right).$$

We may also recover the Laplace transform of the associated perpetuity as follows. Assume that $\nu < 0$ and replace λ by $\lambda\varepsilon$ to obtain:

$$\begin{aligned}&\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathbb{E}_0\left[e^{-\alpha\int_0^t\left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds}\right]dt \\ &= \frac{1}{\varepsilon}\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-\alpha\int_0^{t/\varepsilon}\left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds}\right]dt.\end{aligned}$$

Now letting $\varepsilon \to 0$, we deduce that:

$$\begin{aligned}&\frac{1}{\lambda}\mathbb{E}_0\left[e^{-\alpha\int_0^{+\infty}\left(e^{2\beta(B_s+\nu s)}-1\right)^+ ds}\right] \\ &= \lim_{\varepsilon\to 0}\frac{2\varepsilon}{w_{\lambda\varepsilon}}\left(\int_0^{+\infty} e^{\nu y}K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}e^{\beta y}\right)dy + \frac{K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)}{\nu+\sqrt{2\lambda\varepsilon+\nu^2}}\right)\end{aligned}$$

$$= \frac{2|\nu|}{\lambda} \frac{K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)}{|\nu| K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right) - \sqrt{2\alpha} K'_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)},$$

which agrees with [14, Section 4.4, p.107]. Unfortunately, it does not seem easy to invert this Laplace transform.

4.2 The positive sojourn time of Brownian motion with drift

Let first $\alpha \to 0$. From the asymptotics (see Section A.1):

$$K_\nu(z) \underset{z\to 0}{\sim} \frac{2^{\nu-1}\Gamma(\nu)}{z^\nu} \quad \text{and}$$
$$zK'_\nu(z) = -zK_{\nu-1}(z) - \nu I_\nu(z) \underset{z\to 0}{\sim} \frac{\nu 2^{\nu-1}\Gamma(\nu)}{z^\nu}$$

we deduce that

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-at - a\int_0^t \left(e^{2b(B_s+\nu s)}-1\right)^- ds}\right] dt$$
$$= \frac{2}{\omega_\lambda}\left(\int_{-\infty}^0 e^{\nu y} I_{2\gamma}\left(\frac{\sqrt{2a}}{b}e^{by}\right) dy + \frac{I_{2c}\left(\frac{\sqrt{2a}}{b}\right)}{\sqrt{2(\lambda+a)+\nu^2}-\nu}\right)$$

where the Wronskien ω_λ equals:

$$\omega_\lambda = \sqrt{2a} I'_{2c}\left(\frac{\sqrt{2a}}{b}\right) + \sqrt{2(\lambda+a)+\nu^2}\, I_{2c}\left(\frac{\sqrt{2a}}{b}\right).$$

We now let further $b \to +\infty$. The left-hand side yields :

$$t + \int_0^t \left(e^{2b(B_s+\nu s)} - 1\right)^- ds$$
$$= t + \int_0^t e^{2b(B_s+\nu s)} 1_{\{B_s+\nu s<0\}} ds -$$
$$\int_0^t 1_{\{B_s+\nu s<0\}} ds \underset{b\to+\infty}{\longrightarrow} \int_0^t 1_{\{B_s+\nu s\geq 0\}} ds$$

To compute the limit as $b \to +\infty$ in the right-hand side, we rely on the following integral formula

$$I_\mu(z) = \frac{z^\mu}{2^\mu\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})} \int_0^\pi e^{z\cos(\theta)} \sin^{2\mu}(\theta) d\theta$$

which gives :

$$I_{\frac{\sqrt{2\lambda+\nu^2}}{b}}\left(\frac{\sqrt{2a}}{b}e^{by}\right) \underset{b\to+\infty}{\longrightarrow} e^{y\sqrt{2\lambda+\nu^2}}$$

and

$$I'_{\frac{\sqrt{2\lambda+\nu^2}}{b}}\left(\frac{\sqrt{2a}}{b}\right) = I_{\frac{\sqrt{2\lambda+\nu^2}}{b}+1}\left(\frac{\sqrt{2a}}{b}\right) +$$
$$\frac{\sqrt{2\lambda+\nu^2}}{\sqrt{2a}} I_{\frac{\sqrt{2\lambda+\nu^2}}{b}}\left(\frac{\sqrt{2a}}{b}\right) \underset{b\to+\infty}{\longrightarrow} \frac{\sqrt{2\lambda+\nu^2}}{\sqrt{2a}}.$$

Therefore, we deduce that

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-a\int_0^t 1_{\{B_s+\nu s>0\}}ds}\right] dt$$
$$= \frac{2}{\omega_\lambda}\left(\int_{-\infty}^0 e^{\nu y} e^{y\sqrt{2\lambda+\nu^2}} dy + \frac{1}{\sqrt{2(\lambda+a)+\nu^2}-\nu}\right)$$
$$= \frac{2}{\omega_\lambda}\left(\frac{1}{\nu+\sqrt{2\lambda+\nu^2}} + \frac{1}{\sqrt{2(\lambda+a)+\nu^2}-\nu}\right),$$

with

$$\omega_\lambda = \sqrt{2\lambda+\nu^2} + \sqrt{2(\lambda+a)+\nu^2}.$$

We now study two further simplifications :

i) When $\nu = 0$, this expression simplifies to:

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-a\int_0^t 1_{\{B_s>0\}}ds}\right] dt = \frac{1}{\sqrt{\lambda}\sqrt{\lambda+a}}$$

and this double Laplace transform may be inverted to recover the celebrated Arcsine law of Brownian motion:

$$\mathbb{P}_0\left(\int_0^t 1_{\{B_s>0\}}ds \in dz\right) = \frac{1}{\pi\sqrt{z}\sqrt{t-z}} 1_{\{0<z<t\}} dz.$$

ii) When $\nu < 0$, we may obtain the Laplace transform of the associated perpetuity as before:

$$\mathbb{E}_0\left[e^{-a\int_0^{+\infty} 1_{\{B_s+\nu s>0\}}ds}\right]$$
$$= \lim_{\varepsilon\to 0} \frac{2\lambda\varepsilon}{w_{\lambda\varepsilon}}\left(\frac{1}{\nu+\sqrt{2\lambda\varepsilon+\nu^2}} + \frac{1}{\sqrt{2(\lambda\varepsilon+a)+\nu^2}-\nu}\right)$$
$$= \frac{2|\nu|}{|\nu|+\sqrt{2a+\nu^2}},$$

which may be inverted to give (see [6, Formula 4, p.233]):

$$\mathbb{P}_0\left(\int_0^{+\infty} 1_{\{B_s+\nu s>0\}}ds \in dz\right)$$
$$= \frac{|\nu|\sqrt{2}}{\sqrt{\pi}}\left(\frac{1}{\sqrt{z}}e^{-\frac{\nu^2}{2}z} - |\nu|\sqrt{2}\int_{\frac{|\nu|\sqrt{z}}{\sqrt{2}}}^{+\infty} e^{-t^2}dt\right) 1_{\{z>0\}} dz.$$

4.3 Yor's functional

Take $a = \alpha$ and $b = \beta$. Then, from (11), the Wronskien simplifies to

$$\omega_\lambda = \beta$$

and from the formula (see [8, p.712]) :

$$2I_\gamma(x)K_\gamma(y) = \int_0^{+\infty} e^{-\frac{t}{2}-\frac{x^2+y^2}{2t}} I_\gamma\left(\frac{xy}{t}\right)\frac{dt}{t}, \qquad \text{for } y \geq x,$$

we obtain the expression :

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-\alpha\int_0^t e^{2\beta(B_s+\nu s)}ds}\right] dt$$
$$= \frac{1}{\beta}\int_0^{+\infty}\frac{dt}{t}\exp\left(-\frac{t}{2}-\frac{\alpha}{t\beta^2}\right)\int_{\mathbb{R}}\exp\left(-\frac{\alpha}{t\beta^2}e^{2\beta y}\right)$$

$$e^{\nu y} I_{2\gamma}\left(\frac{2\alpha e^{\beta y}}{t\beta^2}\right) dy$$
$$= \frac{1}{\beta}\int_0^{+\infty} \frac{dz}{z} e^{-\alpha z} \exp\left(-\frac{1}{2z\beta^2}\right)\int_{\mathbb{R}} \exp\left(-\frac{1}{2z\beta^2}e^{2\beta y}\right)$$
$$e^{\nu y} I_{2\gamma}\left(\frac{e^{\beta y}}{z\beta^2}\right) dy.$$

We may therefore invert the Laplace transform in α to obtain:

$$\mathbb{P}_0\left(\int_0^{\tau_\lambda} e^{2\beta(B_s+\nu s)} ds \in dz\right)$$
$$= \frac{\lambda}{z\beta} e^{-\frac{1}{2z\beta^2}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2z\beta^2}e^{2\beta y}\right) e^{\nu y} I_{\frac{\sqrt{2\lambda+\nu^2}}{\beta}}\left(\frac{e^{\beta y}}{z\beta^2}\right) dy$$

where τ_λ denotes an exponential random variable with parameter λ independent from B.
Note that this last Laplace transform may be also inverted thanks to the Hartman-Watson function θ_r given by:

$$I_{\sqrt{2\lambda}}(r) = \int_0^{+\infty} e^{-\lambda t}\theta_r(t) dt$$

which, from Yor [19], admits the representation:

$$\theta_r(t) = \frac{r}{\sqrt{2\pi^3 t}}\int_0^{+\infty} e^{\frac{\pi^2-y^2}{2t} - r\cosh(y)} \sinh(y)\sin\left(\frac{\pi y}{t}\right) dy.$$

4.4 One-sided Yor's functional

Take $a = \alpha$ and let $b \to +\infty$. We obtain :

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-\alpha\int_0^t e^{2\beta(B_s+\nu s)} 1_{\{B_s+\nu s>0\}} ds}\right] dt$$
$$= \frac{2}{\omega_\lambda}\left(\frac{K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)}{\nu+\sqrt{2\lambda+\nu^2}} + \int_0^{+\infty} e^{\nu y} K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}e^{\beta y}\right)\right)$$

with

$$\omega_\lambda = K_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right)\sqrt{2\lambda+\nu^2} - \sqrt{2\alpha}K'_{2\gamma}\left(\frac{\sqrt{2\alpha}}{\beta}\right).$$

This allows to recover the associated perpetuity, for $\nu < 0$:

$$\mathbb{E}_0\left[\exp\left(-\alpha\int_0^{+\infty} e^{2\beta(B_s+\nu s)} 1_{\{B_s+\nu s>0\}} ds\right)\right]$$
$$= \frac{2|\nu| K_{\frac{|\nu|}{\beta}}\left(\frac{\sqrt{2\alpha}}{\beta}\right)}{\sqrt{2\alpha}K_{\frac{|\nu|}{\beta}+1}\left(\frac{\sqrt{2\alpha}}{\beta}\right)}.$$

We refer to Salminen & Yor [16, 17] for a comprehensive study of this family of perpetuities.

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A Appendix on Bessel functions and Bessel processes (with drift)

A.1 Modified Bessel functions [10, Chapter 5]

For $\nu \in \mathbb{C}$, let I_ν denote the modified Bessel function defined by:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)} \qquad x > 0,$$

and K_ν the McDonald function defined, for $\nu \notin \mathbb{Z}$, by:

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)} \qquad x > 0,$$

and for $\nu = n \in \mathbb{Z}$ by :

$$K_n(x) = \lim_{\substack{\nu\to n\\ \nu\neq n}} K_\nu(x).$$

It is known that these functions generate the set of solutions of the linear differential equation:

$$u'' + \frac{1}{x}u' - \left(1 + \frac{\nu^2}{x^2}\right)u = 0.$$

Their derivatives are seen to satisfy several recurrence relations:

$$\begin{cases} xI'_\nu(x) = xI_{\nu-1}(x) - \nu I_\nu(x) = xI_{\nu+1}(x) + \nu I_\nu(x), \\ \\ xK'_\nu(x) = -xK_{\nu-1}(x) - \nu K_\nu(x) = -xK_{\nu+1}(x) + \nu K_\nu(x), \end{cases}$$

and their Wronskien takes a particularly simple form:

$$W(I_\nu(x), K_\nu(x)) := I_\nu(x)K'_\nu(x) - I'_\nu(x)K_\nu(x) = -\frac{1}{x}. \quad (11)$$

Note that they both simplify when $\nu = \pm 1/2$:

$$I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cosh(z), \qquad I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sinh(z)$$

and

$$K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}.$$

For real and strictly positive $\nu > 0$, I_ν is a positive increasing function and K_ν is a positive decreasing function. In this case, they both admit some useful integral representation:

$$\begin{cases} I_\nu(x) = \dfrac{x^\nu}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\displaystyle\int_0^\pi e^{x\cos(\theta)}\sin^{2\nu}(\theta)d\theta \\ \\ K_\nu(x) = \dfrac{1}{2}\left(\dfrac{x}{2}\right)^\nu \displaystyle\int_0^{+\infty}\frac{1}{t^{\nu+1}}\exp\left(-t - \frac{x^2}{4t}\right)dt \end{cases}$$

from which we may deduce the following equivalents (still for $\nu > 0$):

$$I_\nu(x) \underset{x\to 0}{\sim} \frac{x^\nu}{2^\nu\Gamma(\nu+1)} \quad \text{and} \quad K_\nu(x) \underset{x\to 0}{\sim} \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu},$$

and the asymptotic formulae when $x \to +\infty$:

$$I_\nu(x) \underset{x\to+\infty}{\sim} \frac{e^x}{\sqrt{2\pi x}} \quad \text{and} \quad K_\nu(x) \underset{x\to+\infty}{\sim} \sqrt{\frac{\pi}{2x}}e^{-x}.$$

A.2 Bessel processes with drift [11]

Let $\delta \in \mathbb{N}\backslash\{0\}$ and $c > 0$. The Bessel process of dimension δ (or equivalently of index $\nu = \frac{\delta}{2} - 1$) and drift c is the diffusion $(R_t^{(\delta,c)}, t \geq 0)$ with generator:

$$\mathcal{G} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{2\nu+1}{2x} + c\frac{I_{\nu+1}}{I_\nu}(cx)\right)\frac{\partial}{\partial x}. \qquad (12)$$

We denote by $\mathbb{P}_x^{(\delta,c)}$ its law when started from x. This process may be obtained as the euclidean norm of a δ-dimensional Brownian motion $\overrightarrow{B}$ with drift $\overrightarrow{\mu} \in \mathbb{R}^\delta$ such that $\|\overrightarrow{\mu}\| = c$:

$$R_t^{(\delta,c)} = \|\overrightarrow{B}_t + \overrightarrow{\mu}\,.\,\overrightarrow{t}\|.$$

The law of the first passage times of $(R_t, t \geq 0)$ is given by the following Laplace transform:

$$\mathbb{E}_x^{(\delta,c)}\left[e^{-\lambda T_a}\right] = \begin{cases} \dfrac{I_\nu(x\sqrt{2\lambda+c^2})}{I_\nu(a\sqrt{2\lambda+c^2})}\dfrac{I_\nu(ca)}{I_\nu(cx)} & \text{if } x \leq a, \\ \\ \dfrac{K_\nu(x\sqrt{2\lambda+c^2})}{K_\nu(a\sqrt{2\lambda+c^2})}\dfrac{I_\nu(ca)}{I_\nu(cx)} & \text{if } x \geq a. \end{cases}$$

In particular, when $\delta = 3$, (i.e. $\nu = \frac{1}{2}$), these formulae simplify to:

$$E_x^{(3,c)}\left[e^{-\lambda T_a}\right] = \begin{cases} \dfrac{\sinh(x\sqrt{2\lambda+c^2})}{\sinh(a\sqrt{2\lambda+c^2})}\dfrac{\sinh(ca)}{\sinh(cx)} & \text{if } x \leq a, \\ \\ \exp\left(-(x-a)\sqrt{2\lambda+c^2}\right)\dfrac{\sinh(ca)}{\sinh(cx)} & \text{if } x \geq a. \end{cases}$$

Note that this Laplace transform may be inverted, see [6, p.258]:

$$\mathbb{P}_x^{(3,c)}(T_a \in dt)/dt = \begin{cases} \dfrac{\sinh(ca)}{\sinh(cx)} \dfrac{1}{\sqrt{2\pi t^3}} \displaystyle\sum_{n=-\infty}^{+\infty} ((2n+1)a - x) \exp\left(-\frac{((2n+1)a-x)^2}{2t} - \frac{c^2}{2}t\right) & \text{if } x \leq a, \\ \dfrac{\sinh(ca)}{\sinh(cx)} \dfrac{(x-a)}{\sqrt{2\pi t^3}} \exp\left(-\dfrac{(x-a)^2}{2t} - \dfrac{c^2}{2}t\right) & \text{if } x \geq a. \end{cases}$$

A.3 Bessel processes

Letting $c \to 0$ informally in (12), we obtain the generator of the classic Bessel process with index ν:

$$\mathcal{G} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{2\nu+1}{2x}\frac{\partial}{\partial x}. \tag{13}$$

These processes are deeply related with geometric Brownian motion thanks to the following Lamperti's relation:

Theorem 7 ([18]). *Let $(B_t + \nu t, t \geq 0)$ be a Brownian motion with drift $\nu > 0$ starting from $\log(x)$. Then, there exists $(R_t^{(\nu)}, t \geq 0)$ a Bessel process of index ν started from x such that:*

$$\exp(B_t + \nu t) = R^{(\nu)}_{\int_0^t \exp(2(B_s+\nu s))ds}.$$

In particular, for $\nu > 0$, $(R_t^{(\nu)}, t \geq 0)$ is transient and $\lim_{t\to+\infty} R_t^{(\nu)} = +\infty$ a.s. In the framework of perpetuities, we may mention the following theorem, which allows to compute the law of some perpetuities involving squared Bessel processes, see [15, Theorem 1.7 p.444]:

Theorem 8. *Let φ be a positive and measurable function such that $\int_0^{+\infty}(1+x)\varphi(x)dx < +\infty$. Then:*

$$\mathbb{E}_x^{(\delta)}\left[\exp\left(-\int_0^{+\infty} R_t^2\varphi(t)dt\right)\right] = F(+\infty)^{\delta/2}\exp\left(\frac{x}{2}F'(0)\right)$$

where F is the unique solution on $[0, +\infty[$ of :

$\mathcal{F}'' = \varphi(x)F$ *such that F is positive, non increasing, and*

$F(0) = 1$.

A.4 The three-dimensional Bessel process [15, Chapter VI.3]

We now take $\delta = 3$ (i.e. $\nu = \frac{1}{2}$) in (13) to obtain the classic three-dimensional Bessel process. This process enjoys many important properties and is, in some sense, very close de Brownian motion. In particular, there is a weak absolute continuity formula between Brownian motion and the three-dimensional Bessel process :

$$\mathbb{P}^{(3)}_{x|\mathcal{F}_t} = \frac{B_t}{x}1_{\{t>T_0\}} \bullet \mathbb{P}_{x|\mathcal{F}_t}. \tag{14}$$

The paths of a three-dimensional Bessel process admit a useful decomposition as follows:

Theorem 9. *Let $(R_t, t \geq 0)$ be a three-dimensional Bessel process started from 0 and define $G_x = \sup\{t \geq 0, R_t = x\}$ it last passage time at level x. Then:*

i) conditionally on G_x, the processes $(R_t, t \leq G_x)$ and $(R_{t+G_x}, t \geq 0)$ are independent,

ii) the process $(R_{t+G_x}, t \geq 0)$ has the same law as $(x + R_t, t \geq 0)$,

iii) the process $(R_t, t \leq G_x)$ has the same law as $(B_{T_0-t}, t \leq T_0)$ where $(B_t, t \geq 0)$ is a Brownian motion started from x and $T_0 = \inf\{t \geq 0,\ B_t = 0\}$.

This result was first proven by Williams [18] in its decomposition of the Brownian paths, see also Pitman [12] for a related study.

We conclude this appendix by stating the third Ray-Knight theorem, which describes the dependence in the space variable of the total local time of the three-dimensional Bessel process :

Theorem 10 (Ray-Knight). *Let $(R_t, t \geq 0)$ be a three-dimensional Bessel process started from 0 and denote by $L_\infty^y(R)$ its total local time at level y. Then:*

$$(L_\infty^y, y \geq 0) \overset{(law)}{=} (Z_y^{(2)}, y \geq 0)$$

where $Z^{(2)}$ is a two-dimensional squared Bessel process started from 0.

We refer to [2, Chapter V] for other similar Ray-Knight theorems.