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Three Key Concepts of the Theory of Objectification: Knowledge, Knowing, and Learning

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Abstract

In this article I sketch three key concepts of a cultural-historical theory of mathematics teaching and learning—the theory of objectification. The concepts are: knowledge, knowing and learning. The philosophical underpinning of the theory revolves around the work of Georg W. F. Hegel and its further development in the philosophical works of K. Marx and the dialectic tradition (including Vygotsky and Leont'ev). Knowledge, I argue, is movement. More specifically, knowledge is a historically and culturally codified fluid form of thinking and doing. Knowledge is pure possibility and can only acquire reality through activity—the activity that mediates knowledge and knowing. The inherent mediated nature of knowing requires learning, which I theorize as social, sensuous and material processes of objectification. The ideas are illustrated through a detailed classroom example with 9–10-year-old students.

Keywords: objectification; knowledge, knowing, learning, consciousness.

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Tres Conceptos Clave de la Teoría de la Objetivación: Saber, Conocimiento y Aprendizaje

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Resumen

En este artículo presento tres conceptos claves de un teoría histórico-cultural de enseñanza-aprendizaje de las matemáticas—la teoría de la objetivación. Los conceptos en cuestión son: saber, conocimiento y aprendizaje. Las bases filosóficas de la teoría se encuentran en el trabajo de Georg W. F. Hegel y su desarrollo posterior en la filosofía de K. Marx y la tradición dialéctica (que incluye a Vygotsky y a Leont'ev). El saber, sostengo, es movimiento. De manera más específica, el saber esta constituido de formas siempre en movimiento de reflexión y acción histórica y culturalmente codificadas. El saber es pura posibilidad y puede adquirir realidad a través de la actividad concreta—la actividad que mediatiza el saber y el conocimiento. La naturaleza inherente mediatizada del conocimiento requiere la intervención del aprendizaje, que teorizo como procesos sociales, sensibles y materiales de objetivación. Estas ideas son ilustradas a través de un detallado ejemplo con

Palabras Clave: objetivación, saber, conocimiento, aprendizaje, conciencia.

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In this article I sketch three key concepts of a cultural-historical theory of mathematics teaching and learning—the theory of objectification. The concepts are: knowledge, knowing and learning. The theory rests on the fundamental idea that learning is both about knowing and becoming. It considers the goal of mathematics education as a dynamic political, societal, historical, and cultural endeavour aiming at the dialectical creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted and always evolving mathematical discourses and practices. The philosophical underpinning of the theory revolves around the work of the German philosopher Georg Wilhelm Friedrich Hegel (1977, 2009) and its further development in the philosophical works of Karl Marx (1973, 1998) and the dialectic tradition—Ilyenkov (1997), Mikhailov (1980), and Vygotsky (1987-1999), among others.

Most of the article is devoted to the concept of learning. I start, however, with a discussion about the concepts of knowledge and knowing. Although a discussion about knowledge and knowing may seem esoteric and even futile, I claim that if mathematics education theories want to provide suitable accounts of learning they need to clarify what they believe constitutes knowledge and knowing in the first place. Learning is, indeed, always about something (e.g., learning about probabilities, about geometric properties of figures, etc.). As a result, we cannot understand learning if we do not provide a satisfactory explanation of what learning is about. The next section starts with a discussion of knowledge as construction, followed by a discussion of knowledge as it is understood in the theory of objectification.

Knowledge

Knowledge as Construction

It is now common in mathematics education discourse to talk about knowledge as something that you make or something that you construct. The fundamental metaphor behind this idea is that knowledge is somehow similar to the concrete objects of the world. You construct, build or assemble knowledge, as you construct, build or assemble chairs. This idea of knowledge as construction is relatively recent. It emerged slowly in the course of the 16th and 17th centuries, when manufacturing and the commercial production of things became the main form of human production in Europe. Hanna Arendt summarizes this conception of knowledge as follows: a "I 'know' a thing whenever I understand how it has come into being." (Arendt, 1958, p. 585) It is within the general 16th and 17th centuries' outlook of a manufactured world that knowledge is first conceived of as a form of manufacture as well. A limpid exposition of this view appeared at the end of the 18th century in Kant's *Critique of Pure Reason*. In this monumental book whose influence has not ceased to affect us Kant presents mathematics as the most achieved way of knowing and tells us that "Mathematics alone (...) derives its knowledge not from concepts but from the construction of them" (Kant, 2003, p. 590 [A 734/ B 762]). This conception of knowledge as construction was featured by Piaget in his genetic epistemology and was widely adopted in mathematics education where an emphasis was put on the personal dimension of knowledge construction: You and only you construct your own knowledge. For, in this view, knowledge is not something that I can construct and pass on to you; what you know is what results from your own experience.

As many scholars have pointed out, such a view of knowledge is problematic on several counts. For instance, it leaves little room to account for the important role of others and material culture in the way we come to know, leading to a simplified view of cognition, interaction, intersubjectivity and the ethical dimension. It removes the crucial role of social institutions and the values and tensions they convey, and it de-historicizes knowledge (see, e.g., Campbell, 2002; Lerman, 1996; Otte, 1998; Roth, 2011; Valero, 2004; Zevenbergen, 1996).

As we shall see in the next subsection, there are other ways in which to consider knowledge and the students' relationship to it.

Sociocultural Approaches

How do sociocultural approaches conceive of knowledge? We have to bear in mind that, like constructivist approaches, sociocultural approaches move away from knowledge transmission as a model for learning (sociocultural and constructivist approaches diverge widely but converge certainly on this point). In sociocultural and constructivist approaches, to conceive of learning as the transmission and reception of knowledge amounts to a kind of behaviourism. Dogs learn how to successfully react to certain stimuli; mice learn how to get out of a maze through specific inputs. The human mind by contrast is much more complex; the behaviourist model of stimulus-response is decidedly insufficient. In a now very famous passage, Vygotsky and Luria argued that material and spiritual culture mediate human behavior and suggested replacing the stimulus–response segment (S—R) by a triangle (Figure 1) that, despite its apparent simplicity, adds an unimaginable layer of complexity to the study of learning and the human psyche. Humans carry out operations through signs that alter in a fundamental way the manner in which we come to think and know. Vygotsky and Luria said: "With the transition to sign operations we not only proceed to psychological processes of the highest complexity, but in fact leave the field of the psyche's natural history and enter the domain of the historical formation of behavior" (Vygotsky and Luria, 1994, p. 144).

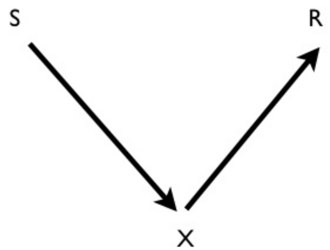

Figure 1. Vygotsky's famous triangle. External signs and other components of material and spiritual culture, X, alter the psyche's natural history (Vygotsky and Luria, 1994, p. 144)

Now, if knowledge is neither something that you merely construct nor something that you transmit, what is it? I would like to develop here a cultural-historical conception of knowledge. In a nutshell the idea is to consider knowledge not as an object but as a process.

Knowledge as encoded movement.

Notice that when we say that knowledge is a *process*, we are saying that knowledge is *movement*. This is how Hegel (2009) considered it. Let me go further and suggest that *knowledge is an ensemble of culturally and historically constituted embodied processes of reflection and action*. In the case of arithmetic, those processes would be processes of reflection, expression, and action that arose in Mesopotamia from specific human activities, such as counting cows or grains, or measuring fields. In the case of music, knowledge would be processes of aesthetic and aural expression that arose in ancient civilizations from specific human activities such as ceremonies to convey bonding, meanings and intentions.

To develop in some detail the idea of *knowledge as an ensemble of culturally and historically constituted embodied processes of reflection and action* I would like to resort to a simple example: nut-cracking in chimpanzees.

Nut cracking in chimpanzees is not an obvious process. As primatologists note, it comprises the following steps: (1) the chimp picks up a nut; (2) puts it on a particular surface: an anvil stone; (3) holds another stone (the hammer stone); (4) hits the nut on the anvil stone with the hammer stone, and (5) eats the kernel of the cracked nut (see Figure 2).

Figure 2. Yoyo cracking a nut while two young chimps watch her attentively. (From Matsuzawa, Biro, Humle, Inoue-Nakamura, Tonooka, & Yamakoshi, 2001, p. 570)

Studies in the wild suggest that it takes 3 to 7 years for the chimp infants to learn the nut-cracking process. Infants do not necessarily start by using a hammer stone and the anvil. The proper attention to the objects, their choice (size, hardness, etc.), and subsequently the spatial and temporal coordination of the three of them (nuts, anvil and hammer), is a long process. Often, young chimps of about 0.5 years manipulate only one object (either a nut or a stone). They may choose a nut and step on it. As chimps grow older, they may resort to the three objects, but not in the correct sequence of nut-cracking behavior, resulting in failed attempts. A key aspect of the process is the appearance of suitable cracking skills—for example “the action of hitting as a means to apply sufficient pressure to a nut shell to break it.” (Hirata, Morimura, & Houki, 2009, p. 98)

Nut-cracking is learned as a social process. The young chimps, who usually remain with their mother until the age of 4 to 5 years, observe attentively how the mother cracks nuts and then try to do it by themselves, even without apparently understanding the goal of the process at first[1].

Not all chimpanzee groups crack nuts, and those groups where nut-cracking occurs do not all crack the same variety of nuts. Primatologists believe that nut-cracking developed somewhere in West Africa and was subsequently conveyed socially from one generation to the next. The nut-cracking practice eventually spread out among neighboring groups as a result of chimps’ immigration (Hirata et al., 2009, p. 88; Matsuzawa et al., 2001, pp. 569-70).

I would like to suggest that “knowledge”—in this case knowledge of how to crack nuts—is a *culturally codified ensemble of actions*. That knowledge is a cultural codification of ways of acting and doing means that knowledge is something general: it cannot be equated to this or that particular sequence of coordinated actions with *these* or *those* stones. Another way to say this is that knowledge is *crystillized labour*. We can think of knowledge as an ideal form of actions, as opposed to the actions themselves. Knowledge as crystillized labour or ideal form goes beyond each one of its concrete instances or realizations. It is nut-cracking as an ideal form that lends the generality to each one of its specific realizations.

Let me notice that knowledge as an ideal form (here, knowledge of how to crack nuts) does not have anything to do with Platonic forms. Rather than considering the nut-cracking Seringbara community of chimps that inhabits the mountain forests of Mt. Nimba in the Republic of Guinea as resorting to Platonic forms or to Kantian things-in-themselves, they would be resorting, I wish to argue, to culturally and historically constituted embodied processes of reflection and action. The nut cracking "ideal form" is to be understood as a *general prototypical* way of doing things. Rather than sitting in an eternal realm of ideas, this ideal form is codified in cultural memory as a pattern or sequence of actions. As opposed to the Platonic forms, which are supposed to exist regardless of what species do on earth, knowledge as an ideal form cannot exist if it is not carried out in practice.

I am almost ready to define knowing. But it might be better that I first give a classroom example. Let me refer to pattern generalization. Like many of my colleagues, in my classroom research I have resorted to pattern generalization to introduce students to algebra. The basic idea is to present the students with simple geometric or numeric sequences (usually artihmetic sequences that can be expressed in a linear form: $y = ax+b$). We give the students a few terms (see Figure 3) and then ask them to come up with ideas about how to calculate "remote" terms (e. g., Terms 10, 25, 100, etc.).

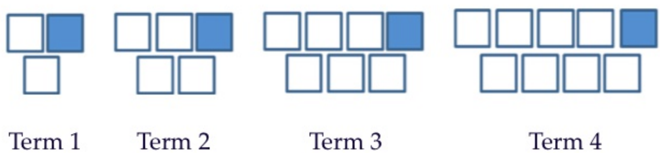

Figure 3. The first terms of a sequence that Grade 2 students investigated in an algebra lesson.

In so doing, we expect the students to enter into a relationship with a historically constituted form of knowledge about arithmetic sequences. More specifically, we expect the students to become aware of an algebraic form of perceiving, reflecting and investigating sequences that goes back to ancient times. Indeed, the investigation of arithmetic sequences appeared in ancient civilisations (for instance Mesopotamia) and was a very popular subject in Late Antiquity in neo-Pythagorean circles (Lawlor & Lawlor, 1979; Nicomachus of Gerasa, 1938, Tarán, 1969). Neo-Pythagoreans were particularly interested in polygonal numbers—that is, numbers represented by pebbles arranged in the shape of a regular polygon. For instance, the first triangular numbers are 1, 3, 6, 10; the first square numbers are 1, 4, 9, 16; the first pentagonal number are 1, 5, 12, 22; see Figure 4).

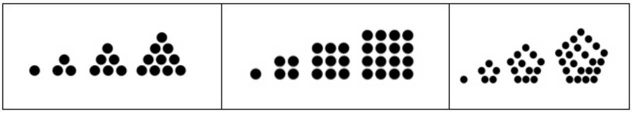

Figure 4. The first triangular, square and pentagonal numbers.

As far as I know, the investigation of theoretical properties of arithmetic sequences first appeared in Hypsikles' text known as *Anaphorikos* (see Radford, 2006)[2]. Proposition 1 is stated as follows:

> If any number of terms is considered such that <starting from the greatest> every two successive ones have the same difference, {the terms} being even in number, then, the difference between {the sum of} half the number of terms [starting from the greatest], from {the sum of} the remaining ones, is equal to the multiple of the common difference by the square of half the number of terms. (Manitius, 1888, p. 2)

In modern symbolism, the proposition asserts that if a number of $2n$ terms, $a_1, a_2, \dots, a_{2n}$, are such that $a_1 > a_2 > \dots > a_{2n}$, and $a_i - a_{(i+1)} = d$ for $i = 1, \dots, 2n-1$, then:

$$(a_1 + a_2 + \cdots + a_n) - (a_{n+1} + a_{n+2} + \cdots + a_{2n}) = n^2 \times d$$

Hypsikles' proposition states a *property* of what we now call an arithmetic sequence. It is still not a *formula* to calculate terms in an arithmetic sequence. Diophantus (ca. 250 AD), in his short text *On polygonal Numbers*, offers a *formula* to calculate any polygonal number, S_n, when the side, *n*, of the polygonal number and the angle *a* are known. The formula is:

$$S_n = \frac{[(2n-1)(a-2)+2]^2 - (a-4)^2}{8(a-2)}$$

Let suppose that we want to calculate the third term of the pentagonal numbers. In this case, $n = 3$ and $a = 5$.

$$S_3 = \frac{[(2\times3-1)(5-2)+2]^2-(5-4)^2}{8(5-2)}$$ yield $S_3 = 12$.

Naturally, Dipohantus did not express this formula through modern symbolism. What he tells us is this: “Take twice the side of the polygonal number; from this subtract one unit; multiply the result by the number of angles minus 2; then add 2 units. Take the square of the resulting number. From this, subtract the square of the number of angles minus 4. Divide the result by 8 times the number of angles minus 2 units. This gives us the polygonal number we are looking for” (based on the translation of Ver Eecke, 1959, pp. 290-291).

Much like nut-cracking in our chimpanzees example, forms of algebraically reflecting, perceiving, and dealing with sequences are codified forms of thinking and doing. And as in the case of chimpanzees and their cultural history, these forms of thinking and doing have been codified and refined in human cultural history. This refinement entails a successive *determination* of knowledge. Drawing on historical records, historians think that the investigation of sequences was in the beginning carried out through pebbles (Lefèvre, 1981). From a Hegelian perspective, the resulting pebble-mediated codified knowledge is considered to have become subsequently embedded or *sublated* into something more specific (e.g., an analytic investigation of theoretical properties of arithmetic sequences, like Hypsikles’), passing hence from something abstract into something more determinate or more concrete. This is what in Hegelian dialectic is called *the ascension from the abstract to the concrete*, and it occurs through a process where new determinations of knowledge do not merely replace new ones, but carry out, in a condensed manner, the meanings of previous theoretical formations. As Marx put it, “The concrete is concrete because it is the concentration of many determinations, hence unity of the diverse. It appears in the process of thinking, therefore, as a process of concentration, as a result, not as a point of departure.” (1973, p. 101)

Within the conception of knowledge that I am outlining, the cultural evolution of knowledge, its rising from the abstract to the concrete, should by no means be understood as something that occurs as if it is pushed by an invisible hand or by rational knowledge’s own logic. The evolution of knowledge is to be understood not as a natural phenomenon but as a cultural one. Much as capital can only be understood as a historical concretion of abstract concepts, such as division of labour, money, value, etc. mathematical knowledge can only be understood as a concretion of prior abstract embodied, linguistic, perceptual and artifactual forms of mathematical thinking and doing.

My example of knowledge about arithmetic sequences does not contain anything special. Similar examples could be given about any topic in school mathematics. The point is hence that through a lengthy process of refinements and concretions, mathematical knowledge has been expressed in different ways (natural language, alphanumeric symbolism, graphs, etc.) and codified in cultural memory and practices and is now present in many educational curricula around the world. It is this knowledge that the students encounter in the school and that would lead them to see that Term 100 of the sequence shown in Figure 3, for example, has $1+2\times 100$ squares.

Now we are ready to define *knowing*.

Knowing

Knowledge, I just argued, is crystallized labour—culturally codified forms of doing, thinking and reflecting. Knowing is, I would like to suggest, the *instantiation* or *actualization* of knowledge.

Now, when we state that knowing is the instantiation or actualization of something already there, the risk of being misread is certainly high. Knowing may appear as a simple repetition. Of course, this is not true. If knowing were a simple repetition, knowledge would be something statistic. There would not be the slightest chance for knowledge to evolve. Yet, as our example of Hypsikles and Diophantus shows, the latter was not simply repeating the former. So when I suggest that knowing is the instantiation or actualization of knowledge, what has to be understood is:

(1) the meaning of knowledge as something general;
(2) the process of its actualization, and
(3) the result of its actualization.

In order to understand these three interrelated aspects of knowledge and knowing, we have to bear in mind that to assert that knowledge is something *general* that cannot to be equated with any of its instantiations or actualizations is to assert that knowledge is mere *possibility*: The possibility of cracking this or that nut; the possibility of finding out a property of arithmetic sequences or the 100th term in a given sequence. This possibility *qua* possibility is simply something *inexistent,* mere *potentiality* that “has not yet emerged into Existence” (Hegel, 2001, p. 36). In order to emerge into existence and to become *actuality*, knowledge has to be instantiated through actualization.

Actualization is a *process*—what Hegel calls a *particular.* It is an event or activity: “activity and relations between people” (Blunden, 2009, p. 103), “the activity of man [*sic*] in the widest sense” (Hegel, 2001, p. 36). What Hegel means by this is that, in order to be instantiated, knowledge has to show itself in the *activity* through which it acquires its content. “It is only by this activity that … abstract characteristics generally, are realized, actualized; for of themselves [i.e., as generals] they are powerless.” (Hegel, 2001, p. 36)

Let me note that the activity of which the particular consists is not a simple channel through which knowledge makes its appearance. The particular as activity *impresses its mark* in knowledge’s instantiation. This instantiation is what Hegel calls the *singular* or *individual*, and corresponds, in our terminology, to what we have been calling *knowing*.

Knowing, hence, is the concrete conceptual content through which knowledge is instantiated. Its concrete conceptual content appears and can only appear through an activity —the activity that mediates knowledge and knowing. There is no such a thing as unmediated knowing. Knowing is indeed the result of a mediation. The meaning of such mediation is the following: knowing bears the imprint of the activity that mediates it (Ilyenkov, 1977). In other words, the particular as activity demarcates the manner in which knowing instantiates knowledge. In even simpler terms, the manner by which we come to know something (like how to solve equations) is consubstantial of the specificities of the process of knowing. The mediating activity does so through the historical and cultural material forms, means and modes of active human intercourse that define it (Mikhailov, 1980).

To sum up, the particular is the activity through which the general appears in the singular —or, on other words, how knowledge is instantiated in knowing. This activity actualizes knowledge, bringing it into life.

We can express the relationship between knowledge and knowing in the following terms. Knowledge's mode of existence turns out to be its practical appearance through one or more of its singulars in the concrete world —i.e., as knowing. And vice-versa: every instance of knowing (nut-cracking or sequence generalization; in short, every singular) is possible insofar as it appears as the *manifestation* or the *incarnation* of knowledge. It is only through this singular developed form that knowledge can be an object of thought and as such to be modified and expanded.

Let me give a historical example to illustrate this last idea. Some Babylonian clay tablets show problems about measuring objects. They are vestiges of activities at the interior of which codified forms of measuring became materialized and instantiated. One of the metrological units of length is the *foot*. While foot might have been a useful unit to measure some objects in the world, it might not took long for the Babylonian scribes to realize that sometimes adding feet was not enough. Adding feet would end up being a bit shorter or larger than the measured object. The encoded forms of measuring appeared in the concrete world and had to be expanded to measure those "difficult" objects. Subdivisions of the foot (or "fractions" of it) could only be envisaged in the concrete world through the actualization of knowledge. The inclusion of fractions gave rise to new forms of measuring, which, trough activity, became encoded, thereby constituting a *modification* of previous knowledge. The new practice of measuring became new knowledge. Without the possibility of actualization, knowledge would remain general and hence unable to be modified.

Figure 5 tries to capture the relationship between the general, the particular, and the singular. The general (knowledge) is pure possibility. The singular (knowing) is the concrete conceptual content (e.g., the theoretical reflection on the material circles in Figure 3; the fleshy and kinesthetic actions of a chimp cracking a concrete nut in Figure 2) that conveys, in its materiality, the abstract nature of the general. It is the content of the general that shows up in sensuous theoretical reflection; the manner in which the general has actuality (Maybee, 2009). As activity, the particular is the mediation between the general and the singular. This mediation is fundamental, as it stresses the unmediated nature of knowing[3].

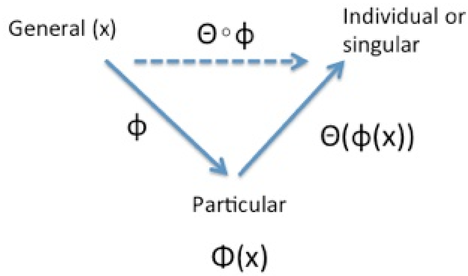

Figure 5. The relationship between the general, the particular, and the singular.

Notice, however, that because the actualization of the general is a singular, the actualization cannot capture the general in its entirety. Hence, by incarnating the general, the actualization affirms it; and, at the same time, by being unable to fully capture the general, it negates it. When my Grade 6 students solve the equation shown in Figure 6, they actualize a cultural form of action and reflection (a pure *possibility*) which becomes materialized in the sensuous theoretical activity (*particularity*) of reflecting on what is required to solve the aforementioned specific equation (this *reflection* on specific equations is the *singular* or *individual*). They do it within a particular and unrepeatable classroom activity —a particular, which is the unique event of solving *that* equation at a *certain* moment and place and through a certain relation between people.

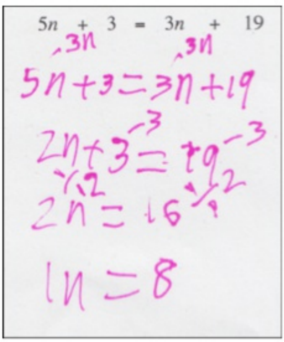

Figure 6. Grade 6 students actualizing an encoded algebraic form of thinking (a general) through a singular reflection (a singular) on an equation mediated by a classroom activity (a particular).

The actualization of the general leads to one of its possible instantiations from where results the awareness of how to solve a specific (individual) equation (or a certain finite number of them). As such, the actualized movement cannot capture the general (how to solve equations) in its entirety: it cannot because the general can only become object of consciousness through particulars and singulars. As a result, the actualization ostensively embodies the general and at the same time negates it. This is why the actualization (as event) is always deficient. But its deficiency is the bearer of new possibilities, for it is only through actualization that something new can arise.

Learning

In the previous section I have dealt with knowledge and knowing as conceived of in the theory of knowledge objectification. Let me address now the third chief concept of the theory: learning.

Some sociocultural approaches theorize learning as a form of *participation* in a social practice; other sociocultural approaches resort to the theoretical idea of internalization. The ideas of participation and *internalization* are certainly interesting. Yet, they bear intrinsic difficulties that we have to overcome.

Participation

In sociocultural theories that resort to participation to provide an account of learning, the basic idea is that individuals come to know as they participate in social practices. There is an explicit intention to move beyond the individualist conceptions of mainstream psychology and philosophy, where the individual is the unit of analysis and the focus of research. The idea of participation was developed by Rogoff (1990), Lave (1988), and Lave and Wenger (1991), among others. Rogoff, for instance, conceives of knowing as apprenticeship in a context of guide participation. She says: “The concept of guided participation attempts to keep the roles of the individual and the sociocultural context in focus” (Rogoff, 1990, p. 18). She goes on to say that she uses the analogy of apprenticeship “to focus on how the development of skill involves active learners observing and participating in organized cultural activity with the guidance and challenge of other people.” (Rogoff, 1990, p. 19) To account for learning and thinking as apprenticeship, Rogoff shows how infants and parents undergo subtle processes of shared attention, and how through adults' support, children gain insights into social referencing, manners to solve problems and to cope with social demands. Learning, however, remains in the end a process whose goal is to adapt oneself to social practices. Despite the tremendous array of new concepts that the participation account brings in, learning appears to be a kind of adaptation, much as in Piaget's account. The difference is that while for Piaget adaptation is carried out through general (universal) cognitive mechanisms and the environment is seen as something "natural," in the participation paradigm learning is the adaptation through social mechanisms to a cultural world of practices. Intersubjectivity is no more than a relation founded by communication, shared meanings and joint attention. In the theory of objectification, intersubjectivity and learning are deeply related; communication, shared meanings and joint attention will play a crucial role. But, as we shall see in a moment, the crucial concept is consciousness in a Hegelian-Marxist-Vygotskian sense. But before I get there, let me comment on learning as internalization.

Internalization

The idea of internalization was elaborated by psychologists such as Pierre Janet (1929) and Vygotsky (1929) in the first part of the 20th century. It is a theoretical construct to account for the link between the individual and his or her environment. Janet, for instance, articulated it in his investigations of personality and argued that all psychological laws have two aspects—one exterior (dealing with other people) and one interior (dealing with us). Almost always," he said, "the latter is posterior to the former" (Janet, 1929, p. 288).

Internalization constitutes one of the central ideas of Vygotsky's cultural-historical theory formulated in the early 1930s – although implicit versions of it can be found in earlier articles, such as the 1929 article "The cultural development of the child" (Vygotsky, 1929). Internalization is deeply related to Vygotsky's own concept of human development and the role that signs play therein. Internalization makes operational another key theoretical construct of the cultural-historical theory, namely the "genetic law of cultural development." Vygotsky stated this law as follows: "Every [psychic] function in the child's cultural development appears twice: first, on the social level, and later, on the individual level" (Vygotsky, 1978, p. 57). Internalization as a process mediated by signs is precisely what ensures the passage from the social to the individual level: "The internalization of cultural forms of behavior involves the reconstruction of psychological activity on the basis of sign operations." (Vygotsky, 1978, p. 57)

The idea of internalization has its own problems. Thus, casting the relationship between the individual and her context in terms of internalization can be said to still keep traces of a form of individualistic thinking that fails to resolve the famous dichotomy between the internal and the external. As Veresov asks, "Where is the difference or even the border between external and internal then?" (Veresov, 1999, p. 225)

We need to recall that Vygotsky's theory was developed as an attempt to go beyond the reflexologist and idealist research of his time. He often complained that psychology inspired by reflexology was a psychology of behaviour without mind, and that psychology inspired by subjective idealism (introspection, for instance) was a theory of the mind without behaviour. In the footsteps of Spinoza (1989), he was trying to overcome dualist theories (theories based on two systems, the internal and the external) and to formulate a monist theory of consciousness. But this was not without contradictions. Veresov —considered one of the greatest contemporary Vygotskian scholars— has this to say:

> What essentially does it mean to abandon the postulate of two system existence and to what conclusions and logical effects does it lead? This logically leads to a full rejection of the idea of the existence of the internal and the external and, consequently, to the radical refusal of the concept of internalization as a mechanism of the origin of internal structures of consciousness. Actually, the concept of internalization becomes senseless in this case. (Veresov, 1999, p. 226)

Vygotsky's last works show his effort to overcome these difficulties (in particular his search for an encompassing account of meaning). I am not going to discuss these ideas here, as my intention is only to show that Vygotsky's theory, based on the idea of internalization, is not exempt from theoretical difficulties that have implications for our conceptions of learning.

Objectification

If we conceive of knowledge as *movement* as I suggested previously —more precisely as a culturally and historicaly codified sequence of actions that are continuously instantiated in social practice— knowledge turns out to be neither something to be "possessed" nor to be "attained." Knowledge appears rather as something that is not us, something that we encounter, wherein it *objects* (i.e., opposes) us. *Objectification* is precisely the process of recognition of that which objects us—systems of ideas, cultural meanings, forms of thinking, etc.[4]

Objectification, as we can see, emphasizes the idea of *otherness*—the quality of not being us. Contrary to the standard accounts of ideas according to which they are born in us and are part of our mental life, for the theory of objectification, ideas and forms of thinking are considered to exist independently of each one of us. From a philogenetic viewpoint, "Knowledge, skills and abilities," Mikhailov notes, "exist without me" (1980, p. 200). We encounter them in the course of our life as external objects.

In the *Shorter Logic*, Hegel says:

> It is a mistake to imagine that the objects which form the content of our mental ideas come first … Rather the notion [i.e., the concept] is the genuine first; and things are what they are through the action of the notion, immanent in them, and revealing itself in them. (Hegel, 2009, p. 329)

The encounter and recognition of systems of thinking, cultural significations, etc. —in short, their objectification— is not a straigthforward process. In Figure 2, we see an adult chimpanzee named Yo cracking Coula nuts. With her right hand Yo places the nut over an anvil and, in a coordinate manner, she holds the stone hammer with her left hand, while the young chimps to her left and right watch her attentively. The young chimps do not yet master the relatively sophisticated motor and conceptual skills that are required to accomplish the cracking of the nut. These skills already exist in their chimp culture and will become part of the young chimps' repertoire of action and reflection after a long period of intense practice and observation.

Much in the same way, my Grade 2 students do not necessarily master the relatively sophisticated motor and conceptual skills needed to extend arithmetic sequences. For example, mathematicians would attend without difficulty to those aspects of the terms shown in Figure 3 above that are relevant for the generalizing task: they would, for instance, see the terms as divided into two rows and notice the immediate relationship between the number of the term and the number of squares in each one of the rows. The perception of those variational relationships usually moves so fast that mathematicians virtually do not even notice the complex work behind it. They would also extend without difficulty the noticed property of the rows to other terms that are not present in the perceptual field, like Term 100, and conclude that this term has100+101 squares, that is 2001 (see Figure 7). Or even better, that the number of squares in any term, say Term *n*, is *2n+1*.

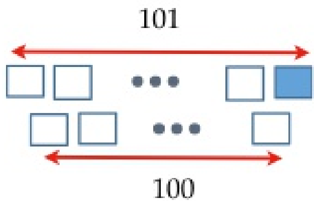

Figure 7. A frequently reported quick imagination of Term 100 by the trained eye.

Yet, the novice eye does not necessarily see the sequence in this way. Figure 8 shows an example of how a Grade 2 students extends the sequence beyond the four given terms shown in Figure 2.

Figure 8. Terms 5 and 6 as drawn by a Grade 2 student.

The student focuses on the *numerosity* of the squares, leaving in the background the *spatiality* of the terms (Radford, 2011). We cannot say, I think, that the student's answer shown in Figure 8 is wrong. The answer makes sense for the student, even if it is probably true that by focusing on the numerosity of the terms of the sequence, it might be difficult later on to end up with a general formula like $2n+1$. This is in fact what we have observed again and again in our research with older students (13-17-year-old students). In the latter case, the students tend to rely on trial-and-error methods that, as I have argued elsewhere, are not algebraic, but arithmetic in nature (Radford, 2008, 2010).

The issue is not that the students do not see the two rows of the terms. In Figure 9, we see a Grade 2 student pointing with his pen to the top row, then to the bottom row, after moving the pen across the top and bottom rows to properly distinguish between them. However, when the student draws Term 5, the *spatial* dimension of the terms is relegated to a second plane and does not play an organizing role in the drawing of the term. He draws a *heap* of rectangles. The issue is rather about not realizing yet that the spatiality of the terms provides us with clues that are interesting from an algebraic viewpoint.

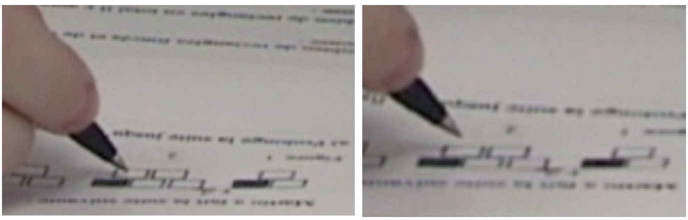

Figure 9. A student pointing to the top row (left) and to the bottom top (right) of Term 2.

The cultural objective encoded forms of action and reflection remain separated from the students. They are forms of action and reflection "in itself." That is to say, they exist but remain unacknowledged and unnoticed by the students. They remain possibility without actualization. Learning is the subjective and idiosyncratic transformation of the "in itself" knowledge into a "for itself" knowledge, that is, a transformation of cultural objective knowledge into an object of consciousness. This transformation is what I term *objectification*.

Let me dwell upon the meaning of these Hegelian terms. Hegel uses "in itself" to refer to something merely potential, unreflective. These are the ideal forms I mentioned previously. They are what they are, mere possibility of action and reflection at a certain historical and cultural point. They may be the mathematician's encoded forms of action and reflection or the chimps' encoded forms of cracking nuts. When we encounter and become conscious of the "in itself" knowledge, consciousness goes outside itself and captures now the "in itself" knowledge as something determinate from consciousness' viewpoint, as something *for us*. The "in itself" becomes actuality, a "being-for-consciousness," and this is what Hegel calls a "being-for-itself."

In the course of learning, the ideal form (the "*in itself*") is enacted or actualized, becoming a particular or individual. In learning we have the merging of the "in itself" and the "for itself." The "in itself" appears in a developed form "both at home with itself, and finding itself in the other" (Gardener, n. d.).

I can now attempt a more operational definition of objectification.

Learning as Objectification

In the theory of objectification, learning is theorized as *processes of objectification*, that is to say, those social processes of progressively becoming critically aware of an encoded form of thinking and doing—something we gradually take note of and at the same time endow with meaning. Processes of objectification are those acts of meaningfully noticing something that unveils itself through our sensuous activity with material culture. It is the noticing of something (the "in itself") that is revealed in the emerging intention projected onto the signs or in the kinesthetic movement in the course of practical concrete activity— the disclosing of the "in itself" that becomes "for itself" in the course of its appearance and is hence transformed into knowledge *for us.*

But in the course of this transformation of the "in-itself" into the "for itself," consciousness is transformed as well. This is why within the theory of objectification learning is not simply about knowing, but also about becoming. Learning is not a mere imitation or participation consistent with a pre-established practice. Learning is the fusion between cultural modes of reflecting and doing and a consciousness which seeks to perceive them (Radford, 2007, pp. 1790-91). In the course of this fusion, consciousness emerges and is continuously transformed. In other terms, processes of objectification are entangled in *processes of subjectification*—processes of creation of a particular (and unique) self.

We see that underneath the concept of learning the theory of objectification brings in, there is a particular concept of consciousness. Consciousness is not considered a metaphysical construct hidden somewhere in an alleged interiority with which we would all have been born. This metaphor of interiority was invented towards the end of Antiquity. It was developed by Augustine in a religious context and later articulated by Descartes and his famous dichotomic view of the world: the one of interiority (mind, ideas, consciousness, etc.) and the one of exteriority (the concrete world) (Taylor, 1989). Within the theory of objectification, consciousness is rather considered as a subjective reflection of the external world. Consciousness is the subjective process through which each one of us as individual subjects reflect on, and orient ourselves in, the world. This reflection is not a contemplative one. The individual consciousness is a specifically human form of subjective reflection of concrete reality in the course of which we come to form cultural sensibilities in order to ponder, reflect, understand, dissent, object and feel about others, ourselves and our world. Consciousness can only be understood as the product of historical-cultural and emergent contingent relations and mediations that, rather than being given, "arise in the course of the establishment and development of society." (Leont'ev, 1978, p. 79) Within this view, consciousness appears in concrete life, not as its origin, but as its result.

To sum up, in this section I introduced the concept of objectification. I started by introducing it as a form of recognition of concepts, systems of thinking, cultural significations, etc. that predate our appearance in the world. Then, I refined the concept as the transformation of "in itself" knowledge into "for itself" knowledge, and noted that this transformation amounts to the creation and continuous growth of the individual's consciousness: objectification is a social process of progressively becoming critically aware of encoded forms of thinking and doing in the course of which consciousness is formed and transformed. In the following section I focus on some aspects of the practical investigation of objectification.

Investigating Objectification

The investigation of objectification focuses on the manner in which the historically and culturally encoded forms of thinking and doing become objects of recognition or objects of consciousness. Given the mediating role that the particular plays between knowledge and its concrete individual instantiation, the particular (as the activity through which knowledge appears in an embodied and sensuous manner) is a key component in the investigation of processes of objectification. Indeed, the actualized general, that is the individual or singular, is a bearer of the general's conceptuality. Yet, in its materiality, that is, in the concrete material culture that mediates it, the singular in itself cannot disclose such conceptuality. This is why in general, concrete materials and artifacts cannot disclose the conceptuality they are supposed to individuate. They need to be embedded in an *activity* (a particular) that makes apparent the conceptuality they are bearers of.

Here is an example. In a Grade 4, we gave to the students (9–10-year-old) a problem where they had to deal with an arithmetic sequence. The context was stated as follows:

> For his birthday, Marc receives a piggy bank with one dollar. He saves two dollars each week. At the end the first week he has three dollars; at the end of the second week he has five dollars and so on.

We provided the students with bingo chips of two colors (blue and red) and numbered plastic goblets that stand for the piggy bank at week 1, 2, etc., so that the students could model the saving process until week 5. Then, they were required to generalize: they were required to answer questions so as to find the amount of money saved at the end of weeks 10, 15, and 25.

The students began modeling the saving process in the manner of a "real situation": they started placing the bingo chips in the goblets (three bingo chips in the goblet that corresponded to the piggy bank of week 1, etc.). Although it was interesting, the model proved to be of limited use to answering the questions about the amount of money saved in some distant weeks (like week 25). Indeed, the bingo chips were piled up inside the glass, making it hard to discern any structure, let alone a mathematical one. The students' attention was directed to the sequential additive actions (adding two bingo chips) that remained unsynthesized in a more abstract multiplicative structure. The artifacts were insufficient to help the students disclose the general's conceptuality we aimed for. The artifacts were rather bearers of a conceptual quotidian content that was distant from the algebraic one. At that time that the students finished putting the bingo chips in the goblets without noticing any algebraic structure, the teacher was in the process of talking to another group at the other end of the class. I removed my earphones, left the camera with which I was videotaping this group of three students and went to talk to them. The group was formed by Albert (Fig. 10, to the right), Krysta (in the middle), and Manuel (to the left). I suggested putting the bingo chips in front of the goblets. The students accepted the suggestion and started piling them up without distinguishing between colors. Then, I proposed to use a blue bingo chip to signify the initial dollar in the piggy bank. Following this suggestion, the three students created a model of the saving process (see Figure 10).

Figure 10. The modeling arithmetic sequence through bingo chips.

The new arrangement of concrete materials helped the students to better understand the saving process. Yet, despite the new bingo chip arrangement the students were not able to come up with a formula to calculate the savings in remote weeks (e.g., week 15, 25) right away (see Radford & Roth, 2011). The singular's conceptuality (the algebraic content it embodies) was not revealed.

The problem is that the encoded forms of movement (in this example, algebraic encoded forms of thinking related to numerical sequences) cannot be instantiated directly into singular instances. The actualization of the general is mediated by a particular activity (this is what the diagram in Figure 5 asserts). In order for the students to perceive the general, its content has to be deliberately recognized in accordance with the structural place it occupies in the students' activity (Leont'ev, 1978). This structural place is what the particular offers, for as mentioned previously, the only way for the conceptual generality to be disclosed is through the particularity of the particular, that is to say, the activity in which the general appears in a developed, actual form.

The activity of which the particular consists has to be understood as entailing much more than people interacting between themselves. It is more than a milieu of interaction with people and artifacts. It is a form of life, something organic and systemic, something emergent, driven by a common search that is at the same time cognitive, emotional and ethical. For learning to occur, the realm of the possible and the virtual has to appear in a concrete manifestation in the students' consciousness. This in turn requires that the general be mediated by the particular —a specific activity that makes the general appear in the concrete world, to become endowed with a particular conceptual content (see Figure 5). If the general is a form of thinking algebraically about sequences, the particular is the activity that would require the teacher and the students to engage in some type of reflection and action that features the target algebraic conceptual content, so that the general finds itself embodied in the resulting singular —maybe even in novel ways.

I can now present the structure of the particular as follows.

The Structure of the Particular

The Relation Φ

At the most general level, let us bear in mind, the particular is the way in which the general *shows up*. If the general consists of culturally encoded forms of algebraically thinking about sequences, the particular may take a variety of forms: for instance, an activity that feature thinking of figural or numerical sequences in an algebraic way through alphanumeric symbolism, or through graphs, or through natural language, etc. If the general consists of culturally encoded forms of thinking about motion, the particular may be an activity that features thinking about space and time in qualitative manners or in a Cartesian co-variational sense; it can also be thinking through infinitesimals and derivatives, etc. In all cases, while the general is mere possibility (the realm of the virtual), the particular is a step forward towards the concretion of the general. It concretizes the general by *particularizing* it (in our second example, through a focus on co-variation, derivatives, areas, etc.). But the particular is not static: its link to the general is a *morphism*, and as such preserves the general's most basic structure: *movement*. This is why the particular is activity —joint activity between people carried out through material culture.

The particular is hence particularizing activity. This particularization of the general by the particular considered as activity is what Leont'ev (1978) calls the *object* of the activity. The particular as activity moves towards its object through the identification of *goals*. These goals can be, if we continue with our algebra example, to solve problems about sequences algebraically. To reach the goals of the activity, specific *tasks* have in turn to be envisioned. They may appear as a sequence of related problems of increasing conceptual difficulty.

The *object—goal—task* structure corresponds to the relation Φ that we can add to our Figure 5 (see Figure 11). The relation Φ relates to the pedagogical intention of the classroom activity. In the case of the theory of objectification it involves an epistemological analysis of the target mathematical content that we complement with an a *priori analysis* (Artigue, 1995, 2009).

Let me note that the relation Φ applied to the general x (e.g., algebraically thinking about sequences) may take several "values" $\Phi(x)_1, \Phi(x)_2, \ldots$ depending on the implementation of the pedagogical intention of the activity. In research on early algebra, we find cases where the values of $\Phi(x)$ revolve around: (1) problems that require expanding figural terms, (2) a functional approach, (3) the use of symbolism to designate qualitative relationships, etc. (see Cai and Knuth, 2011).

The Relation Θ

The Particular as an activity that *actualizes* the general in the form of an individual or singular instance is what the relation Θ expresses in Figure 11: activity as actualized concrete movement, leading hence to a singular instantiation of the general.

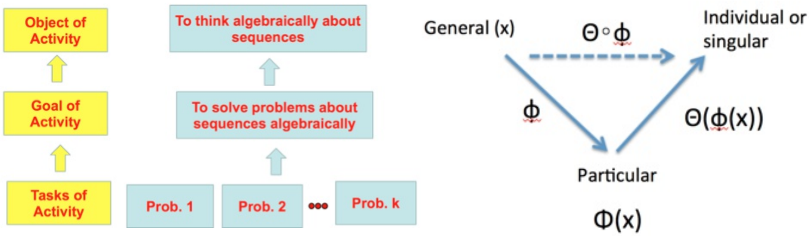

Figure 11. The structure of the particular: The particular as particularizing activity is made up of two relations, Φ and Θ.

Let us have a more detailed look at the relation Θ. The relation Θ as an activity that actualizes the general through the particular can be envisioned in various ways. Within the theory of knowledge objectification, the actualization of the general is articulated as an emergent process of instantiation of the general. The adjective "emergent" means that the classroom is envisioned as a system that evolves through "states" and that this evolution cannot be determined in advance. Teachers and researchers may have an idea, but the process is not a mechanical one. It will depend on how students and teachers engage in the activity, how they respond to each other, etc.

In the case of the theory of objectification, we usually divide the class into small groups of two to three or four students. The first state of Θ is a presentation of the activity by the teacher (see Figure 12). Then, the students are invited to work in small groups (see "Small Group Work" in Figure 12). Then, the teacher visits the various groups and asks questions to the students, gives feedback, etc. (see "Teacher-students Discussion" in Figure 12). At a certain point, the teacher may invite the class to a general discussion where the groups can present their ideas and other groups can challenge them or improve and generalize them (see "General discussion" in Figure 12). The lesson may end there or continue with additional small group, etc.

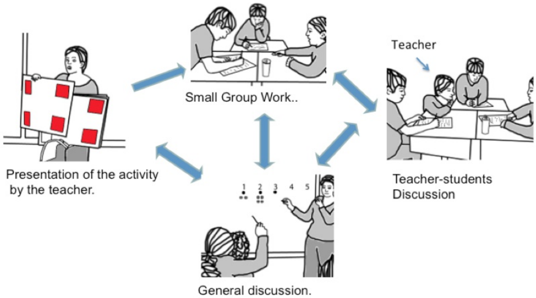

Figure 12. The relation Θ as classroom activity goes into different states.

The arrow $\Theta(\varphi(x))$ in Figure 11 goes into those states that are related to the manner in which the class has been divided and the tasks of the activity defined.

Objectification occurs when the students and the teacher, through their joint sensuous and practical activity, make apparent in the singular the target conceptuality of the general. Here the objectification occurs when the singular actualizes a form of looking at the saving sequence that is algebraic in nature. For objectification is that moment of the activity where the general, mediated by the particular, shows up through the singular in the students' consciousness. In our example, after that the students finished modeling the bingo chips as shown in Figure 10, they tackled the question of the savings in week 10; they suggested doubling the savings of week 5 and removing one of the blue bingo chips (see Figure 13).

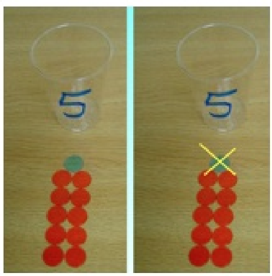

Figure 13. The students strategy to calculate the savings in week 10.

So instead of the expected expression $10 \times 2 + 1$, the students suggested $11 + 10$. When the teacher, Mrs. Giroux, went to see the group, Manuel was receiving some help from Krysta and was busy writing on his activity sheet. Mrs. Giroux grabbed the fifth glass (pic 1 in Figure 14; note that the first glass is not shown), and talking to Albert, said:

1. Mrs Giroux: What did you do here? 5… (*Pointing now to the red bingo chips; see pic 2 in Figure 14*) times…?
2. Albert: … 2
3. Mrs. Giroux: (*Pointing to the blue bingo chip; Pic. 3*) Plus?
4. Albert: 1

Then Mrs. Giroux took the glass of week 5, moved it to her left to a place where one would expect to find week 10 if the sequence would materially be extended and asked:

Mrs. Giroux: What would you do for week 10, if week 10 was here? (*See Pic 4*).

Albert did not utter the expected expression. Both the teacher and the student were very tense at this point (see Pic 5). She invited Albert to start anew:

5. Mrs. Giroux: (*Grabbing the glass of week 5 again*) What did you do here? (*Pic. 6*)
6. Albert: (*Taking a deep breath and hitting the desk with the back of the pen, while Mrs. Giroux holds the glass of Week 5; see Pic. 7*) Ok.
7. Mrs. Giroux: (*Still holding the glass, she utters softly*) 5…
8. Albert: (*In synchronization with Mrs. Giroux' gesture that points to the side of the red bingo chips; Pic. 8*) Times 2…
9. Krysta: (*Who has been following the discussion for a while*) Times 2 equal…
10. Mrs. Giroux: (*Pointing now to the blue bingo chip; Pic. 9*) plus 1.
11. Albert: (*Almost at the same time*) Plus 1.

12. Mrs. Giroux: (*Pointing now to an empty space where Week 10 would hypothetically be; Pic. 10*) 10?
13. Albert: (*Mrs. Giroux points silently to the place where the red bingo chips would be; Pic 11*) Times 2.
14. Krysta: (*At the same time*) Times 2.
15. Mrs Giroux: (*Silently pointing now to where the blue bingo chip would be; Pic. 12.*)
16. Krysta: Plus 1.
17. Albert: (*Looking at the teacher*) Minus 1? Times 2, minus 1? Plus 1?

In turn 5 the teacher makes an invitation to Albert to recommence the search of the formula or sequence of calculation to calculate the savings. She asks: "What did you do here?" (turn 5, Pic. 6). Albert exhibits acceptance of the teacher's invitation with his entire body: He takes a deep breath and hits the desk with the back of his pen (Pic. 7). The way the teacher asks the question is encouraging: it conveys the idea that Albert knows but has not yet sufficiently attended to what is marked in the bingo chip configuration and what is intended to be remarked —that is, the mathematical structure from an algebraic viewpoint.

It is implicit that the teacher knows this algebraic structure. But knowing it is not enough. It is not enough because the teacher cannot inject such a structure into the student's consciousness. For the general to appear in the singular *both the student and the teacher have to work together*. The teacher and the student have to engage in a process of objectification. It will happen when the sought-after general incarnated into the singular leaves the realm of latent attention, ceases to be "in itself" knowledge, and crosses the threshold of explicit attention in Albert's consciousness to become "for itself" knowledge. But Albert and the teacher are not there yet. Despite the inconclusive result of interaction in turns 1-4, in turn 5 the teacher engaged again in joint action with a soft and inviting word: "Five," that she uttered while holding the fifth glass. Without talking, she moved the hand to point to the red bingo chips (Pic. 8). Albert's voice filled the space left behind by the teacher's silence. He said "Times 2." The teacher moved the pointing gesture to the blue bingo chip (Pic. 9) and said, almost at the same time as Albert, "plus 1." She then moved her hand to an empty space where the model of week 10 would be (Pic. 10) and softly said "10?" Without speaking she pointed to the imagined position of the red bingo chips (Pic. 11), while Albert looked at the hand and said "Times 2" (Turn 13). She moved again in silence and made the pointing gesture toward the imagined position of the blue chip (Pic. 12) and Albert hesitantly said "Minus 1? Times 2 minus 1? Plus 1?"

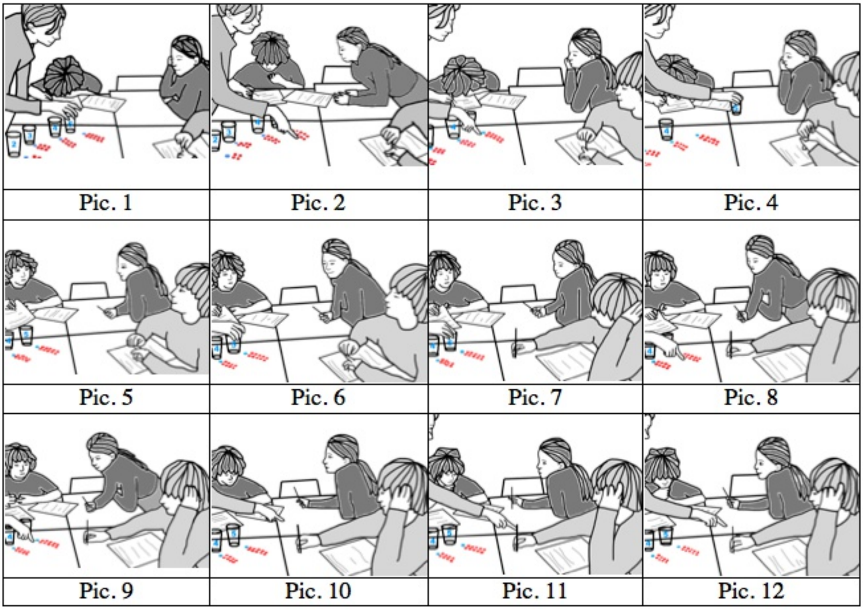

Figure 14. Pics. 1-12. Mrs. Giroux and Albert working together.

At this point of the activity, the objectification has almost succeeded. Albert still has to better secure the various elements of the formula. That does not take long. A few minutes later, the teacher organized a general discussion. She invited several students to present their ideas. At a certain point she asked Albert to explain the calculations to determine the amount of money at the end of week 2.

18. Albert: It's 2, the second week, it's times 2 because you add ... 2 euh, dollars…
19. Mrs. Giroux : Okay . . .

20. Albert: And one, plus one, like one.
21. Mrs. Giroux : Ok. . .. Do it for [Week] 4. Same idea. 4.
22. Albert: 4 times 2...
23. Mrs. Giroux : 4 times 2 because it's the double…
24. Albert: Plus one. 4 times 2, plus 1 equals… 9.

The lesson ended at this point. On the following day, the students in this class worked on an isomorphic problem. This time the piggy bank had \$6 when Marianne received it and she saved \$3 per week, so that at the end of the first week she had \$9, at the end of the second week she had \$12, and so on. Talking to his group-mates about how to calculate the savings at the end of week 10, Albert said: "She adds 3 dollars each week. So I will do it like this, 'kay, 3 times 10 is 30 [plus 6] it's 36. Okay, it's 36."

Through a lengthy process of objectification, Albert progressively grasped the general mathematical structure behind the saving process. Albert was able to extend the culturally encoded form of knowledge that was the target of these lessons to new situations during a test that the class had to write more than one week after we finished the algebra lessons. In the test there was a question about finding an expression for Term 25 of the sequence shown in Figure 15.

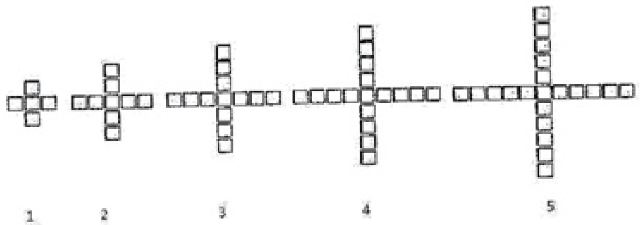

Figure 15. A sequence featured in a test that the students wrote more than one week later.

Albert's answer was: $25 \times 4 + 1$ He went even further and suggested the following formula for whatever term of the sequence:

$$\_\_\_ \times 4 + 1 = \_\_\_\_\_$$

The first line, he explains, "is to put the number of the term." The number 4 means “the number of squares that you add each time. The number 1 is the first [square] you started with.” The second line “is the answer.”

Learning has occurred. The “in itself” cultural knowledge has been transformed into knowledge “for itself” and transformed into knowledge *for Albert*. This transformation results in a new form of perceiving, talking, and conceptually dealing with sequences —a new form of consciousness whose emotional content appears clearly in the hesitation that Albert displays in turn 17 and which is subsequently replaced by an assured way of calculating things. Objectification, or the transformation of the “in itself” knowledge into an object of consciousness, is not the result of solitary deeds, nor is it the result of contemplation. The transformation is the result of sensuous joint material activity —an activity where Albert and Mrs. Giroux put themselves at risk. Learning, indeed, is always a risky endeavour. It is risky in that it requires that we leave the comfort of our own solipsistic niche to go towards something that is not us, an unknown region where we can nonetheless make ourselves at home.

Of course, there are still many things to learn, for learning is not a state; learning is a process. This is why we talk about objectification as a moment in the constitution of consciousness, not as a “stage.”

Synthesis

In this article I presented three key concepts of the theory of objectification, namely knowledge, knowing, and learning. I suggested that knowledge is a culturally and historically encoded form of reflecting. These encoded forms present us with mere potentiality. Through actualization, they acquire a conceptual content. This actualized or instantiated conceptual content is what knowing consists of. But the conceptual content is not something that is unmediated. To acquire actuality, to be real, the conceptual content can only appear through *activity*. In other words, how we come to know is shaped by, and consubstantial with, the activity through which knowledge is instantiated. This consubstantiality of knowing and activity is reflected in the manner in which the historical and cultural material and ideal forms and modes of social intercourse that underpin the activity impress their mark in the instantiated conceptual content. This is one of the central ideas exposed here and one that makes activity theory in general (Leont'ev, 1978) and the theory of objectification in particular distinctive.

But because of the inherent mediated nature of knowing, knowing is not a straightforward process. It is here that learning enters the scene. Knowing requires learning. In the theory of objectification, learning is thematized as the conscious and deliberate encounter with historically and culturally encoded forms of thinking and doing. More precisely, learning is accounted for in terms of processes of objectification. The latter we defined as activity-bound processes through which the "in itself" knowledge becomes an object of consciousness, and hence knowledge for us, or "for itself" knowledge (knowledge for *consciousness*).

The example discussed in the previous section illustrates the previous ideas. We presented a Grade 4 class with a series of tasks (here piggy bank problems) of increasing difficulty whose goal was to instantiate or actualize an encoded form of thinking that we recognize as algebraic. This form of thinking is mere potentiality. It cannot simply appear out of the blue. It can only be instantiated, that is, filled with theoretical content, through an activity that particularizes it. Our didactic design favoured a theoretical content where a generalized formula was targeted through the mediation of goblets, bingo chips, paper, pencil, and elaborated forms of social interaction —our relationships Φ and Θ (see Figure 11). The excerpts presented here show that the encoded form of thinking remained in the beginning unnoticed by the students, who resorted rather to arithmetic forms of generalization. To notice the algebraic forms of thinking the classroom activity had to evolve in such a way that the algebraic forms of thinking become objects of consciousness, that is to say *recognized*. First, it entails the recognizance of a difference between "I" and "It." Then, it entails the overcoming of the difference in the coming together of the "I" and the "It." "Recognition," Heidegger says, is "to *re-cognize* = to *differ*entiate, that is, something as that and that, and thus to grasp *it* as '*it*self'" (Heidegger, 2004, p. 16; italics in the original).

This "It" that is-not-us-yet appears faintheartedly in turns 5 and 6, where Albert starts noticing that there might be a different manner in which to see the bingo chips. The subsequent intense joint endeavour of Mrs. Giroux and Albert, where they truly work together, leads to the an instantiation of the encoded form of thinking. Through a joint process of objectification, where the teacher's gestures and Albert's words come together and form a single unity, the encoded form of algebraic thinking appears now in consciousness endowed with a specific theoretical content. This singular theoretical content does not apply to this or that piggy bank question or problem only. Albert is capable of applying it to other problems as well, as the one referred to in the test that does not have anything to do with savings. The singular that incarnates the general is indeed a *totality*. And it is when it is a totality that learning occurs.

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Notes

[1] For instance, they play with the stones; see http://www.youtube.com/watch?v=bpRu1Zg-128.

[2] Hypsikles lived in Alexandria. Historians are uncertain about much of his life, which they think occurred between the 2nd century BC and the 2nd century AD.

[3] I would like to take advantage of this discussion to point out the theoretical differences between activity theories that draw from Hegel and the ensuing dialectical tradition (the theory of objectification is an example) and some contemporary theories of action. As Figure 5 shows, the particular is a joint activity framed by material and spiritual historical and cultural forms of production and modes of social interaction. It is not just a sequence of individuals' actions occurring in interaction.

[4] I offer a more operational definition of objectification later on, once the key required concepts are introduced.

References

Arendt, H. (1958). The modern concept of history. *The Review of Politics, 20*(4), 570-590. doi:10.1017/S0034670500034227.

Artigue, M. (1995). The role of epistemology in the analysis of teaching/learning relationships in mathematics education. In Y. M. Pothier (Ed.), *Proceedings of the 1995 annual meeting of the Canadian mathematics education study group* (pp. 7-21). University of Western Ontario.

Artigue, M. (2009). Didactical design in mathematics education. In C. Winsløw (Ed.), *Nordic research in mathematics education. Proceedings of NORMA08*. Rotterdam: Sense Publishers.

Blunden, G. (2009). Foreword. In *Hegel's logic* (pp. 7-101). Pacifica, CA: MIA.

Cai, J., & Knuth, E. (2011). *Early algebraization*. New York: Springer.

Campbell, S. (2002). Constructivism and the limits of reason: Revisiting the Kantian problematic. *Studies in Philosophy and Education, 21*, 421-445.

Gardener, S. (n.d.). Hegel: Glossary. Retrieved from http://philosophyfaculty.ucsd.edu/faculty/ewatkins/Hegel-Glossary.pdf.

Hegel, G. W. F. (1977). *Phenomenology of spirit*. Oxford: Oxford University Press (First edition, 1807).

Hegel, G. (2001). *The philosophy of history*. Kitchener, ON: Batoche Books. (Original work published 1837)

Hegel, G. (2009). *Logic*. (W. Wallace, Trans.). Pacifica, CA: MIA. (Original work published 1830)

Heidegger, M. (2004). *On the essence of language*. New York: SUNY. (Original work published 1999)

Hirata, S., Morimura, N., & Houki, C. (2009). How to crack nuts: Acquisition process in captive chimpanzees (pan troglodytes) observing a model. *Animal Cognition, 12*, 87-101. doi: 10.1007/s10071-009-0275-3

Ilyenkov, E. V. (1977). *Dialectical logic*. Moscow: Progress Publishers.

Janet, P. (1929). *L'évolution psychologique de la personnalité* [The psychological evolution of personality]. Paris: Masson.

Kant, I. (2003). *Critique of pure reason*. (N. K. Smith, Trans.) New York: St. Marin's Press. (Original work published 1787)

Lave, J. (1988). *Cognition in practice*. Cambridge: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

Lawlor, R., & Lawlor, D. (1979). *Theon of Smirna: Mathematics useful for understanding Plato*. San Diego: Wizards Bookshelf.

Lefèvre, W. (1981). Rechenstein und sprache [computing stones and language]. In P. Damerow & W. Lefèvre (Eds.), *Rechenstein,* *experiment, sprache. Historische fallstudien zur entstehung der exakten wissenschaften* [Computing stones, experiment, language. Historical case studies on the emergence of the exact sciences] (pp. 115-169). Stuttgart: Klett-Cotta.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality.* Englewood Cliffs, NJ: Prentice-Hall.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education, 27*(2), 133-150.

Manitius, K. (1888). *Des hypsikles schrift anaphorikos nach überlieferung und inhalt kritisch behandelt* [Hypsikles' text "anaphorikos" according to transmission and content dealt with in a critical way]. Dresden: Lehmannsche Buchdruckerei.

Marx, K. (1973). *Grundrisse: Introduction to the critique of political economy.* Baltimore: Penguin Books.

Marx, K. (1998). The German ideology. New York: Prometheus Books.

Matsuzawa, T., Biro, D., Humle, T., Inoue-Nakamura, N., Tonooka, R., & Yamakoshi, G. (2001). Emergence of culture in wild chimpanzees: Education by master-apprenticeship. In T. Matsuzawa (Ed.), *Primate origins of human cognition and behavior* (pp. 557-574). Tokyo: Springer.

Maybee, J. (2009). *Picturing Hegel.* Lanham, MD: Lexington Books.

Mikhailov, F. T. (1980). *The riddle of the self.* Moscow: Progress Publishers.

Nicomachus of Gerasa. (1938). *Introduction to arithmetic.* (M. L. D'Ooge, Trans.). Ann Arbor: University of Michigan Press.

Otte, M. (1998). *Limits of constructivism : Kant, Piaget and Peirce*. Science & Education, 7, 425-450.

Radford. (2006). Variables, unknowns, and parameters of mathematical generality. In *Mini-Workshop on studying original sources in mathematics education*. Oberwolfach, April 30th-May 6th, 2006. Report No. 22/2006, 16-17.

Radford, L. (2007). Towards a cultural theory of learning. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the fifth congress of the european society for research in mathematics education (CERME – 5)* (pp. 1782-1797). Larnaca, Cyprus.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM - the International Journal on Mathematics Education, 40*(1), 83-96. Doi:10.1007/s11858-007-0061-0

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12*(1), 1-19. Doi: 10.1080/14794800903569741

Radford, L. (2011). Embodiment, perception and symbols in the development of early algebraic thinking. In *Proceedings of the 35th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 17-24). Ankara: PME.

Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment: An activity perspective on classroom interaction. *Educational Studies in Mathematics, 77*(2-3), 227-245. Doi: 10.1007/s10649-010-9282-1

Rogoff, B. (1990). *Apprenticeship in thinking*. Oxford: Oxford University Press.

Roth, W.-M. (2011). *Passibility: At the limits of the constructivist metaphor.* New York: Springer.

Spinoza, B. (1989). *Ethics including the improvement of the understanding.* (R. Elwes, Trans.). Buffalo: Prometheus. (Original work published 1667)

Tarán, L. (1969). *Asclepius of Tralles: Commentary to Nichomacus' introduction to arithmetic* (Vol. 59, part 4). Transactions of the American Philosophical Society.

Taylor, C. (1989). *Sources of the self*. Cambridge, MA: Harvard University Press.

Valero, M. (2004). Postmodernism as an attitude of critique to dominant mathematics education research. In P. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 35-54). Greenwich, CT: Information Age Publishing.

Ver Eecke, P. (1959). *Diopante d'Alexandrie. Les six livres arithmétiques et le livre des nombres polygones* [Diophantus of Alexandria: The six books on arithmetic and the book on polygonal numbers]. Liège: Desclée de Brower. (Original work published 1926)

Veresov, N. (1999). *Undiscovered Vygotsky. Etudes on the pre-history of cultural-historical psychology*. Frankfurt: Peter Lang.
Vygotsky, L. (1929). The problem of the cultural development of the child. *Journal of Genetic Psychology, 36,* 415-434.
Vygotsky, L. S. (1978). *Mind in society*. Cambridge, Ma: Harvard University Press.
Vygotsky, L. S. (1987-1999). *Collected works*. New York: Plenum Press.
Vygotsky, L. S., & Luria, A. (1994). Tool and symbol in child development. In R. van der Veer & J. Valsiner (Eds.), *The Vygotsky reade*r (pp. 99-174). Oxford: Blackwell Publishers.
Zevenbergen, R. (1996). Constructivism as a liberal bourgeois discourse. *Educational Studies in Mathematics, 31*, 95-113.

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