MISSING VALUE AND COMPARISON PROBLEMS: WHAT PUPILS KNOW BEFORE THE TEACHING OF PROPORTION

Ana Isabel Silvestre and João Pedro da Ponte

This paper analyses grade 6 pupils' mathematical processes and difficulties in solving proportion problems before the formal teaching of this topic. Using a qualitative methodology, we examine pupils' thinking processes at four levels of performance in missing value and comparison problems. The results show that pupils tend to use scalar composition and decomposition strategies in missing value problems and functional strategies in comparison problems. Pupils' difficulties are related to a lack of recognition of the multiplicative nature of proportion relationships.

Keywords: Comparison problems; Difficulties; Missing value problems; Proportion problems; Strategies, Thinking processes

Problemas de valor ausente y de comparación: qué saben los alumnos antes de la enseñanza de la proporción

En este trabajo se analizan los procesos matemáticos y las dificultades en la resolución de problemas de proporción de alumnos de sexto grado antes de la enseñanza formal de este tema. Usando métodos cualitativos, se examinan los procesos de pensamiento de los alumnos, en cuatro niveles de desempeño, en problemas de valor ausente y comparación. Los resultados muestran la tendencia de los alumnos a usar una composición escalar y estrategias de descomposición en los problemas de valor ausente, y estrategias funcionales en los problemas de comparación. Las dificultades de los alumnos se relacionan con la falta de reconocimiento de la naturaleza multiplicativa de las relaciones proporcionales.

Términos clave: Dificultades; Estrategias; Problemas de comparación; Problemas de proporción; Problemas de valor ausente; Procesos de pensamiento

Pupils' ability in proportional reasoning is essential for their mathematical development. This reasoning is fundamental to solve daily life problems and also for learning advanced mathematical topics as well as other fields of study, including natural and social sciences (Post, Behr, & Lesh, 1988). Pupils' difficulties in this aspect of mathematical reasoning are well known (Bowers, Nickerson, & Kenehan, 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). Furthermore, as Lesh, Post, and Behr (1988) note, there are many people that solve direct proportion problems without using proportional reasoning.

The Portuguese mathematics syllabus indicates that pupils at grades 5-6 must understand the notion of proportion and develop proportional reasoning. This document points out that pupils at grades 1-4 already work on mathematical tasks involving proportional relationships. In their planning, teachers must take into account such pupils' prior informal knowledge. Thus, it is important to know pupils' ability to solve proportion problems before the teaching of this topic. In this paper, we discuss pupils' mathematical processes, including representations and strategies, in solving missing value and comparison problems, as well as their difficulties.

DIRECT PROPORTION

Proportional relationships may be investigated under different perspectives. Firstly, from a psychological perspective, Vergnaud (1983) stresses the isomorphism of measures. In this model, variables remain independent and the transformations within or between variables keep proportional relationships between numerical values (Figure 1).
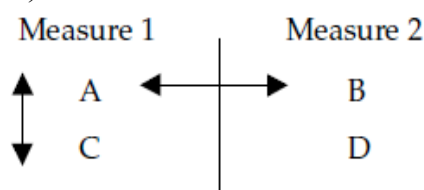

Figure 1. Vergnaud's model for simple proportion

Secondly, from a mathematical perspective, a proportional relationship between two variables is represented as an equality of two ratios $a/b = c/d$ (a and c are values of a variable, b and d values of another variable) or as a linear function $y = m \cdot x$ with $m \neq 0$. Finally, a curriculum perspective stresses the use of representations, leading pupils to learn first to solve problems using equalities between ratios and then to use linear functions. Generally, work with these two representations remains unconnected. Stanley, McGowan, and Hull (2003) argue that the usual teaching approach for the development of proportional thinking in which pupils solve proportions is outdated and should be replaced by another in which pupils engage in activities that help them discover that proportion is the variation of two quantities related to each other.

Types of Problems and Pupil's Strategies

Missing value problems present three numerical values and ask for the fourth value, whereas comparison problems present two or more pairs of numerical values and request their comparison. Several studies identified pupils' strategies in solving these problems. For example, Post, Behr, and Lesh (1988) and Cramer, Post, and Currier (1993) identified the following strategies:

The ratio unit. It is the most intuitive strategy that pupils use since the early years of schooling (computation of ratio units on division problems and computation of multiple ratios in the multiplication unit).

Factor or factor of change scale[1]. Known as "often like" strategy, it is related to the numerical aspects of the problems but is used by many children.

Comparison of ratio problems associated with comparison. It allows comparing ratio units through two divisions.

Cross product algorithm. Also known as "rule of three", while effective, it is a mechanical process devoid of meaning in the context of the problems.

In addition, Post et al. (1988) identified the strategy of *graphic interpretation*. Another rather informal strategy that appears both in additive and multiplicative reasoning is *composition/decomposition* (Christou & Philippou, 2002; Hart, 1984).

Lamon (1993) classifies reasoning strategies as within and between variables, distinguishing between scalar reasoning (concerning the transformations within the same variable) and functional reasoning (establishing relationships between variables). In her view, the distinction between these two types of relationships is important because they involve different cognitive processes.

METHODOLOGY

This study follows a qualitative approach (Denzin & Lincoln, 1998). The participants were six 4th-grade pupils, 11 years old, belonging to two different classes. Before the proportion chapter, all pupils in these two classes took a diagnostic test on the topic. One pupil with satisfactory performance and another one with difficulties in solving problems were selected from each class. Semi-structured interviews, video and audio recordings, were conducted with these four pupils. Based on the strategies for solving proportional problems identified in the literature, we created a system of categories of analysis (Table 1). This repertoire of strategies was complemented by a pictorial strategy that was detected in this study.

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[1] Hart (1984).

Table 1
Categories of Analysis for Pupils' Strategies

Strategy	Description
Multiplicative	Establishes a multiplicative relationship between variables. Understands the meaning of ratio. Establishes a multiplicative co-variation relationship among variables.
Additive and multiplicative	Computes the unit ratio and uses it in additive processes. Composes and decomposes numbers involving addition, multiplication and division.
Additive	Composes numbers using addition.
Pictorial	Represents pictorially objects or sets of objects and counts them.

RESULTS

A diagnostic test on proportionality was administered to four pupils (S1, S2, S3, and S4) before the teaching of this topic. Missing value problems as well as comparison problems were included in the test. Here we provide the pupil's answers to all the problems along with discussion of their solution approach.

Solving Missing Value Problems

The following problem (Problem 1), posed in an interview, has a simple context and its data involves multiples of 3.

> *Margarida bought three books from the collection "Once upon a time" for 12 euro. If Margarida has 48 euro, how many books can she buy?*

The following text shows some of the pupils' answers to the Problem 1:

S1: 1 book costs 4 €. If 10 books is 40 € more 2 (8 €) is 12 books. She can buy 12 books. (see Figure 2)

S2: I multiplied by 2. It is the double. …I think that this way will not do. ...48 euros, is here. (Points between 36 and 72)

S3: 4 euro is one book, right? ...I do 48 divided by 4 [euro] of one book (computes mentally.) It gives... 40 [euro] is 10 [books]. 44 [euro] is 11 [books]. 48 [euro] is 12 [books]. She can buy 12 books.

S4: I'll do 3 on 3 [books] until it gives? (see Figure 3)

Figure 2. S1's strategy to Problem 1

S1, S3 and S4 give correct answers. S1 calculates the unit ratio but she does not use this functional relationship to determine the missing value, choosing to compose numerical values, as ten times, the double and their sum. S2 uses a composition strategy which involves the successive computation of the doubles of the numerical pairs. Using this strategy, she cannot find the missing value. S3 develops a multiplicative functional strategy, using the price of each book, computing mentally the missing value (12 books), starting with a reference value (40 euro correspond to 10 books).
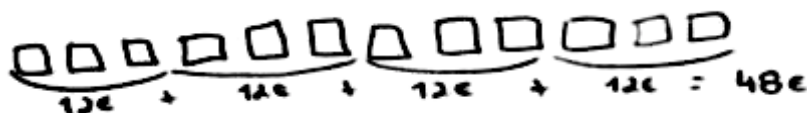

Figure 3. S4's strategy to Problem 1

Finally, S4 uses a strategy which involves the pictorial representation of the books and the successive addition of the price of 3 books until 48 euro, providing the missing value (12 books).

The problem that follows (Problem 2), included in the diagnostic test, has a simple context and data that involve multiples of 5.

> *A car takes 30 minutes to go 50 miles. At the same speed, how long it takes to go 125 miles?*

In Figure 4 we present some of the pupils' answers. S1 determines the missing value using numerical composition/decomposition strategy. However, her written record does not allow us to understand if the numerical composition/decomposition involves additive or multiplicative reasoning. To obtain the values 100 kilometers and 60 minutes, respectively, the pupil may have done: (a) the additions $50+50$ and $30+30$ or (b) the multiplications 2×50 and 2×30. Then, she may have decomposed the initial values, building a new representation in columns (upper right corner). The numerical values that are written by S1 on the third row of her first table seem to come from the addition of the numerical values of the second lines of the two representations, $125\text{km}=25\text{km}+100\text{km}$ and $75\text{min}=15\text{min}+60\text{min}$. S2 and S3 answered in a similar way. S4 answered incorrectly computing the difference between the distances as the value of the travelling distance (missing value).
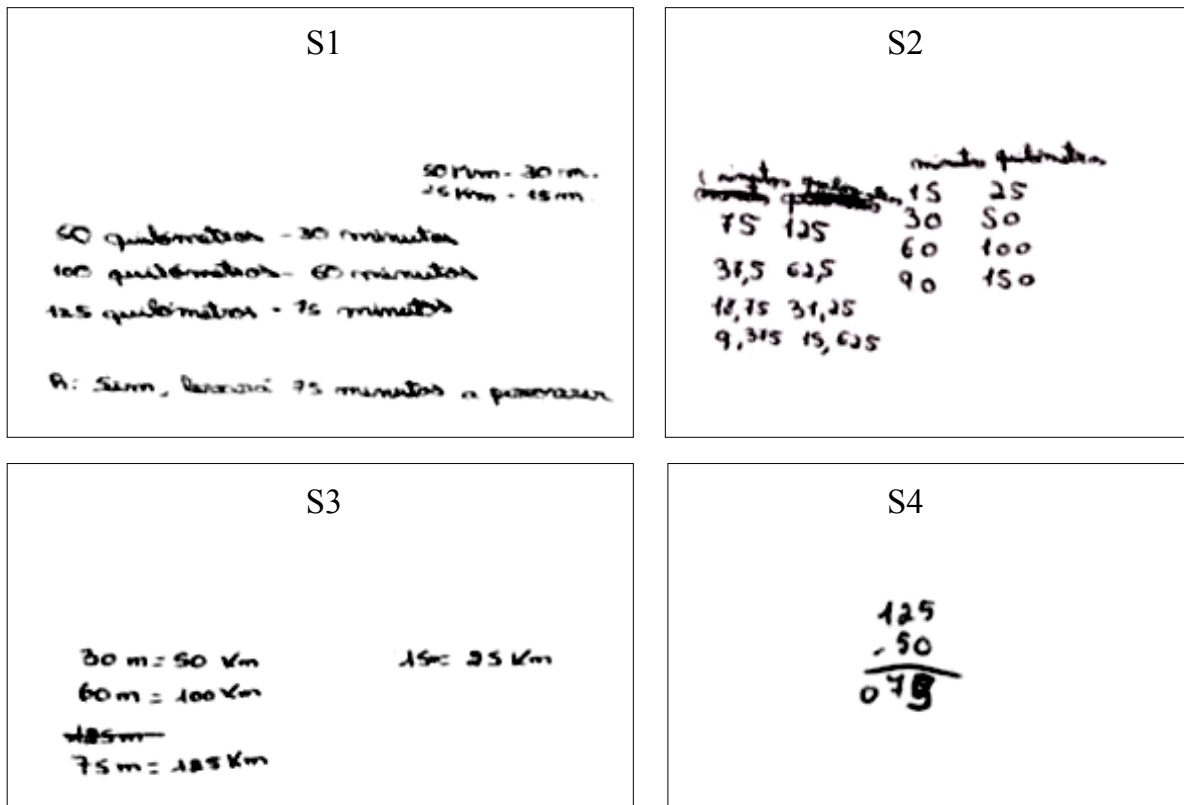
S1	S2
50 Km - 30 m. 25 Km - 15 m. 60 quilómetros - 30 minutos 100 quilómetros - 60 minutos 125 quilómetros - 75 minutos R: Sim, levará 75 minutos a percorrer	15 25 75 125 30 50 60 100 37,5 62,5 90 150 18,75 31,25 9,375 15,625

S3	S4
30 m = 50 Km 15 = 25 Km 60 m = 100 Km 125 m 75 m = 125 Km	125 - 50 075

Figure 4. Pupil's answers to Problem 2

Solving Comparison Problems

The following comparison problem (Problem 3), proposed on the diagnostic test, has a simple context and the values of one variable are multiples of the values of the other variable.

> *Luis and Rosa will make chocolate shakes for their brothers and cousins. The tables represent their recipes. In which one the chocolate flavour is stronger?*

Luis' recipe		Rosa's recipe	
Milk (cups)	*12*	*Milk (cups)*	*20*
Chocolate (spoons)	*3*	*Chocolate (spoons)*	*5*
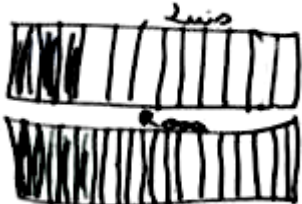
Pupils' answers to this problem are shown in Figure 5.

S1

Luis' recipe – 3:12=0.25 amount of chocolate in a cup…

The milk tastes the same.

S2

Rosa's recipe has a stronger chocolate flavour.

S3

None, they had all the same flavour.

S4

It is in Luis recipe (because it takes less milk, # of cups) that the milk tastes more to chocolate.
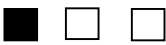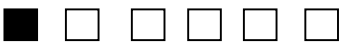

Figure 5. Pupil's answers to Problem 3

S1 and S3 answer correctly. S1 represents the data as a division—using the symbol ":"—and computes the missing value. This is a functional strategy because it involves the relationship between the two variables. It shows that the pupil understands the meaning of the invariant (0.25) which she does not yet see as the proportion constant but that she uses to make a qualitative judgement about the flavour of the mixtures. S3 begins by representing part of the data by an equation ($3x? = 12$). Then he becomes involved on the exploration of the relationship between the quantity of chocolate and the quantity of milk, and he seems to have obtained mentally the invariant factor that he uses to judge the flavour of the mixture. This is a functional strategy.

S2 and S4 answer incorrectly. S2 makes an incorrect pictorial representation of the data, because it is not a whole:part relationship but a part:part relationship. Then, she writes another representation (a fraction) that does not correspond to the pictorial representation. Based on this representation she says that Rosa's mixture has a stronger taste of chocolate. S4 establishes a relation based on subtracting the quantities of milk and chocolate.

The next problem (Problem 4) shows a mixture of liquids, on a simple context, involving small numbers.

At art class, Inês and Maria mixed black ink (black square) with white paint (white square) to produce gray ink. Which girl prepared the darker shade of gray ink?

Ines: ■ ☐ ☐

Maria: ■ ☐ ☐ ☐ ☐ ☐

The following text shows some of the pupils' answers to the Problem 4:

S1: It's like there's only one in black... (Make the next record. Corresponds to two attempts.) (See Figure 6)

S2: It's the same colour [The grey is similar?] It is... Or not? This [Maria mix] has more one [cup] white. It has the same black, it is equal.

S3: It is this one [Inês mix] because Maria's ink has more white, is clearer.

S4: I don't know... Well... (Points to Maria's mix) This has more ink? And... It has the same quantity of black ink... But which is more gray... I have no idea!
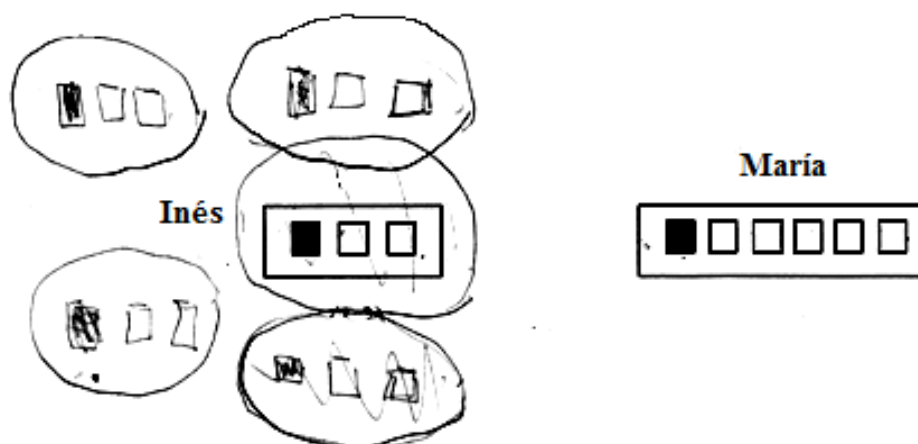
Figure 6. S1' strategy to Problem 4

S3 answers correctly because he establishes a comparison between the mixtures attending to the quantities of white and black ink, knowing that the darker mixture is the one with less white ink. S1 and S2 answer wrongly saying that the colour of the mixtures is equal, arguing that they have the same amount of black ink, which means they are using only a part of the information they were given. However, S1 seems to be focused on the amount of black ink, because with her representations (Inês' mixture) she tries to find the exact quantity of white ink that Maria used. S4 says that he does not know which of the mixtures is the darkest and shows that he is thinking about the total amount quantities of ink instead of the tone.

CONCLUSION

The results show that pupils tend to solve missing value problems correctly by using composition/decomposition strategies which involves additive and multiplicative relationships at the same time. They tend to calculate doubles of the initial and intermediate values, getting more accurate values, as the solutions to Problem 2 show. Then, they decompose the initial values on halves and find the missing value using addition. We may think that this strategy involves specific relationships using halves, as in Problem 2 (two and a half times); however, pupils also use this strategy in relationships involving integer numbers. This is what happens in S1's solution to Problem 1, where she uses reference values that she knows well, obtained by multiplication, and gets the missing value through addition. Pupils' use of composition/decomposition strategies faces a difficulty when it is not possible to determine the missing value by numerical composition through doubling. In fact, in Problem 1, S2 could not use other knowledge to find the missing value. Another difficulty involves the lack of understanding that the difference between the given values of a variable does not correspond to the numerical value of the other variable.

On comparison problems, pupils show different performances, with more difficulties on the problem that involves the tone of the colour. It is likely that they never had to think deeply about this situation, and they focus their attention on data in a partial way, despite the fact that the operations described on the problem are simple to carry out on an art class. On the problem about chocolate flavour, pupils tend to use a functional strategy. The representation of the relation between the values of the variables involves the fraction, indicated by ":", but also the use of numerical columns where pupils explore the multiplicative relationship. One pupil reveals that she is able to explain the meaning of the missing value that is constant in this problem. The difficulties identified relate to interpreting the part:part relationship as a part:whole relationship as well as using different representations of the relationship part:whole. Another difficulty concerns the lack of understanding of the multiplicative relationship, establishing incorrect additive relationships between the variables.

We suggest that pupils, before the formal teaching of direct proportion, tend to solve missing value problems using strategies that involve simultaneous addition and multiplication, and it is possible to identify rudimentary strategies that involve pictorial elements and unitary count. On comparison problems, pupils are able to use multiplicative functional strategies. However, correct answers depend on pupils' comprehension of representations, of the nature of the relationship between the variables and of the context described in the problem.

This knowledge about grade 6 pupil's mathematical processes and difficulties in solving proportion problems before its formal teaching is important to teachers as it allows them to help pupils to develop proportional reasoning from their prior knowledge. On the one hand, teachers may promote the sophistication of pupil's strategies. For instance, composition/decomposition strategies may evolve and become scalar multiplicative strategies, reinforcing the comprehension of the multiplicative nature of proportions. On the other hand, knowing the difficulties and helping to identify the concepts that are not well understood and may lead to poor pupil performance, provide teachers with hints about the work they need to propose to their pupils.

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Ana Isabel Silvestre   João Pedro da Ponte
Universidade de Lisboa   Universidade de Lisboa
anaisabelsilvestre@gmail.com   jpponte@fc.ul.pt