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Horizon Content Knowledge: Shaping MKT for a Continuous Mathematical Education

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Abstract

Mathematics learning is a continuous process in which students face some abrupt episodes involving many changes of different natures. This work is focused on one of those episodes, transition from primary to secondary school, and targets teachers and their mathematical knowledge. By characterising the mathematical knowledge that teachers of mathematics need to smooth transition processes, we aim to highlight teachers' impact in the continuity of mathematics education. The concept of Mathematical Knowledge for Teaching (MKT) developed by Ball among others and, within this framework, the construct Horizon Content Knowledge (HCK) emerge as our theoretical response to the knowledge needed for teaching mathematics in a continuous way, particularly relevant during transition to secondary school. The enrichment of the idea of HCK and the characterisation of its expression in the teaching practice intends to develop a theoretical tool to approach transition from teachers' mathematical knowledge perspective.

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Horizonte de Contenido de Conocimiento: Dando Forma al MKT para la Continuidad de la Educación Matemática

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Resumen

El aprendizaje de las matemáticas es un proceso continuo en el que los estudiantes afrontan algunas transiciones abruptas. Este trabajo se centra en una de estas transiciones, la del paso de la educación primaria a la secundaria, desde la perspectiva del conocimiento matemático del profesorado. Nuestro objetivo es caracterizar cuál es el conocimiento que necesitan los maestros de matemáticas para hacer esta transición más fluida. El concepto de Conocimiento del Horizonte Matemático emerge como conceptualización teórica para analizar la transición desde la perspectiva del conocimiento matemático de los maestros y atender a la enseñanza de las matemáticas de una forma continua.

Palabras clave: transición, continuidad, conocimiento matemático para la enseñanza, horizonte de contenido de conocimiento

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urrent trends in mathematics education tend to consider the roles of the teacher and the students from a holistic perspective, considering the whole process of learning and, particularly, the acquisition of mathematical competences from a long life learning perspective. Mathematics learning is then conceived as a long life enterprise for both teachers and students. It is a continuous process in which students face some abrupt episodes involving many changes of different nature that derive in a variety of alterations in their educational path.

Compulsory transitions within the specific educational structure, called systemic transitions by Rice (1997), result as especially problematic for students since during them the external influences are particularly significant and affect students' academic achievement and adjustment to mathematics education (Ding, 2008; McGee, Ward, Gibbons & Harlow, 2004; New Zealand Ministry of Education, 2008; Noyes, 2006; Rice, 1997). In particular, transitions from primary to secondary and secondary to tertiary education have been studied the most in the past (Anderman & Midgley, 2004; Coad & Jones, 1999; Ferguson & Fraser, 1998; Jourdan, Cretchley & Passmore, 2007; Kajander & Lovric, 2005; Sdrolias & Triandafillidis, 2008; Rice, 1997; Zanobini & Usai, 2002; Zeedyk et al., 2003). The former's importance arises from its compulsory character for every pupil. The latter obviously affects only students who choose to study university degrees, but socio-political and economical interests explain part of its relevance in educational research.

Albeit systemic transitions are processes with a broad range of effects and consequences in the whole students' educational experience, we want to centre our work in the specificity of mathematics in transition. Which characteristics describe transition in mathematics? What makes it different? Are there any factors specifically linked to mathematics that are relevant during transition? Looking at previous research foci of attention and conclusions the content emerges as a determinant factor. For instance, the step from arithmetic to algebra (Boulton-Lewis et al., 1997; Boulton-Lewis, Cooper, Atweh, Pillay & Wilss, 1998; Cooper et al., 1997; Flores, 2002; Gonzales & Ruiz Lopez, 2003), the learning of integer numbers (Gallardo, 2002; Pujol, 2006) or the development of the need of proofs in geometry (Berthelot & Salin, 2000-2001; Sdrolias & Triandafillidis, 2008) and other fields appear as particular well-known problems embedded in the teaching and learning of mathematics that involve transition to secondary school.

Also, the mathematical content is highlighted in previous investigations regarding transition to university mathematics (Guzmán, Hodgson, Robert & Villani, 1998; Jourdan, Cretchley & Passmore, 2007; Kajander & Lovric, 2005). Since teachers shape the access of students to this content, we consider their role in transition as crucial.

However, the scarce number of investigations of our literature review in which the teacher plays a relatively relevant role at some point are mainly focused on the teacher's role in the classroom during transition and his/her influence on the classroom environment (Fraser, 1998; Ferguson & Fraser, 1998; Queen, 2002), teachers' views and opinions about transition (Akos & Galassi, 2004; Lewbel, Haskins, Spradling & Thompson, 1996) and the importance of teachers' knowledge in primary-secondary transition when it is applied to specific students' special needs (Zhang, Ivester & Katsiyamis, 2005; Benitez, Morningstar & Frey, 2009). Also, investigations specifically referred to mathematics have been mainly focused on students-teachers interaction (Ding, 2008; Noyes, 2006) the effects of teaching strategies as team teaching (Alspaugh & Harting, 1997; Alspaugh & Harting, 2008) and the role of the teacher in the development of longitudinal projects designed to smooth transition (Simpson & Goulder, 1998). In other words, investigations of teachers' knowledge have been commonly referred to pedagogic knowledge, leaving aside mathematical content knowledge, which we consider fundamental to successfully accompany students through the mathematical content during transition.

Our work on transition is centred in the transition from primary to secondary mathematics and knowledge for teaching will be our main focus of research. Reasons to choose the access to secondary education include the following: a) it is a systemic transition; b) it takes place within the compulsory education and thus affects every student; c) it emphasises the learning difficulties encountered by the students when they face some crucial mathematical contents, like algebra, integer numbers or generalisation and proof, with more presence in secondary than in primary; and d) it involves many differences which target teacher knowledge and teacher practice as key issues for the investigation. Besides the broad number of previous studies centred in students during transition and in the development of educational policies designed to smooth this process, it is also necessary to pay attention to teachers' mathematical knowledge and practice during transition. For this we require a theoretical frame that suits

our purpose of gaining a better understanding of the role of the teacher and, particularly, the relationship of his or her mathematical knowledge and practice with primary to secondary transition.

Within the practice-based theory of content knowledge for teaching developed by Ball, Thames, and Phelps (2008) we choose the theoretical concept of Mathematical Knowledge for Teaching (MKT) (Ball, Thames & Phelps, 2008; Hill, Rowan & Ball, 2005; Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008), which is a combined knowledge of the subject itself and pedagogy that is necessary to carry out the job of teaching successfully. MKT is divided by the authors in different domains and sub-domains that aim to cover the complexity of this subject. One of these sub-domains, Horizon Content Knowledge (HCK), emerges as the theoretical response to the knowledge for teaching mathematics in a continuous way, particularly relevant during transition to secondary school. HCK refers to the general knowledge of the previous and the forthcoming. Recently, they have referred to HCK as a "mathematical peripheral vision needed in teaching" (Hill, Rowan & Ball, 2005, p. 70) that they have lately detected and in which they are working at the moment.

Now we are ready to place our research focus on the enrichment of the idea of HCK in the particular context of primary to secondary transition. We will deal with questions such as: What characteristics must a primary or secondary school teacher have in order to ensure a smooth transition? Which ones refer specifically to the mathematical knowledge? To what extent do these questions help to define HCK and its place within MKT frame? How is HCK reflected in teachers' everyday practice at any grade? Do teacher training programs need to consider HCK specifically? Far away to offer complete responses to these questions, our aim is to give some hints to address in future research and enlarge the conceptualisation of HCK in order to use it as a theoretical tool to approach transition.

Transition And Horizon Content Knowledge

Our purpose of investigating transition looking at teachers' knowledge requires a theoretical frame in which to look for elements that allow us to attend continuity in the teaching of mathematics.

The theoretical model of the Mathematical Knowledge for Teaching (MKT) developed by Hill, Rowan and Ball (2005) is a solid framework on

the subject of teachers' knowledge that suits our purpose. Ball, Thames and Phelps (2008) distinguish two domains within the MKT, namely Subject Matter Knowledge and Pedagogical Content Knowledge. These are not independent from each other, but it is their combination which defines the knowledge needed for teaching mathematics. Pedagogical content knowledge refers to "the special ways in which teaching demands a simultaneous integration of key ideas in the content with ways in which students apprehend them" (Ball, Thames & Phelps, 2008, p.393) and subject matter knowledge concerns the knowledge of mathematics itself. Each one of them is also sub-divided in more specific domains. One of the sub-domains recently included in this framework, the Horizon Content Knowledge (HCK), emerges as particularly relevant to transition since it concerns the longitudinal view of mathematical instruction that teachers need. However, as Ball, Thames and Phelps (2008) point out, HCK's place in this framework is not clear yet and thus, our subsequent aim is to refine and position the construct of HCK so that it can be applied to attend transition. In the following section we continue by addressing the main features of MKT theory to the extent that they affect our development of HCK (see Figure 1).

Mathematical Knowledge for Teaching (MKT) overview

Pedagogical content knowledge is subdivided attending three foci of attention within the teaching practice: students, methodology and curriculum. Knowledge of content and students (KCS) is a combination of "specific mathematical understanding and familiarity with students and their mathematical thinking" (Ball, Thames & Phelps, 2008, p.401) and therefore involves students' expected difficulties, questions, motivations, etc. and teacher's preparation and responses to those. Knowledge of content and teaching (KCT) is a combination of "specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (Ball, Thames & Phelps, 2008, p.401) that concerns methodology issues such as the design of the sequence of a topic or the use of appropriate tasks, representations and examples, etc. Finally, knowledge of content and curriculum (KCC) is the curricular knowledge needed for teaching.
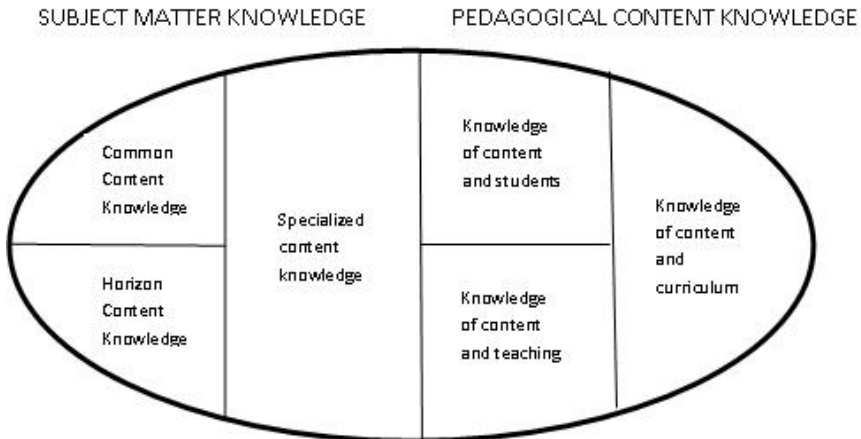
SUBJECT MATTER KNOWLEDGE   PEDAGOGICAL CONTENT KNOWLEDGE

Common Content Knowledge

Specialized content knowledge

Horizon Content Knowledge

Knowledge of content and students

Knowledge of content and teaching

Knowledge of content and curriculum

Figure 1. Categories of Mathematical Knowledge for teaching (Ball, Thames & Phelps, 2008, p.403)

Subject matter knowledge is sub-divided in three categories: common content knowledge (CCK), which is the mathematical knowledge that is common to other professions and specialised content knowledge (SCK), which is the specific mathematical knowledge needed for the teaching practice (Ball, Thames & Phelps, 2008). Our main interest is focused on a third sub-domain provisionally included within subject matter knowledge: the horizon content knowledge (HCK). About HCK the authors say:

> We are not sure whether this category is part of subject matter knowledge or whether it may run across the other categories. (Ball, Thames & Phelps, 2008, p.403)

In the following sections we will explore the concepts of MKT and HCK in order to clarify the latter's place in the diagram and introduce a different approach to the MKT's organisation above, that will conclude with the refinement and placement of HCK in this framework and thus, with the inclusion of the notion of continuity in MKT's theory. The practical use of this new arrangement will be shown by analysing some examples of teaching practice.

Re-thinking Horizon Content Knowledge (HCK)

Within the approach of Ball, Thames and Phelps (2008), HCK refers to the general awareness of the previous and the forthcoming, and from this perspective it reminds us of the dimensions of curricular knowledge that Schulman (1986) labelled as vertical curriculum knowledge, which refers to the familiarity with the topics and issues during the preceding and later years (Shulman, 1986).

Enlarging the authors' approach, HCK requires an overview of students' mathematical education so that it can be applied to the mathematics taught in the classroom. Teachers' consciousness of the past and the future within their subject is actually very closely related to continuity in mathematics education and thus, our view of HCK comprises this teacher's longitudinal perspective required for continuity. Thus, HCK refers specifically to the teaching of mathematics in a particular (continuous) way. However, this longitudinal view that we understand as HCK encompasses a complex combination of pedagogical and mathematical knowledge, skills and experience that must be clarified in order to successfully approach transition and continuity issues in mathematics.

Firstly, despite the fact that HCK may be related to the knowledge of the curriculum (KCC), we consider that it is independent from the curriculum itself. HCK is not only an awareness of how mathematical topics are related over the span of mathematics included in the curriculum but it also refers to the global knowledge of the evolution of the mathematical content and the relationship among its different areas needed for the teaching practice. This general knowledge does not depend on the curriculum context and it is different to the curriculum awareness that a teacher must have in order to teach the appropriate topics at a particular grade. In other words, a teacher could have good level on KCC but fail on approaching this knowledge from a long-term perspective.

Secondly, whereas the other sub-domains included in MKT theoretical framework have clear definitions and positions in the diagram (although in some cases their boundaries might not be very clear as they influence each other), HCK has a different nature. Responding to the authors (Ball, Thames & Phelps, 2008) HCK is not just another sub-domain within the framework since it is related to the KCS, the KCT and the SCK because it

must include the ability of the teacher to find out students' previous mathematical ideas and to prepare them for the future. This ability involves KCS (for instance the knowledge of previous, current and future students' difficulties, misconceptions or questions) and KCT (for example, different ways students might have seen that represent the same idea or types of tasks that facilitate students' learning in the future). Also, the SCK of a teacher for a particular grade depends on whether that teacher currently teaches in that level. If so, his/her SCK for that grade will be obviously greater than for the other grades. From a continuous mathematical education perspective, we cannot ask a secondary teacher to be aware of how pupils learn the numbers for the first time as well as we cannot ask a primary teacher to know different ways to approach the concept of derivative. And even if those were reasonable demands, they would not be helpful (nor harmful) to reach continuity as the levels are too far apart. Therefore, what are the reasonable demands? Is it possible to organise them under big ideas of mathematical knowledge? What sounds reasonable for each level is that SCK must be extended to those topics that may really have an effect in what students are learning at the moment or to those future topics for which a teacher is setting up the basis. This thought implies a relationship between HCK and SCK.

Thirdly, it does not seem sensible to find the presence of HCK in a particular teaching situation if there is a previous absence of KCS, KCT or SCK. In fact, these are the required bases that allow the posterior gradual inclusion of HCK in the professional practice. From this perspective HCK is not another category in the diagram but it adds a more sophisticated (continuous) perspective to the teaching practice. For instance, we would not expect to observe a teacher recognising previous misconceptions in a particular topic, dealing with students' difficulties or conveying a prospective mathematical view of the future if that teacher does not know the topic and its methodological issues at first. The following example, taken from a primary lesson observation, illustrates this idea by showing how teacher's lack of mathematical knowledge about the perpendicular bisector entails the absence of HCK from the start:

> The teacher asks a student to read a definition from the textbook: "A perpendicular bisector of a segment is the line that divides the segment in two halves and is perpendicular to it." Then, the teacher

> shows on the board how to construct the perpendicular bisector of a given segment with a ruler and a compass and specifies that the compass must be open more than half the segment's length. While the teacher explains the steps of the construction, one student interrupts: "I don't understand… why do you have to open the compass more than half the length of the segment?" The teacher answers: "In order to draw the perpendicular bisector you have to open the compass a bit more than a half. I don't' know, I don't know why, but that's how it is."

Thus HCK influences KCS, KCT and SCK and it is independent from the curriculum (KCC) and the mathematical knowledge common to other professions (CCK). Moreover, observing all these categories we detect that the first three sub-domains have a different character than the last two since the KCS, KCT and SCK arise and are expressed only during the teaching practice in mathematics or in the observation of other's teaching practice, while the KCC and the CCK are not necessarily linked to the teaching practice. We highlight this observation by considering KCC and CCK as foundation knowledge and KCS, KCT and SCK as having an in-action nature. It is important to remark here that the word foundation denotes our idea of this theoretical knowledge being a basis for the rest and it is not related to the more complex construct of Foundation knowledge included in the Knowledge Quartet of Mathematical Knowledge in Teaching (MKiT) framework (Rowland, Huckstep & Thwaites, 2005). Figure 2 shows our interpretation of the categories of MKT from this approach, not considering yet HCK.

As we have seen, HCK must be present in every in-action category in order to attend transition. We maintain that, in order to be used as a tool to guide and analyse the role of the teacher practical knowledge in transition, any in-action knowledge must be approached from a HCK perspective. Hence, HCK is expressed only in the teaching practice and its characterisation must be done according to this theoretical assumption.
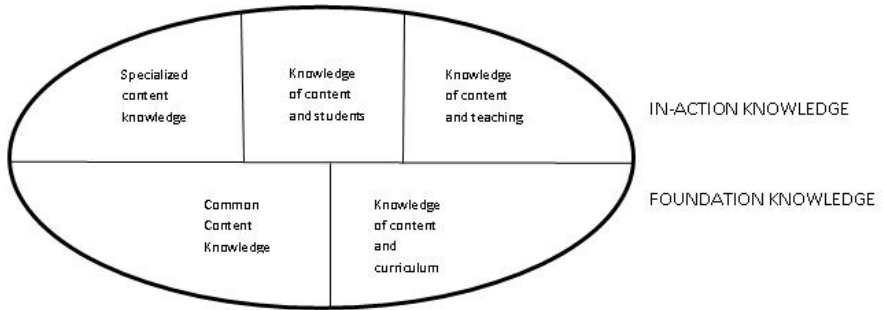
Figure 2. Categories of Mathematical Knowledge for teaching without the HCK

The previous ideas lead us to consider HCK, not as another sub-domain of MKT, but as a mathematical knowledge that actually shapes the MKT from a continuous mathematical education point of view. Figure 3 shows our interpretation of the framework with the inclusion of the HCK. The idea of shaping the diagram when including the HCK intends to indicate: a) the difference in the expression of each in-action category in the teaching practice with or without HCK; b) the connection with past and future mathematical levels, particularly important from a transition (or a continuous) point of view and c) the fact that HCK has a different nature than the rest of categories since it does not appear in the diagram but its presence modifies the teaching practice.

From this perspective, this refinement of the construct HCK becomes our theoretical way in to investigate the role of teachers' professional knowledge during transition to secondary school. Summarising, HCK is a mathematical knowledge specific for the teaching practice that requires both a longitudinal perspective of the mathematical topics and also the ability to communicate this perspective in the teaching practice. Hill, Rowan and Ball (2009) describe HCK as "a kind of peripheral vision" but we consider this horizon picture from a whole temporal perspective which includes and relates the past, the present and the mathematical future. In fact, it is this temporal perspective which differentiates and represents the essence of the HCK that led us to show it graphically by the shape of Figure 3 and becomes now the scaffold of our characterisation.
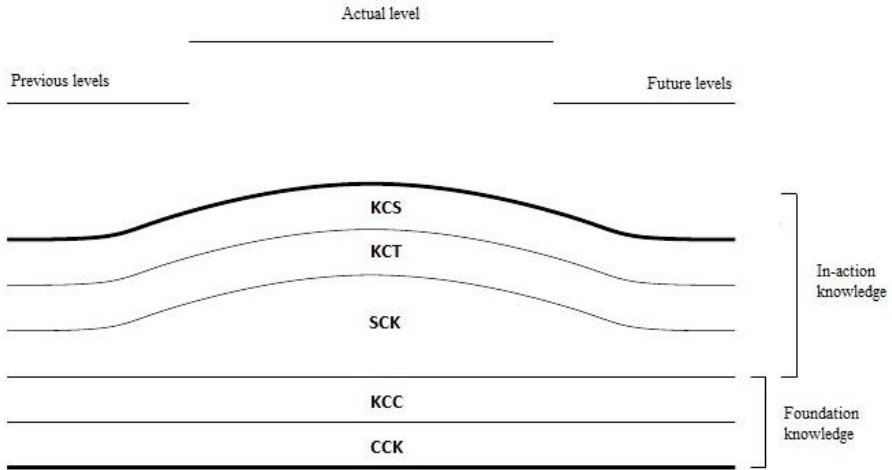
Figure 3. HCK shapes MKT and outlines its nature

At this point where the importance of the HCK has been underlined, our consequent aim now is to conceptualise those reasonable demands that characterise more specifically our idea of HCK. Since the MKT is a practice-based theory and because the HCK shapes the in-action part of teachers' MKT, we devote the rest of this paper to describe the HCK in mathematics teaching as well as offer some examples that illustrate the expression of HCK in the teaching practice.

Characterisation of HCK in the Teaching Practice

Hill et al (2008) highlight the need of clarifying how teachers' knowledge affects classroom instruction by carrying out an investigation in which the relationship between teachers' MKT and the quality of their practice is analysed. Since we do not consider HCK as a theoretical cluster itself but a type of knowledge that shapes the in-action knowledge needed for teaching and also because our purpose is to obtain a useful tool for future research on transition, we adopt a practical perspective from which the following question emerges immediately: how does HCK get expressed in teaching practice? We need to guarantee that HCK is present in the

principal tasks of teaching, but how does HCK make its presence evident in these tasks?

Table 1 shows the three main practice events which help us to concrete and identify HCK in the teaching practice. These require professional knowledge about the actual level where a teacher teaches as well as about previous or future levels. For each of these events we have established some indicators to identify that particular event.

Table 1
The expression of HCK in the teaching practice

Preparation of activities *from a continuity perspective*.
The teacher considers the sequence of the content when planning activities, increases the rigor in mathematical language and the level of generalisation in mathematical problems
And his/her sequence plan includes exercises that target the emergence of typical mistakes and difficulties
And recognises the potential of tasks or extra-mathematical information that lead to the construction of mathematical activities in different levels.
Identification, prevention and reorientation of misconceptions and difficulties *from a continuity perspective*.
The teacher knows previous students' ideas, knowledge and difficulties
And identifies and analyses students' mistakes and locates them on (educational) time
And shares with the student the origin of a mistake and returns to the student or the class the responsibility of correcting it.
Adaptation of the classroom activity from students' contributions and level.
The teacher interprets students' mathematical knowledge from a continuity perspective
And integrates students' knowledge by adapting the language and rigor without modifying the previously planned task
And establishes connections among tasks and uses new examples.

Concreting HCK in an Example of the Teaching Practice

The following example is rich enough to allow us to clarify how teacher's HCK is expressed in practice, to interpret this practice and to support our consideration of HCK as a conceptual tool for future research focused on investigating the role of teacher's knowledge in the continuity

of mathematical instruction. We will use this one example along with two students' work to illustrate each HCK competence.

The telephone boxes' problem

You are given a map where there are three points- one yellow, one red and one blue- that represent telephone boxes. We want to divide the map in coloured zones according to the colour of the closest telephone box. How would you find these regions?
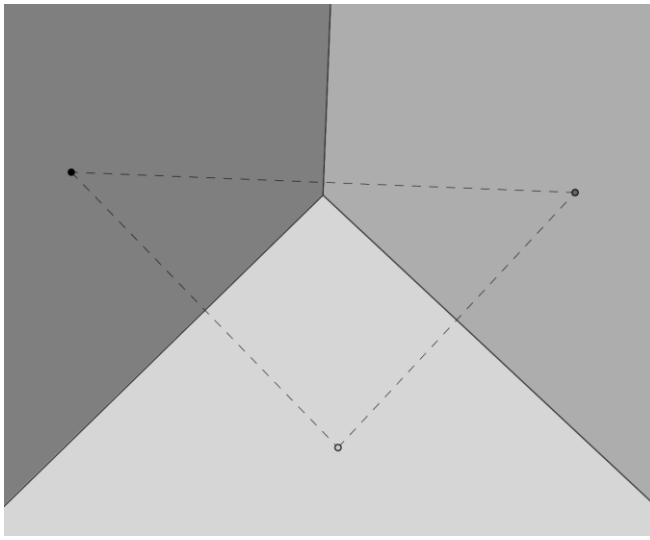

Figure 4. The telephone boxes' problem mathematical construction

The most frequent solution to this problem involves the construction of two or the three perpendicular bisectors. The point of intersection of the three bisectors is equidistant to the three telephone boxes as those lines are the loci of the points that are equidistant – respectively- to each pair of points (see Figure 4).

Preparation of activities from a continuity perspective.

This highlights the influence of HCK in KCT, since it involves teachers' planning before the lesson and the methodology used in it.

The fact that a mathematics teacher uses the telephone boxes' problem in a primary or secondary geometry lesson may involve various objectives, such as emphasising problem solving strategies or making sense of the perpendicular bisector of two points as the locus of the points which are all at the same distance to two initial given points. We consider that including a problem like this in the sequence plan would already be a sign of the presence of HCK in the teaching practice at this first degree because the rigor, level of generalisation and understanding required to solve it is significantly greater than those required to solve a simple problem of construction of the three perpendicular bisectors or the circumcenter of a triangle.

The telephone boxes' problem targets the confusion between the different properties and characteristics of the medians and the perpendicular bisectors. If a teacher poses this problem to a class, he or she should be aware of this previous misconception and thus, be ready to act upon it.

This competence requires a more complex mathematical knowledge in order to detect the possibilities that a given problem may offer and moreover, to distinguish mathematical content in (apparently) non-mathematical activities.

The potential of a problem like the telephone boxes' problem, which could be introduced with a real city map avoiding an obvious mathematical context, begins with making sense of the perpendicular bisector of a segment and identifying the difference between that construction and the medians of a triangle. Moreover, a strategist teacher would also see the potential of this problem to introduce, for instance, the idea of locus by defining the perpendicular bisector as the locus of the points that are equidistant to two given points and broadening this idea to other simple loci.

In the following, the HCK competences concern teacher-student interaction and therefore, we use two examples of students' work to illustrate possible reactions of a teacher.

Anna's example

Figure 5 shows the work of a student, Anna, who drew the medians of the triangle in order to solve the problem. Her work shows a common difficulty about finding equidistant points to two given points. Drawing the first equidistant point is usually straightforward for students, it appears clear to them that the midpoint of the segment that joins the two points is at the same distance of both. However, when students are asked to find more equidistant points, the difficulty arises (Dickson, Brown and Gibson, 1991). If the teacher knows this fact, he or she would be expecting some students drawing the medians of the triangle, highlighting their only understanding of the midpoint and their confusion with the geometrical meaning of a perpendicular bisector.
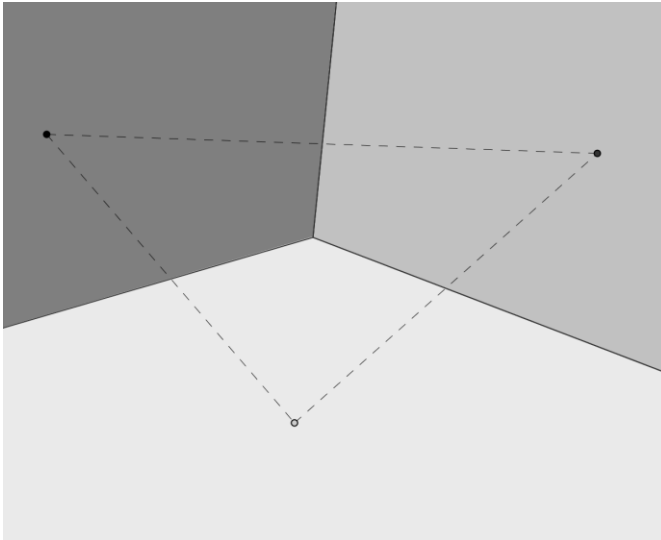

Figure 5. The telephone boxes' problem "solved" using the medians of the triangle

Identification, prevention and reorientation of misconceptions and difficulties from a continuity perspective.

This competence emphasises the effect of HCK in KCS, especially in relation to students' misconceptions and difficulties.

At the first degree, the teacher would just be ready to explain the mistake to the student showing that the medians construction does not work. For instance, in the lesson observed the teacher showed a counterexample: a point whose closest telephone box was not the one indicated by the colour of the region where the point was. This way Anna realised that her approach was wrong although she did not understand completely why it did not work

A possible interpretation of Anna's misconception could be originated by the consideration of the triangle at once and not considering an extension of the approach to the simplest version of the problem for only two points. Another teacher's interpretation of Anna's mistake could be the intuitive intention of trying to find regions of equal area, since the median divides the triangle into three equal area triangles. In any case, the second degree of strategist should include the analysis and positioning of the mistake, moving forward the basic previous explanation of the fact that it is wrong.

The last degree of sophistication in this competence is to find out and share with the student and the class the origin of the mistake. Depending on the previous analysis, the teacher in this case could, for instance, simplify the problem to two points and pose it to the student or the class for discussion, in order to reach a generalisation of their findings for the three points' problem. Another option could be to show clearly a counter example with another appropriate triangle (see Figure 6) with the intention of using it to discuss the geometrical properties of the medians of a triangle.
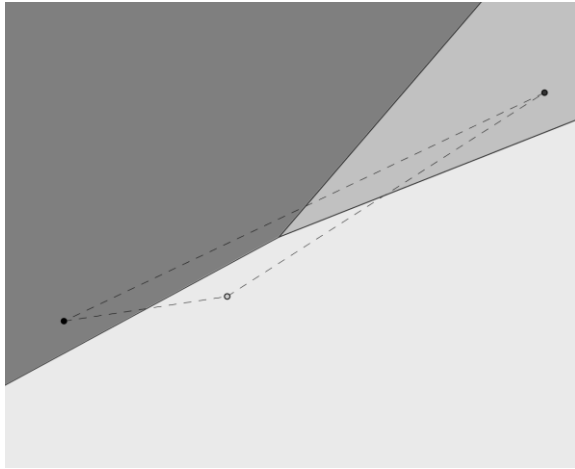
Figure 6. A clear counterexample of the telephone boxes' problem "solved" using the medians

Baltasar's example

To show more in detail the last competence we use the work of another student, Baltasar, who suggested drawing the regions with the same area by using circumferences centred in each of the three given points and he accompanied its response with a gesture of opening his hands, as if a circle was expanded (see Figure 7).
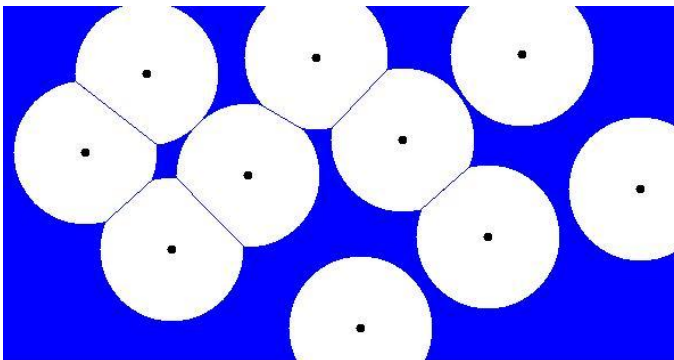

Figure 7: Baltasar's approach to the telephone boxes' problem

Adaptation of the classroom activity from students' contributions and level.

This competence draws special attention to the ability of the teacher to improvise in the lesson, which we consider a determinant sign of the presence of HCK and is inspired in the concept of Contingency included in the MKiT framework (Rowland, Huckstep & Thwaites, 2005).

At this first degree we expect the teacher to wonder whether there is mathematical potential in Baltasar's idea. Hence, this would mean that the teacher intends to interpret the mathematical knowledge involved in the student gestural representation (the intersection of two of the circles leads to the perpendicular bisector of the segment determined by their centres). In case of uncertainty, we expect the teacher to find out about it in order to confirm or disregard any students' mathematical idea.

In the particular case of Baltasar's idea the second degree would mean to show its mathematical potential in order to discuss whether the telephone boxes' problem could be solved by expanding circles centred on the three points. This could be done, for instance, by the inclusion of specialised software to show that both computational geometry algorithms (namely expanding circles and drawing perpendicular bisectors) lead to the solution. This way the teacher gives the opportunity to connect the current level with more sophisticated mathematical ideas that perhaps students will face later, such as computational geometry algorithms.

The last degree involves the extension of the problem or its connections with other problems of branches of mathematics. In Baltasar's case this would mean the introduction of Voronoi diagrams and how to draw them by expanding circles, as shown in Figure 7 (Abellanas, 2006-2007). The teacher could explain, for example, that if the problem had to be solved for more than three points, comparing the efficiency of both algorithms is a problem of convergence. The students could also be asked to investigate other possible algorithms. Although considering regions of the same area is a wrong solution for the problem, the student's intervention could be conducted to emphasise its mathematical potential and even more, it would leave the problem open to reconsider it in the future for a greater number of points.

These examples of real classroom practice provide a better understanding of our theoretical approach intending to move towards a

conceptualisation of continuity in mathematics education. Firstly, we have suggested a re-definition of HCK which emerges as a noteworthy construct within MKT framework that allows approaching continuity questions in mathematical education by looking at teachers' professional knowledge. Developing tools in order to systematically investigate if teachers' HCK has consequences on transition will show a path to the potential inclusion of this mathematical knowledge as part of teacher training programs designed to smooth transition.

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