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An APOS Analysis of Natural Science Students' Understanding of Integration

Aneshkumar Maharaj[1]

1) University of KwaZulu-Natal, South Africa.

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Aneshkumar Maharaj
University of KwaZulu-Natal

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Abstract

This article reports on a study which used the APOS (action-process-object-schema) Theory framework and a classification of errors to investigate university students' understanding of the integration concept and its applications. Research was done at the Westville Campus of the University of KwaZulu-Natal in South Africa. The relevant rules for finding antiderivatives, the link between derivatives and antiderivatives, interpreting a definite integral as area under the relevant curve and their context-based applications were taught to undergraduate science students. This paper reports on the analysis of two students' responses to questions on integrals and their applications. The findings of this study suggest that those students had difficulty in applying the rules for integrals and their applications, and this was possibly the result of them not having appropriate mental structures at the process, object and schema levels.

Keywords: integral concept, APOS theory, and student errors

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Un Análisis APOS de la Comprensión del Concepto de Integral de los Estudiantes de Ciencias Naturales

Aneshkumar Maharaj
University of KwaZulu-Natal

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Resumen

Este artículo responde a un estudio en el que se utiliza la teoría APOS (acción-proceso-objeto-esquema) y una clasificación de errores para investigar la comprensión del concepto de integral de un conjunto de estudiantes universitarios, y sus aplicaciones. La investigación se llevó a cabo en el Campus Westville de la University of KwaZulu-Natal en Sudáfrica. Las normas relevantes para encontrar anti-derivadas, la relación entre las derivadas y las anti-derivadas, la interpretación de la integral definida como área bajo la curva y las aplicaciones basadas en el contexto fueron enseñadas a los estudiantes de grado de ciencias. Este artículo presenta el análisis de las respuestas de dos estudiantes a preguntas sobre integrales y sus aplicaciones. Los resultados sugieren que los estudiantes tienen dificultades en la aplicación de las normas de integración, y posiblemente este resultado fue motivado por no disponer de estructuras mentales del proceso adecuadas.

Palabras clave: concepto de integral, teoría APOS, errores de los estudiantes

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In South Africa, students encounter the concept of the integral and related applications during their first year university studies in mathematics. The natural science students at our university are exposed to the following interpretations with regard to the integral concept: (1) $\int f(x)dx$ represents the general antiderivative of $f(x)$ so $\int f(x)dx = F(x) + C$ provided $F'(x) = f(x)$. (2) For a continuous function $f(x) \geq 0$ on the interval $[a, b]$ the definite integral $\int_a^b f(x)dx$ can be interpreted as the area formed between the graph of $y = f(x)$ and the x-axis on the interval $[a, b]$. (3) $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$. (4) If $f(x)$ gives the rate of change of $F(x)$; that is $f(x) = F'(x)$; for x in $[a, b]$, then the total change in $F(x)$ as x goes from a to b is given by $\int_a^b f(x)dx$. Students are also exposed to the rules for standard antiderivatives, the u-substitution technique for integration, and the technique for integration by parts. They are then expected to apply all of these to context based applications, specific to a field of study.

My interactions with first year natural science mathematics students, at the University of KwaZulu-Natal, indicated that many of them find it difficult to evaluate integrals especially when these are given out of the context of a particular section. For example, when integrating $\int\left(5^{-2t} + \sqrt{t}\right) dt$ some students respond with $\frac{1}{-2t}5^{-2t+1} + \frac{2}{3}t^{\frac{3}{2}} + C$. They confuse the standard integral structure for $\int x^n \, dx$ where $n \neq -1$, with that of $\int a^x \, dx$ where $a > 0$ but $a \neq 1$. This indicated that there was a need to engage with a study on students' understanding of the structures of integrands and how this should inform the integration technique to be used. The research questions for this study were: How should the teaching of the concept of integration be approached? What insights would an APOS analysis of students' understanding of the integration concept and related applications reveal?

Literature Review

Various studies (eg. Abdul-Rahman, 2005; Orton, 1983; Sevimli & Delice, 2010; Haciomeroglu, Aspinwall & Presmeg, 2009) have focused on student understanding of integration and what could be done to improve their understanding. Those studies suggest that students face difficulties in the integration concept for two principal reasons: 1) differentiation can be viewed as a *forward* process and the difficulties faced by students in this concept are not as complicated as those in the reverse or backward process of integration, and 2) integration has a dual nature since it is both the inverse process of differentiation and a tool for calculation, of for example area and volume. In his study of student understanding of the integration concept, Orton (1983) tried to categorize student errors as (a) *structural errors* - those errors arising from some failure to appreciate the relationships involved in the problem or to grasping a principle essential to the solution, (b) *arbitrary errors* - those in which the subject behaved arbitrarily and failed to take account of the constraints laid down in what was given, and (c) *executive errors* - those which involved failure to carry out manipulations, though the principles involved may have been understood. He found that some errors involved elements of more than one type.

The above studies imply that when the antiderivative is introduced this should be related to the concept of the derivative. For example $\int f(x)dx$ represents the general antiderivative of $f(x)$ so $\int f(x)dx = F(x) + C$ provided $F'(x) = f(x)$. Visualization in the graphical context can help students to understand the relations between differentiation and integration. So teaching should focus on the development of spatial visualization ability, which could influence and strengthen the relationship between the graphical and the symbolic integral representations, since this “increases the performance of solving definite integral problems” (Sevimli & Delice, 2010, p. 57-58). The implication here is that visualization should be used when the definite integral is introduced, since visualization could be an important aid to students when confronted with a definite integral problem. However, it seems that a student’s use of area under a curve is helpful in problem solving only when a deeper understanding of the structure behind the definite integral is present (Sealey, 2006).

Haciomeroglu, Aspinwall, & Presmeg (2009) illustrated how students’ understanding can be enriched by changing thinking processes and establishing reversible relations between graphs of functions and their derivative or antiderivative graphs. They analysed three students’ thinking processes in the context of those students’ responses and sketches to solving tasks during interviews. Those interviews led to their findings that (1) a student displayed either a preference for analytical thinking or visual thinking, (2) students' visual or analytic interpretations of the derivative graph to be an example of a one-way relationship (differentiation→ integration), not as a reversible two-way (differentiation ↔ integration) relationship. Since differentiation and integration are two fundamental concepts of calculus, and are by their nature inverse processes, the implication is that the reversibility of thinking be emphasized when exploring the relationship between derivatives and antiderivatives. For example the derivative of a^{kx} is $k \ln a \,.\, a^{kx}$, therefore an antiderivative of a^{kx} is $\frac{1}{k \ln a} a^{kx}$ so $\int a^{kx} dx = \frac{1}{k \ln a} a^{kx} + C$.

Theoretical Framework

The design of the teaching and learning experience to which the students were exposed was guided by APOS (action-process-object-schema) Theory (Dubinsky & McDonald, 2001). A more detailed account of this theory can be found in Maharaj (2010, 2013). APOS theory proposes that an individual has to have the appropriate mental structures relating to action, process, object and schema to make sense of a given mathematical concept. So if appropriate mental structures are not present, then learning the concept is likely to be almost impossible. Research based on this theory requires that for a given concept the likely mental structures need to be detected, and then suitable learning activities should be designed to support the construction of those mental structures. The following assumptions underpin APOS theory and its application to teaching practice (Dubinsky, 2010): [1] Assumption on mathematical knowledge: An individual's mathematical knowledge is his/her tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context, and constructing or reconstructing mental structures to use in dealing with the situations. [2] Hypothesis on learning: An individual does not learn mathematical concepts directly. Rather he/she applies mental structures to make sense of a concept (Piaget, 1964). For a given mathematical concept, learning is facilitated if the individual possesses the appropriate mental structures. The descriptions of action, process, object and schema that follow are based on those given by Weller, Arnon & Dubinsky (2009). Action: A transformation is first conceived as an action, when it is a reaction to stimuli which an individual perceives as external. It requires specific instructions, and the need to perform each step of the transformation explicitly. Process: As an individual repeats and reflects on an action, it may be interiorized into a mental process. A process is a mental structure that performs the same operation as the action, but wholly in the mind of the individual. Some do not agree with the latter point. However, my interactions with Dubinsky indicated that this is how he interprets process. Specifically, the individual can imagine performing the transformation without having to execute each step explicitly. Object: If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has encapsulated the process into a cognitive object. Schema: A mathematical topic often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, called a schema. It is coherent in that it provides an individual with a way of deciding, when presented with a particular mathematical situation, whether the schema applies.

A genetic decomposition of a mathematical concept is a structured set of mental constructs which might describe how this concept can develop in the mind of an individual (Asiala, et. al., 1996). If this is accepted then a genetic decomposition postulates the particular actions, processes, and objects that play a role in the construction of a mental schema for dealing with a given mathematical situation. The genetic decomposition arrived at for the integral concept, was as follows.

As part of his or her function schema, the student has developed:

1. a process or object conception of a function, and
2. a process or object conception of product and composition of functions.

As part of his or her integral schema, the student has:

3. an action conception which enables the finding of integrals of simple functions, whose rules are given in the symbolic form. For example, $\int 3\, x^2 dx$.
4. a process conception of integration which enables the finding of integrals of functions. This could involve studying the structure of the function, detecting whether a rule for integration could be applied or whether the function should be written in a standard form which enables the application of the appropriate rules for integrating.

5. an object conception which enables the seeing of strings of processes as a totality and performing mental or written actions on the internal structure of the given function which enables integration. For example, the student views the integrand $h(x)$ as an object which is a product of $g'(x)$ and a composition of two functions, $h(x) = g'(x).f(g(x))$, to which the u-subsitution technique for integration can be applied.
6. organized the actions, processes, and objects related to the integral concept and linked them into a coherent framework. This framework includes various interpretations of the integral in different contexts, and possible techniques for [a] finding integrals of various function types, [b] finding improper integrals, [c] interpreting the area between two curves as a definite integral, [d] setting up a definite integral to represent the volume of a solid of revolution, or [e] determining the total change of a function on an interval when given the rate of change of the function.

The ACE Teaching Cycle is a pedagogical approach, based on APOS Theory and the hypothesis on learning and teaching. It is a repeated cycle consisting of three components: (A) activities, (C) classroom discussion, and (E) exercises done outside of class (Asiala, et. al., 1996). The activities are designed to foster the students' development of the mental structures called for by an APOS analysis. By performing mathematical tasks in a formal setting, for example a classroom, students are guided by the teacher to reflect on the activities and its relation to the mathematical concepts being studied. Students then discuss their results and listen to explanations. Fellow students or the teacher, could provide explanations for the mathematical meanings of what they are working on. Classroom discussion is followed by homework exercises which are fairly standard problems. These reinforce the knowledge obtained by the activities and classroom discussions. Students are required to apply that knowledge to solve standard problems, related to the topic being studied. The implementation of such an approach and its effectiveness in helping students make mental constructions and learn mathematics has been reported in several research studies (eg. Weller et al., 2003; Maharaj, 2010). It is in that context the teaching and learning experiences relating to the integral and its applications were designed. Figure 1 gives an overview of how APOS

Theory impacted on the activities, classroom discussion and the homework exercises. Note that at the tutorials students had to produce their attempts to the homework exercises. This was to focus the discussions between a tutor and his/her students in a group setting.

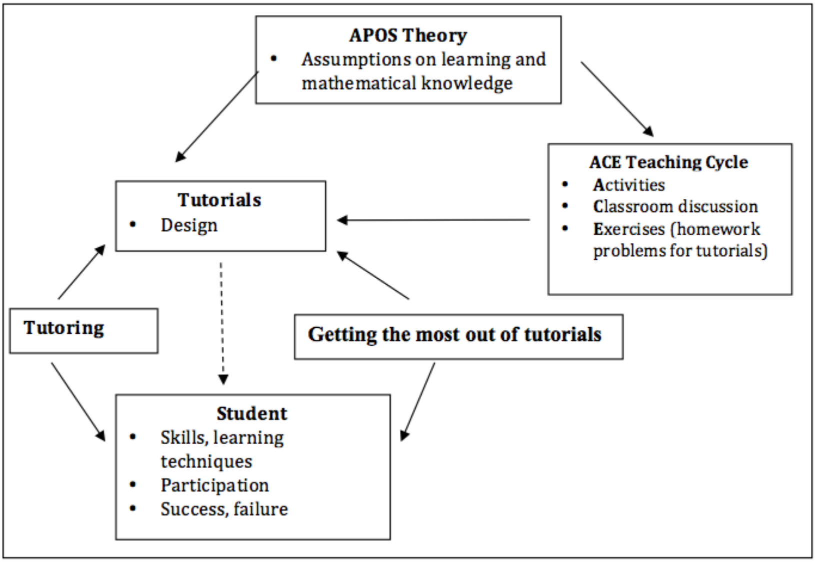

Figure 1. Impact of APOS Theory on the teaching and learning experience

Participants and Methodology

This took into consideration the implications from the literature review and theoretical framework. Both of those influenced the design of the teaching and learning experience that the participants were exposed to, see Figure 1. The participants for this study were two first year natural science students who studied the module *Further Topics in Mathematics*. The written responses of those students to a written test (see Appendix A for some of the questions) were analysed in the context of the genetic decomposition outlined in the theoretical framework and the three error types discussed in the literature review. Then interviews were held with those two students to get further clarity on their written responses and the possible reasons for those responses. So the methodology which was qualitative and interpretative relied on document analysis (written responses) and interviews. The investigation could be viewed as a case study based on two students' understanding of integration and its applications. Writing about case studies, Cohen, Manion and Morrison (2002) noted that: "It provides a unique example of real people in real situations, enabling readers to understand ideas more clearly than simply presenting them with abstract theories or principles" (p. 180). This was the motivation for choosing randomly two students to investigate their mental constructions with regard to the concept of integration and its applications.

Analysis and Findings

These focus of the written and interview responses of two students who are referred to as Student C and Student R.

Written and interview responses of Student C

Figure 2 indicates the following three shortcomings in the written response: (1) incorrect use of the implication sign, Þ, in the second line which seems to be an executive error, (2) incorrect writing of the second integral in the second line, since dt is omitted, and (3) incorrect application of the integral of an exponential function, for the object $\int a^{kx}dx$ where $a > 0$ but $a \neq 1$, second term in the last line which implies a structural error. During the interview Student C indicated he could see nothing wrong with his response. Those errors could be a result of this student inadequately interpreting objects represented in symbolic form. With regard to the incorrect use of the implication sign the following transpired during the interview:

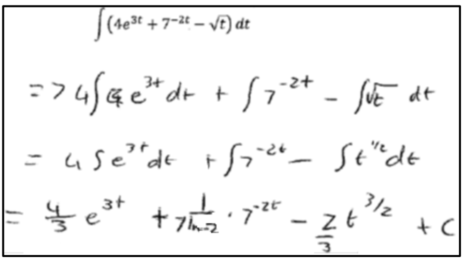

Figure 2. Written response of Student C to question 1.2

> Researcher: What sign should you use here? (pointing to the implication sign in the second line).
> Student C: Therefore, no, equal to sign. No, I don’t know.

After the researcher explained the use of the implication and equal to sign in the context of expressions and equations respectively the student, pointing to the integral in the first line, responded: I understand it is an expression, not a formula. This indicates he made an executive error. When asked to explain his incorrect writing of the second integral, the student indicated he was rushing. With regard to explaining his incorrect application of the integral of the exponential function, see second term in the last line, he responded:

> Student C: I switched them up. It should be, $-2 \ln 7$.

His response indicated he knew the relevant rule for integration. It seems he did not sufficiently unpack the structure of the object $\int 7^{-2t}\,dt$ with the relational structure of the objects in the context of the relevant rule for integration, $\int a^{kx} dx = \frac{1}{k \ln a} a^{kx} + C$.

An analysis of Student C’s response to question 3.3, see figure 3, revealed that he (1) adequately unpacked the question, evidence of this is his underlining of the important words and the first 6 lines of his written response, and (2) had a suitable schema for evaluating integrals since he detected that the integration by parts technique was required, as evident from the 5th and 6th lines of his written response. However, two executive errors are evident in the 7th and 11th lines. During the interview the following question was posed to get an insight regarding the error in the 7th line.

> Researcher: What type of an integral is this? (Pointing to the 7th line).
> Student C: I don’t know. What do you mean?

After explaining to him with examples that during lectures indefinite, definite and improper integrals were discussed, he responded that it was a definite integral. He also mentioned that he was careless in equating a definite integral with an indefinite integral. This indicated that during the interview, after prompting the student was able to detect the structure of the definite integral and realized that he made an executive error. However, it could be argued that he did not correctly interpret the definite integral as the object it represented and this led to a structural error, which led to the executive error.

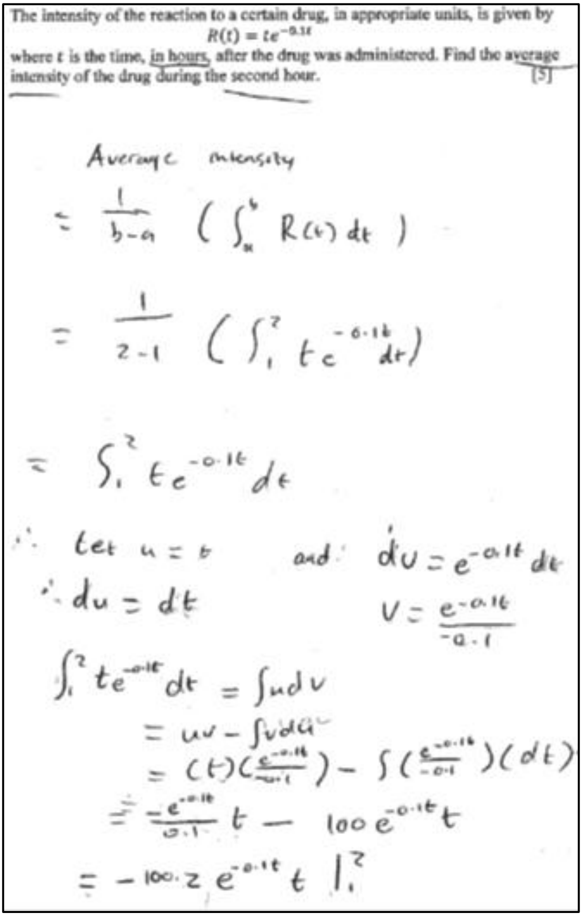

Figure 3. Written response of Student C to question 3.3

The written response of Student C to question 3.2, see Appendix A, was almost perfect with a minor executive error. During the interview he correctly indicated that an improper integral was involved. His written response indicated that he correctly introduced limits to evaluate the improper integral $\int_{-\infty}^{-1} \frac{1}{x^3} dx$. What was confusing was his written response to question 3.4, see figure 4, which also dealt with a similar improper integral. His response indicates that he had a suitable schema for dealing with word problems based on the total change of a function on an interval when given the rate of change of the function. However, the 9th line gives the suggestion of an executive error. This was further probed during the interview.

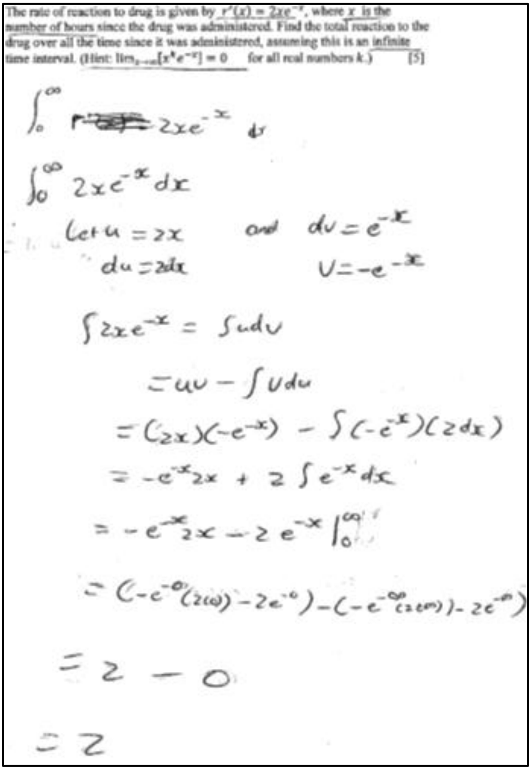

Figure 4. Written response of Student C to question 3.4

> Researcher: What type of an integral is this? (Pointing to the 1st and 2nd lines of his written response)
> Student C: Improper, as well.
> Researcher: Is this correct? (Pointing to the 9th line of his written response)
> Student C: No. Have to introduce limits. (Writes $\lim_{b\to\infty}(\quad)\big|_0^b$.)

Again, the student's response above suggests that the error beginning in the 9th line was a structural error. Once he correctly read the structure represented in symbolic form and categorized the object as an improper integral, he was able to indicate that limits had to be introduced. However, it seems that during the writing of his response he did not go through those required processes which serve as guides to prevent structural errors.

Figures 2, 3 and 4 indicate that Student C had schemata developed for functions and integration. According to my genetic decomposition those enabled him to deal with situations requiring the (1) determining of antiderivatives of basic functions [Figure 2], (2) detecting and applying of the integration by parts technique [Figures 3 and 4], and (3) determining the total change of a function on an interval when given the rate of change of the function, in a word problem context [Figure 4]. However his execution of those schemata included the types of errors indicated in the literature review.

Written and Interview Responses of Student R

Figure 5 indicates that Student R was able to correctly interpret the basic function structure in the integrand as an object and correctly apply the relevant process in accordance with the appropriate rule for integration. There is evidence of possibly an arbitrary error since he failed to take account of the constraints laid down in what was given, an indefinite integral in symbolic form. He did not include the integration constant in the 2nd and 3rd lines of his written response. During the interview, he indicated that he should have included this in the 3rd line. However, it was clear that he did not know why that constant was included. This was evident during the interview.

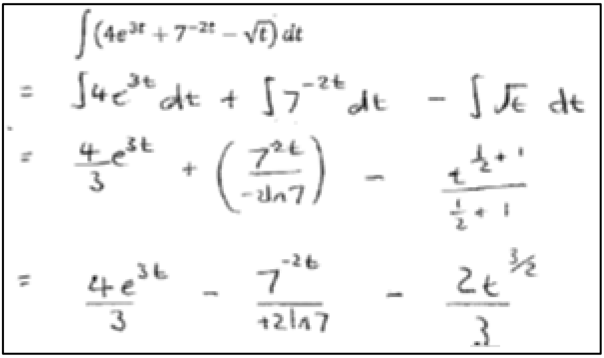

Figure 5. Written response of Student R response to question 1.2

> Researcher: Do you know why we add on a constant?
> Student R: No. I just know that after you integrate you add on C.
> Researcher: What does the symbol [pointing to the indefinite integral symbol in $\int(\quad)dt$] represent?
> Student R: To find the integral with respect to t.
> Researcher: Does that integral symbol represent anything else?

Even after probing he did not indicate that the integral could also be interpreted as an antiderivative. The above imply that Student R only interpreted the structure as an integral, he did not classify the type of integral he was required to react to. This classification is important since it serves as a trigger for caution, to take account of hidden constraints.

The written response of Student R to question 1.6, see figure 6, suggests that he made an executive error. However, a closer examination of the 1st line of his written response and what transpired during the interview suggested an arbitrary or a structural error. Firstly note that in the 1st line of his written response he makes two distinct but unrelated assumptions, an arbitrary error. Here I am referring to his use of the word, let. He did not use his 1st assumption to work out the implication for y in terms of u, a structural error.

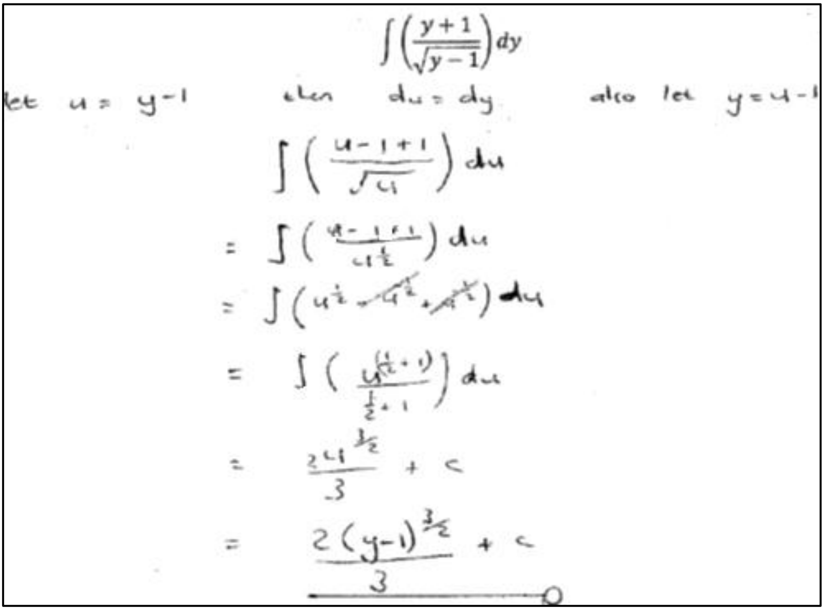

Figure 6. Written response of Student R to question 1.6

> Researcher: Do you know why this is incorrect? [Pointing to $y = u - 1$ in the first line of his response]
> Student R: No.
> Researcher: From $u = y - 1$ what is y equal to? [After a period of silence, writes $u = y - 1, \therefore y =$ ___]
> Student R: $y = u + 1$. I completely didn't see that. …. I just went too fast without thinking it through.

The implication is that students should be more careful in their writing and their use of assumptions. Students need to interrogate what they write and say in the context of objects. This could help to reorganize and refine their mental structures and schemata.

Note that for question 2.3, see Appendix A, the formulation of an integral that represents the area was required. Figure 7 indicates that Student R had a schema to deal with such type of situations. His written response indicates a structural error, which arises from his failure to appreciate the relationships involved in the translation of the graphical representation of the area to a definite integral representation. He failed to grasp an essential principle to the solution, that the finite area can be represented by a definite integral. Further note that the 2nd and 3rd lines of his response to the right of his graph indicate that he was solving an expression, instead of an equation when finding the x-intercepts of the parabola.

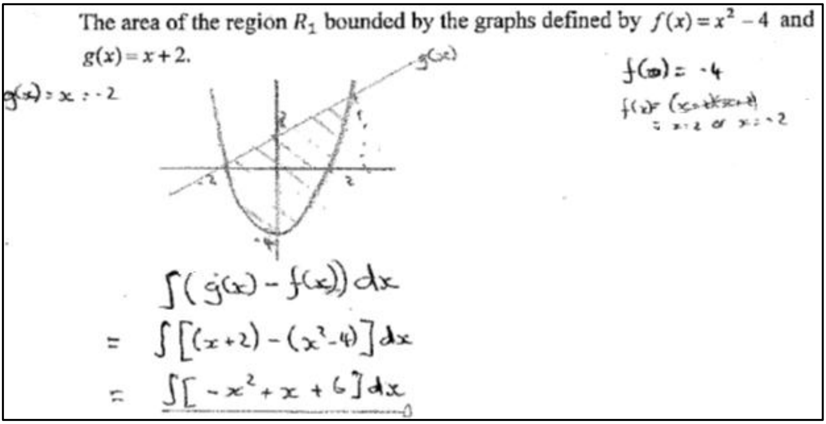

Figure 7. Written response of Student R to question 2.3

> Researcher: On which interval must the integration be done?
> Student R: From −2 to that point. [pointing to the intersection in the first quadrant] …. Which I didn't calculate.

His response during the interview indicates that he knew that a definite integral was required. It seems from his written response in figure 7 and the discussion for figure 5 that he did not have an adequate schema to distinguish between the different types of integrals and what their symbolic notations represented. Figures 5, 6 and 7 indicate that Student R had some sort of schemata, although not adequately developed, for functions and integration. According to my genetic decomposition those enabled him to deal with situations requiring the (1) determining of antiderivatives of basic functions [Figure 5], (2) detecting and applying of the integration by u-substitution technique [Figure 6], and (3) interpreting the area between two curves as a (definite) integral [Figure 7]. However, a closer look at the figures indicated evidence of one of the three types of errors outlined in the literature review.

Conclusions and Recommendations

Although this study dealt with the qualitative interpretation of written and interview responses of two students, the theoretical framework provided useful insight into their understanding of integration and related errors. These could be generalized. Errors made by students could result from their inadequate interpreting of objects represented in symbolic form. For example errors could result if a student does not sufficiently unpack the structure of the object $\int 7^{-2t}\,dt$ with the relational structure of the objects in the context of the relevant rule for integration, $\int a^{kx}dx = \frac{1}{k\ln a}\,a^{kx} + C$.

Errors made by students could also be the result of them not having an adequate schema to distinguish between the different types of integrals and what their symbolic notations represent. If a student does not correctly interpret the definite integral as the object it represents then this could lead to a structural error, which could be the root cause of an executive error. So with regard to the three types of errors discussed in the literature review this study supports the finding of Orton (1983) that some errors could involve elements of more than one type.

This study also suggests that students need be more careful in their writing and use of assumptions. Teaching should therefore focus on the need for students to interrogate what they write and say in the context of objects. This could help them to reorganize and refine their mental structures and schemata. Since derivatives and antiderivatives are related concepts, the teaching of integration should focus on the reversibility of thinking between derivatives and antiderivatives. For example $\int f(x)dx$ represents the general antiderivative of $f(x)$ so $\int f(x)dx = F(x) + C$ provided $F'(x) = f(x)$. During the teaching of integration techniques, to eliminate the types of errors discussed, the focus should be on the object represented by the integrand and the relational structure of the objects in the context of the relevant rule for integration. This requires a focus on the actions and processes that are necessary to interpret the structure of the relevant objects. When the definite integral is introduced, visualization should be used since this could be an important aid to students when confronted with a definite integral problem.

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Aneshkumar Maharaj is a lecturer in the School of Mathematics, Statistics and Computer Science at the University of KwaZulu-Natal, Durban, South Africa.

Contact Address: Westville Campus, Private Bag X54001, Durban 4000, South Africa. **Email:** maharaja32@ukzn.ac.za

Appendix A: Some of the questions

Determine the following:

1.1

$$\int \left(2 + x^3 - \frac{3}{x}\right) dx$$

1.2

$$\int \left(4e^{3t} + 7^{-2t} - \sqrt{t}\right) dt$$

1.6

$$\int \left(\frac{y+1}{\sqrt{y-1}}\right) dy$$

1.7

$$\int x^3 lnx\, dx$$

2.1 Use appropriate geometric figures to evaluate

$$\int_0^2 \left(5 - \sqrt{4 - x^2}\right) dx.$$

2.2 The rate of infection of a disease (in people per month) is given by the function

$$I'(t) = \frac{100t}{t^2 + 1}$$

where t is the time (in months) since the disease first broke out (when $t = 0$).

2.2.1 Interpret the definite integral: $\int_0^4 I'(t)dt$.

2.2.2 Write a definite integral to express the total number of people who will be infected in the second month of the disease.

2.3 For the following, draw an appropriate sketch and then express what is required in integral form. [DO NOT EVALUATE THE INTEGRAL.]
The area of the region R_1 bounded by the graphs defined by
$f(x) = x^2 - 4$ and $g(x) = x + 2$.

3.2 Evaluate, if possible, the improper integral: $\int_{-\infty}^{-1} \frac{1}{x^3}\, dx$

3.3 The intensity of the reaction to a certain drug, in appropriate units, is given by $R(t) = te^{-0.1t}$ where t is the time, in hours, after the drug was administered. Find the average intensity of the drug during the second hour.

3.4 The rate of reaction to drug is given by $r'(x) = 2xe^{-x}$, where x is the number of hours since the drug was administered. Find the total reaction to the drug over all the time since it was administered, assuming this is an infinite time interval. (Hint: $\lim_{x\to\infty}[x^k e^{-x}] = 0$ for all real numbers k.)