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ON THE FIRST AND SECOND Φ-VARIATION IN THE SENSE OF SCHRAMM-RIESZ

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Abstract

In this paper we present a generalization of the concepts of first and second Φ-variation (where Φ is a certain sequence of positive convex function defined on $[0, +\infty)$), in the sense of Schramm-Riesz for normed space valued functions defined on an interval $[a, b] \subset \mathbb{R}$. We characterized the functions of second Φ-variation in the sense of Schramm-Riesz, as those that can be expressed as the integral of a function of bounded Φ-variation in the sense of Schramm-Riesz.
Keywords: Variation of a Function, $\mathcal{N}$-Function, $\mathcal{N}$-sequence.

SOBRE LA PRIMERA Y SEGUNDA Φ-VARIACIÓN EN EL SENTIDO DE SCHRAMM-RIESZ

Resumen

En este trabajo se presenta una generalización de los conceptos de primera y segunda Φ-variación (donde Φ es cierta sucesión de funciones convexas positivas definidas sobre $[0, +\infty)$), en el sentido de Schramm-Riesz para funciones con valores en un espacio normado y definidas sobre un intervalo $[a, b] \subset \mathbb{R}$. Se caracterizan las funciones de segunda Φ-variación en el sentido Schramm-Riesz, como funciones que se expresan como la integral de una función de Φ-variación acotada en el sentido de Schramm-Riesz.
Palabras clave: Variación de una función, $\mathcal{N}$-Función, $\mathcal{N}$-sucesión.

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Introduction

In 1881 C. Jordan [7] introduced the class of real valued functions of bounded variation and established the relation between these functions and the monotonic ones. Thus, the Dirichlet Criterion for the convergence of the Fourier series applies to the class of functions of bounded variation. This, in turn, has motivated the study of solutions of nonlinear equations that describe concrete physical phenomena in which, often, functions of bounded variation intervene.

The interest generated by this notion has lead to some generalizations of the concept, mainly, intended to the search of a bigger class of functions whose elements have pointwise convergent Fourier series. As in the classical case, this generalizations have found many other applications in non-linear functional analysis and in the theory of functions; in particular, in the study of certain differential and integral equations (cf. e.g., [3], [13], [14], [5]).

In 1908, De La Vallée Poussin [8] generalized the concept of function of bounded variation introducing the notion of function of second bounded variation. He showed that if a function f is of second bounded variation on an interval $[a, b]$ then it is absolutely continuous on $[a, b]$ and it can be expressed as a difference of two convex functions. A few years later, in 1911, [16], F. Riesz proved that a function is of bounded second variation on $[a, b]$ if, and only if, it is the definite Lebesgue integral of a function of bounded variation. This result is now known as Riesz's lemma. More recently, in 1983, A. M. Russell and C. J. F. Upton, [17], obtained a similar result for functions of bounded second p-variation ($1 \leq p \leq \infty$), in the sense of Wiener. On the other hand, in 1937, L. C. Young ([19]) gave a generalization of the concept of function of bounded variation by introducing the notion of Φ-variation of a function, this concept, in turn, was generalized by V. Chistiakov, [4], for functions which take values in a linear normed space. In 1985 M. Schramm, [18], introduced a notion of bounded variation that simultaneously generalizes the notions of bounded variations given by C. Jordan, N. Wiener, L.C. Young and D. Waterman.

In this paper we present a generalization, to vector valued functions, of the notions introduced by M. Schramm in [18] and by F. Riesz in [16]. Related (partial) generalizations were given in [6] and [2] where the notions of second Riesz Φ-variation and second bounded variation in the sense of Schramm, both for normed space valued maps, are dealt with respectively. Here, and based on the work done in [6] and [2], we actually combine the two notions. Among other things, we show that a function f is of bounded second Φ-variation in the sense of Schramm-Riesz if it is the integral of a function of bounded (first) Φ-variation in sense of Schramm-Riesz.

Among some of the works related to the present paper, in which the authors are involved, we would like to mention the following:

In [2], it is presented a Riesz-type generalization of the concept of second variation of normed-space-valued functions defined on an interval $[a, b] \subseteq \mathbb{R}$; in particular, it is shown that a function $f \in X^{[a,b]}$, where X is a reflexive Banach space, is of bounded second-variation (in the sense of Riesz) if and only if it can be expressed as the (Bochner) integral of a function of bounded (first) Φ-variation. Moreover, there the authors obtained a Riesz's lemma type inequality to estimate the total second Riesz Φ-variation of a function.

In [6], is introduced the concept of second Φ-variation in the sense of Schramm, for normed-space valued functions defined on an interval $[a, b] \subseteq \mathbb{R}$.

On the other hand, in [9] is generalized the concept of Φ-variation in the sense of Schramm for real valued functions of two variables defined on certain rectangle in the plane. Also in [10] is considered a *superposition operator problem* in the Banach algebra of functions, of two variables, of bounded total Φ-variation in the sense of Schramm. In [11] is characterized uniformly continuous superposition operators in the space of functions of bounded generalized Φ-variation in the sense of Schramm.

Preliminaries and notation

As usual, we shall denote by $\mathcal{N}$ the set of all continuous convex functions $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(\rho) = 0$ if and only if $\rho = 0$ and $\lim_{\rho \to 0^+} \frac{\varphi(\rho)}{\rho} = 0$. Likewise, the notation $\mathcal{N}_\infty$ shall be used to denote the set of all functions $\varphi \in \mathcal{N}$, for which the Orlicz condition holds: $\lim_{\rho \to \infty} \frac{\varphi(\rho)}{\rho} = +\infty$. According to [12] functions in $\mathcal{N}$ are called φ-functions. Any function $\varphi \in \mathcal{N}$ is strictly increasing; besides, the functions $\rho \longmapsto \frac{\varphi(\rho)}{\rho}$.

In this paper we will denote by $\mathfrak{I}[a, b]$ the family of all sequences $\{I_n := [a_n, b_n]\}_{n \geq 0}$ of non-overlapping closed intervals contained in $[a, b]$ such that $|I_n| := b_n - a_n > 0$, for all $n \geq 0$. Given two sets A and B, A^B will denote the set of all functions from B to A. Also, if $\mathbb{A}$ is a linear space, $\mathbb{A}^B$ will denote the linear space of all functions from B to $\mathbb{A}$.

Throughout the paper $(\mathbb{X}, \|\cdot\|)$ will denote a Banach space.

Definition 1 ($\mathcal{N}$-sequence) *A sequence of φ-functions $\Phi = \{\varphi_n\}_{n \geq 1}$ is* *called an $\mathcal{N}$-sequence if for all $t > 0$:*

$$\varphi_{n+1}(t) \leq \varphi_n(t),\ n \geq 1, \quad \text{and} \quad \sum_{n\geq 1} \varphi_n(t) = +\infty.$$

If each $\varphi_n \in \Phi$ belongs to $\mathcal{N}_\infty$ we will write $\Phi \in \mathcal{N}_\infty$.

The space $BV^{SR}_{(\Phi,1)}[a,b]$.

Now we present the space of $\mathbb{X}$-valued functions of bounded variation in the sense of Schramm-Riesz (Cf. [6], [2]).

Definition 2 *Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be an $\mathcal{N}$-sequence and let $[a,b] \subset \mathbb{R}$ a closed interval. A function $f \in \mathbb{X}^{[a,b]}$ is said to be of bounded Φ-variation, in the sense of Schramm-Riesz, if*

$$V^{SR}_{(\Phi,1)}(f;[a,b],\mathbb{X}) := \sup_{\{[a_n,b_n]\}\in\mathfrak{I}[a,b]} \sum_n \varphi_n\left(\frac{\|f(b_n)-f(a_n)\|}{b_n-a_n}\right)(b_n-a_n) < \infty. \quad (1)$$

Remark 1 *In what follows, for the sake of simplicity in the notation, if it is clear from the context that a function f, defined on $[a,b]$ take values in a Banach space $\mathbb{X}$. We will simply write $V^{SR}_{(\Phi,1)}(f)$ instead of $V^{SR}_{(\Phi,1)}(f;[a,b],\mathbb{X})$.*

Let, $D^{SR}_{(\phi,1)} := \left\{ f \in \mathbb{X}^{[a,b]} ;\ V^{SR}_{(\Phi,1)}(f) < \infty \right\}$. From the definitions it is readily seen that the set $D^{SR}_{(\phi,1)}$ is a convex and symmetric subset of $\mathbb{X}^{[a,b]}$ and, as a consequence of the fact that the norm $\|\cdot\|$ and each φ_n in Φ are convex functions, the function $f \longmapsto V^{SR}_{(\Phi,1)}(f;[a,b],\mathbb{X})$, defined on $D^{SR}_{(\phi,1)}$, is a convex function. In particular, if $c_1 > 0$ and $f \in \mathbb{X}^{[a,b]}$ then, for all $0 < c \leq c_1$:

$$V^{SR}_{(\Phi,1)}(cf) \leq V^{SR}_{(\Phi,1)}(c_1 f).$$

Now, *in general $D^{SR}_{(\phi,1)}$ is not a vector space*, therefore it is necessary to consider the linear space, $\langle D^{SR}_{(\phi,1)}\rangle$, generated by this class, to which we will denote by $BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$. It is readily seen that

$$BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X}) = \langle D^{SR}_{(\phi,1)}\rangle = \left\{ f:[a,b]\to\mathbb{X} \,:\, V^{SR}_{(\Phi,1)}(\lambda f) < \infty, \text{ for some } \lambda > 0 \right\}.$$

Remark 2 *A similar argument, as the one given in the real valued case* (see [18], Lemma 1.1 and Theorem 1.2), *shows that in definition* 1 *the* $\sup_{\{I_n\}\in\mathfrak{I}[a,b]}$ *can be replaced by the* sup *over all finite collections $\{I_n\}_{n=1}^{m}$ in $\mathfrak{I}[a,b]$.*

Also, notice that if f is a constant function then $V^{SR}_{(\Phi,1)}(f;[a,b]) = 0$ and therefore the class $BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$ is not empty.

The next proposition follows easily from the definitions. We will omit the details of the proof.

Proposition 1 *Let $f \in \mathbb{X}^{[a,b]}$, then $V^{SR}_{(\Phi,1)}(f) = 0$ if and only if f is a constant function.*

It follows from the general theory of modular spaces presented in [15] (see also [12]) that the functional $f \to V^{SR}_{(\Phi,1)}(f)$ is a convex modular, in the sense of Musielak-Orlicz, on $\{f \in BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X}) : f(a) = 0\}$ and the space $BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$ with the norm

$$\|f\| := \|f(a)\|_{\mathbb{X}} + \inf\left\{\lambda > 0 : V^{SR}_{(\Phi,1)}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

is a Banach space.

Our next result shows that functions in $BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$ are regulated, meaning, they have one-sided limits at every point of $[a,b]$.

Proposition 2 *If $\{\varphi_n\}_{n\geq 1}$ is an $\mathcal{N}$-sequence then any $f \in BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$ can only have simple discontinuities.*

Proof Suppose that $g \in BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$ and let $\lambda > 0$ be such that $V^{SR}_{(\Phi,1)}(\lambda\, g) < \infty$. Put $f := \lambda g$, and suppose there exist an $\widehat{x} \in (a,b]$ such that $\lim_{x\to\widehat{x}^-} f(x)$ does not exist. Then there is a sequence $\{x_i\}_{i\geq 1}$ in $(a,\widehat{x})$ which converges to $\widehat{x}$ and such that $\{f(x_i)\}_{i\geq 1}$ diverges. Thus $\{f(x_i)\}_{i\geq 1}$ is not a Cauchy sequence, and consequently, there exits a real number $\epsilon_0 > 0$ and a subsequence $\{x_{i_n}\}_{n\geq 1}$ such that for all $n > 0$

$$\|f(x_{i_{n+1}}) - f(x_{i_n})\| \geq \epsilon_0. \tag{2}$$

As $\{x_{i_n}\}_{n\geq 1}$ is a Cauchy sequence and is strictly increasing, we can assume without loss of generality that $0 < x_{i_{n+1}} - x_{i_n} < 1$ for each $n \in \mathbb{N}$. Then, from the convexity of φ_n it follows that

$$0 < \varphi_n(\epsilon_0) = \varphi_n\left(\epsilon_0\frac{x_{i_{n+1}} - x_{i_n}}{x_{i_{n+1}} - x_{i_n}}\right) \leq (x_{i_{n+1}} - x_{i_n})\varphi_n\left(\frac{\epsilon_0}{x_{i_{n+1}} - x_{i_n}}\right).$$

Consequently,

$$c\sum_{n\geq 1}\varphi_n\left(\frac{\|f(x_{i_{n+1}})-f(x_{i_n})\|}{x_{i_{n+1}}-x_{i_n}}\right)(x_{i_{n+1}}-x_{i_n})$$
$$\geq \sum_{n\geq 1}\varphi_n\left(\frac{\epsilon_0}{x_{i_{n+1}}-x_{i_n}}\right)(x_{i_{n+1}}-x_{i_n})=+\infty.$$

This contradicts that $f\in BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$. The fact that $\lim_{x\to\widehat{x}^+} f(x)$ exists is proved similarly. □

The space $D^{SR}_{(\Phi,2)}([a,b],\mathbb{X})$

In this section we introduce the concept of second variation in the sense of Schramm-Riesz. We will show that, as in the classical case, there is a fundamental integral-type relationship between functions of first and second Φ-variation in the sense of Schramm-Riesz.

In what follows we will consider collections of intervals in $\mathfrak{I}[a,b]$ that contain a least two subintervals of $[a,b]$. We will denote this class as $\mathfrak{I}_2[a,b]$.

Definition 3 *Let $\Phi=\{\varphi_n\}_{n\geq 1}$ be an $\mathcal{N}$-sequence and $[a,b]\subset\mathbb{R}$ an interval. A function $f\in\mathbb{X}^{[a,b]}$ is said to be of second bounded Φ-variation, in the sense of Schramm-Riesz, if*

$$V^{SR}_{(\Phi,2)}(f;[a,b],\mathbb{X}):=\sup\sum_{n\geq 0}\varphi_n\left(\frac{\left\|\frac{f(b_{n+1})-f(a_{n+1})}{b_{n+1}-a_{n+1}}-\frac{f(b_n)-f(a_n)}{b_n-a_n}\right\|}{b_{n+1}-a_n}\right)(b_{n+1}-a_n)<\infty$$

the supremum being taken over $\{[a_n,b_n]\}_n\in\mathfrak{I}_2[a,b]$.

Just as in the previous section, to simplify notation, we often will write $V^{SR}_{(\Phi,2)}(f)$ instead of $V^{SR}_{(\Phi,2)}(f;[a,b],\mathbb{X})$.

If $V^{SR}_{(\Phi,2)}(f)<\infty$ we will say that the function f is of bounded second Φ-variation, in the sense of Schramm-Riesz, and we will use the notation: $f\in D^{SR}_{(\Phi,2)}([a,b],\mathbb{X})$.

Example 1 *Let $f:[a,b]\to\mathbb{X}:=(c_0,\|\cdot\|_\infty)$ defined as $f(t):=\left(\frac{t}{k}\right)_{k\in\mathbb{N}}$.*

Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be any $\mathcal{N}$-sequence. Then for every $\{[a_n, b_n]\}_{n\geq 1} \in \mathfrak{I}_2[a,b]$

$$\sum_n \varphi_n \left(\frac{\left\| \frac{\left(\frac{b_{n+1}}{k}\right)_{k\in\mathbb{N}} - \left(\frac{a_{n+1}}{k}\right)_{k\in\mathbb{N}}}{b_{n+1}-a_{n+1}} - \frac{\left(\frac{b_n}{k}\right)_{k\in\mathbb{N}} - \left(\frac{a_n}{k}\right)_{k\in\mathbb{N}}}{b_n - a_n} \right\|_\infty}{b_{n+1}-a_n} \right) (b_{n+1}-a_n)$$

$$= \sum_{n\geq 0} \varphi_n \left(\left\| \left(\frac{0}{b_{n+1}-a_n} \right)_k \right\|_\infty \right) (b_{n+1}-a_n) = \sum_{n\geq 0} \varphi_n(0)(b_{n+1}-a_n) = 0.$$

which means that $f \in D^{SR}_{(\Phi,2)}([a,b],\mathbb{X})$ and $V^{SR}_{(\Phi,2)}(f;[a,b],\mathbb{X}) = 0$.

Proposition 3 *Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be an $\mathcal{N}$-sequence, and let $f : [a,b] \to \mathbb{X}$ be a function. Then*

(a) The function $\nu_{\Phi,2}(x) := V^{SR}_{(\Phi,2)}(f;[a,x],\mathbb{X})$ is not decreasing on $[a,b]$.

(b) The functional $V^{SR}_{(\Phi,2)} : D^{SR}_{(\Phi,2)}([a,b],\mathbb{X}) \to \mathbb{R}$, defined as

$$V^{SR}_{(\Phi,2)}(f) := V^{SR}_{(\Phi,2)}(f;[a,b],\mathbb{X})$$

is convex.

(c) If $|\lambda| \leq 1$ then $V^{SR}_{(\Phi,2)}(\lambda f) \leq |\lambda|\, V^{SR}_{(\Phi,2)}(f)$.

Proof The proposition is a consequence of the definition and the properties of the supremum and the convexity of each φ_n. □

Remark 3 *Recall that if $(\mathbb{X}, d)$ is a metric space and $[a,b] \subset \mathbb{R}$ is an interval, a function $f : [a,b] \to \mathbb{X}$ is said to be absolutely continuous if for every $\epsilon > 0$ there is $\delta > 0$ such that $\sum_k d(f(b_k), f(a_k)) \leq \epsilon$ whenever $\{[a_k, b_k]\}_{k\geq 1}$ is a finite sequence of pairwise disjoint subintervals of $[a,b]$ that satisfies $\sum_k (b_k - a_k) \leq \delta$.*

Theorem 1 *Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be an $\mathcal{N}$-sequence. If $f \in D^{SR}_{(\Phi,2)}([a,b],\mathbb{X})$, then f is a Lipschitz function and consequently f is absolutely continuous.*

Proof

Fix $c \in (a,b)$ and let $s,t \in [a,b]$. The proof will depend on the different locations of s,t with respect to a,b and c.

Case 1 $a < s < c < t < b$. Here we have

$$
\begin{aligned}
& (b-a)^{-1}\left\|\frac{f(t)-f(s)}{t-s}\right\| \\
\leq\ & \frac{\left\|\frac{f(t)-f(s)}{t-s}-\frac{f(b)-f(t)}{b-t}\right\|}{b-a}+\frac{\left\|\frac{f(b)-f(t)}{b-t}-\frac{f(c)-f(a)}{c-a}\right\|}{b-a}+\frac{\left\|\frac{f(c)-f(a)}{(c-a)}\right\|}{b-a} \\
=\ & \frac{\left\|\frac{f(t)-f(s)}{t-s}-\frac{f(b)-f(t)}{b-t}\right\|}{b-s}\frac{b-s}{b-a}+\frac{\left\|\frac{f(b)-f(t)}{b-t}-\frac{f(c)-f(a)}{c-a}\right\|}{b-a}+\left\|\frac{f(c)-f(a)}{(c-a)(b-a)}\right\| \\
=\ & \varphi_1^{-1}\left[\varphi_1\left(\frac{\left\|\frac{f(t)-f(s)}{t-s}-\frac{f(b)-f(t)}{b-t}\right\|}{b-s}\frac{b-s}{b-a}\right)\right] \\
+\ & \varphi_1^{-1}\left[\varphi_1\left(\frac{\left\|\frac{f(b)-f(t)}{b-t}-\frac{f(c)-f(a)}{c-a}\right\|}{b-a}\right)\right]+\left\|\frac{f(c)-f(a)}{(c-a)(b-a)}\right\| \\
\leq\ & \varphi_1^{-1}\left[\frac{V^{SR}_{(\Phi,2)}(f)}{b-a}\right]+\varphi_1^{-1}\left[\frac{V^{SR}_{(\Phi,2)}(f)}{b-a}\right]+\left\|\frac{f(c)-f(a)}{(c-a)(b-a)}\right\| \\
=\ & 2\varphi_1^{-1}\left[\frac{V^{SR}_{(\Phi,2)}(f)}{b-a}\right]+\left\|\frac{f(c)-f(a)}{(c-a)(b-a)}\right\|.
\end{aligned}
$$

Thus, in this case we conclude that
$\left\|\frac{f(t)-f(s)}{t-s}\right\| \leq K$
where

$$K=(b-a)2\varphi_1^{-1}\left[\frac{V^{SR}_{(\Phi,2)}(f)}{b-a}\right]+\left\|\frac{f(c)-f(a)}{c-a}\right\|. \tag{3}$$

Case 2 $a < s < c < t = b$. In this case

$$
\begin{aligned}
& (b-a)^{-1}\left\|\frac{f(t)-f(s)}{t-s}\right\| \\
\leq\ & \frac{\left\|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(a)}{s-a}\right\|}{b-a}+\frac{\left\|\frac{f(s)-f(a)}{s-a}-\frac{f(b)-f(c)}{b-c}\right\|}{b-a}+\frac{\left\|\frac{f(b)-f(c)}{b-c}\right\|}{b-a} \\
=\ & \varphi_1^{-1}\left[\varphi_1\left(\frac{\left\|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(a)}{s-a}\right\|}{b-a}\right)\right] \\
+\ & \varphi_1^{-1}\left[\varphi_1\left(\frac{\left\|\frac{f(s)-f(a)}{s-a}-\frac{f(b)-f(c)}{b-c}\right\|}{b-a}\right)\right]+\frac{\left\|\frac{f(b)-f(c)}{b-c}\right\|}{b-a}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \varphi_1^{-1}\left[\frac{V^{SR}_{(\Phi,2)}(f)}{b-a}\right]+\varphi_1^{-1}\left[\frac{V^{SR}_{(\Phi,2)}(f)}{b-a}\right]+\frac{\left\|\frac{f(b)-f(c)}{b-c}\right\|}{b-a} \\
=\ & 2\varphi_1^{-1}\left[\frac{V^{SR}_{(\Phi,2)}(f)}{b-a}\right]+\frac{\left\|\frac{f(b)-f(c)}{b-c}\right\|}{b-a}.
\end{aligned}
$$

Thus, as in the previous case, we have

$$\left\|\frac{f(t)-f(s)}{t-s}\right\| \leq K$$

where K is as in (3).

In the other cases : $a < s < t \leq c < b$, $a = s < c < t < b$, $a < c \leq s < t < b$ and $a = s < c < t = b$, one obtains similar results. □

Corollary 1 *If* $\mathbb{X}$ *is a* reflexive *Banach space and* $f \in D^{SR}_{(\Phi,2)}([a,b],\mathbb{X})$, *then* f *is strongly differentiable almost everywhere with derivative strongly measurable.*

The next theorem shows the relation between functions of first and second bounded variation in the sense of Schramm-Riesz. The proof is based on ideas from [17] (see also [6].)

In what follows the integral of a normed space valued function means the Bochner integral (see [1]).

Theorem 2 *Let* $\Phi = \{\varphi_n\}_{n\geq 1}$ *be an* $\mathcal{N}$*-sequence,* $V^{SR}_{(\Phi,1)}(f;[a,b],\mathbb{X}) < +\infty$ *and define* $F(x) := \int_a^x f(t)dt$. *Then* $F \in D^{SR}_{(\Phi,2)}[a,b]$ *and* $V^{SR}_{(\Phi,2)}(F) \leq V^{SR}_{(\Phi,1)}(f)$.

Proof Let $\{[a_n, b_n]\}_{n\geq 1} \in \mathfrak{I}_2[a,b]$ be a finite collection. Then

$$\begin{aligned}
& \frac{F(b_{n+1}) - F(a_{n+1})}{b_{n+1} - a_{n+1}} - \frac{F(b_n) - F(a_n)}{b_n - a_n} \\
= \;& \frac{\displaystyle\int_a^{b_{n+1}} f(t)dt - \int_a^{a_{n+1}} f(t)dt}{b_{n+1} - a_{n+1}} - \frac{\displaystyle\int_a^{b_n} f(t)dt - \int_a^{a_n} f(t)dt}{b_n - a_n} \\
= \;& \int_{a_{n+1}}^{b_{n+1}} f(t)\frac{dt}{b_{n+1} - a_{n+1}} - \int_{a_n}^{b_n} f(t)\frac{dt}{b_n - a_n} \\
= \;& \int_0^1 f(a_{n+1} + s(b_{n+1} - a_{n+1}))ds - \int_0^1 f(a_n + s(b_n - a_n))ds \\
= \;& \int_0^1 [f(a_{n+1} + s(b_{n+1} - a_{n+1})) - f(a_n + s(b_n - a_n))]ds
\end{aligned}$$

Hence, an application of Jensen's inequality and Minkowski's integral inequality yields

$$\begin{aligned}
& \sum_n \varphi_n \left(\frac{\left\| \frac{F(b_{n+1}) - F(a_{n+1})}{b_{n+1} - a_{n+1}} - \frac{F(b_n) - F(a_n)}{b_n - a_n} \right\|}{b_{n+1} - a_n} \right) (b_{n+1} - a_n) \\
= \;& \sum_n \varphi_n \left(\frac{\left\| \int_0^1 [f(a_{n+1} + s(b_{n+1} - a_{n+1})) - f(a_n + s(b_n - a_n))]ds \right\|}{b_{n+1} - a_n} \right) (b_{n+1} - a_n) \\
\leq \;& \int_0^1 \left[\sum_n \varphi_n \left(\frac{\| f(a_{n+1} + s(b_{n+1} - a_{n+1})) - f(a_n + s(b_n - a_n)) \|}{b_{n+1} - a_n} \right) (b_{n+1} - a_n) \right] ds.
\end{aligned}$$

Now, for $s \in [0,1]$: $\quad a_{n+1} + s(b_{n+1} - a_{n+1}) \in [a_{n+1}, b_{n+1}]$, $\quad a_n + s(b_n - a_n) \in [a_n, b_n]$, and $\mathcal{K} := a_{n+1} + s(b_{n+1} - a_{n+1}) - [a_n + s(b_n - a_n)] \leq b_{n+1} - a_n$. Thus,

$$
\begin{aligned}
& \sum_n \varphi_n \left(\frac{\left\| \frac{F(b_{n+1}) - F(a_{n+1})}{b_{n+1} - a_{n+1}} - \frac{F(b_n) - F(a_n)}{b_n - a_n} \right\|}{b_{n+1} - a_n} \right) (b_{n+1} - a_n) \\
\leq \quad & \int_0^1 \left[\sum_n \varphi_n \left(\frac{\|f(a_{n+1} + s(b_{n+1} - a_{n+1})) - f(a_n + sb_n - a_n)\|}{b_{n+1} - a_n} \right) (b_{n+1} - a_n) \right] ds \\
= \quad & \int_0^1 [\sum_n \varphi_n \left(\frac{\|f(a_{n+1} + s(b_{n+1} - a_{n+1})) - f(a_n + sb_n - a_n)\|}{\mathcal{K}} \frac{\mathcal{K}}{b_{n+1} - a_n} \right) \\
& \qquad (b_{n+1} - a_n)] ds \\
\leq \quad & \int_0^1 [\sum_n \varphi_n \left(\frac{\|f(a_{n+1} + s(b_{n+1} - a_{n+1})) - f(a_n + sb_n - a_n)\|}{\mathcal{K}} \right) (b_{n+1} - a_n) \\
& \qquad \frac{\mathcal{K}}{b_{n+1} - a_n} ds \\
= \quad & \int_0^1 \left[\sum_n \varphi_n \left(\frac{\|f(a_{n+1} + s(b_{n+1} - a_{n+1})) - f(a_n + sb_n - a_n)\|}{\mathcal{K}} \right) \mathcal{K} \right] ds \\
\leq \quad & \int_0^1 V^{SR}_{(\Phi,1)}(f; [a,b], \mathbb{X}) ds \\
= \quad & V^{SR}_{(\Phi,1)}(f; [a,b], \mathbb{X}).
\end{aligned}
$$

Therefore $V^{SR}_{(\Phi,2)}(F) \leq V^{SR}_{(\Phi,1)}(f)$, as claimed. □

By Proposition 2 any function in $BV^{SR}_{(\Phi,1)}([a,b])$ is regulated. This notion will be needed as an hypothesis to get our next result which is a version of the classical Riesz's lemma.

Theorem 3 *Let X be a reflexive Banach space. Suppose $F \in D^{SR}_{(\Phi,2)}[a,b]$, $\Phi \in \mathcal{N}_\infty$ and that F' has left limit $F'(x-)$ at every point $x \in (a,b]$ and right limit $F'(x+)$ at every point $x \in [a,b)$. Then $F' \in BV^{SR}_{(\Phi,1)}([a,b], \mathbb{X})$ and $V^{SR}_{(\Phi,1)}(F') = V^{SR}_{(\Phi,2)}(F)$.*

Proof Theorem 1 guarantees that F is an absolutely continuous function. Hence, it has strong derivative almost everywhere on (a,b) and $F(x) = F(a) + \int_a^x F'(t)dt$, for all $x \in [a,b]$ (see [1]). To show that $F' \in BV^{SR}_{(\Phi,1)}([a,b], \mathbb{X})$ we need to prove, first of all, that $F'(x)$ exists for all $x \in (a,b)$. Indeed, if that where not the case then there would exist at least a point $x_0 \in (a,b)$ such that $F'(x_0+) - F'(x_0-) \neq 0$; consequently, we can choose constants $\epsilon > 0$ and $\delta > 0$ such that $\|F'(x_0+h) - F'(x_0-h)\| \geq \epsilon$ for all $h \in (-\delta, \delta)$. Thus, by the definition of the functional $V^{SR}_{(\Phi,2)}$ and the definition and properties of $\mathcal{N}$ functions, we get

$$
V^{SR}_{(\Phi,2)}(F) \quad \geq \quad \lim_{h \to 0} \varphi_1 \left(\frac{\epsilon}{2h} \right) 2h = \infty.
$$

A contradiction. Therefore F' exists everywhere in (a,b) and we define $F'(a)$ and $F'(b)$ just as the respective one-sided limits $F'(a+)$ and $F'(b-)$.

To prove that $F' \in BV^{SR}_{(\Phi,1)}([a,b]$ let $\{[a_i,b_i]\}_{i=1}^n$ be any finite collection of non-overlapping closed intervals contained in $[a,b]$. Then, for all $i = 1,\cdots,n-1$, we have

$$\begin{aligned}
& \sum_{i=0}^{n-1}\varphi_i\left(\frac{\|F'(b_i)-F'(a_i)\|}{b_i-a_i}\right)(b_i-a_i)\\
= & \sum_{i=0}^{n-1}\varphi_i\left(\lim_{h\to 0}\frac{\left\|\frac{F(b_i+h)-F(b_i)}{h}-\frac{F(a_i+h)-F(a_i)}{h}\right\|}{b_i-a_i+h}\right)(b_i-a_i)\\
= & \sum_{i=0}^{n-1}\lim_{h\to 0}\varphi_i\left(\frac{\left\|\frac{F(b_i+h)-F(b_i)}{h}-\frac{F(a_i+h)-F(a_i)}{h}\right\|}{b_i-a_i+h}\right)(b_i-a_i)\\
= & \lim_{h\to 0}\sum_{i=0}^{n-1}\varphi_i\left(\frac{\left\|\frac{F(b_i+h)-F(b_i)}{h}-\frac{F(a_i+h)-F(a_i)}{h}\right\|}{b_i-a_i+h}\right)(b_i-a_i+h)\\
\leq & V^{SR}_{(\Phi,2)}(F).
\end{aligned}$$

Since the above inequality holds for all n, we conclude that

$$\sum_{n\geq 0}\varphi_n\left(\frac{\|F'(b_n)-F'(a_n)\|}{b_n-a_n}\right)(b_n-a_n) \quad\leq\quad V^{SR}_{(\Phi,2)}(F).$$

Thus $f := F' \in BV^{SR}_{(\Phi,1)}([a,b],\mathbb{X})$ and $V^{SR}_{(\Phi,1)}(f) \leq V^{SR}_{(\Phi,2)}(F)$, and by Theorem 2 $V^{SR}_{(\Phi,1)}(f) = V^{SR}_{(\Phi,2)}(F)$. □

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