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Agency as Inference: Toward a Critical Theory of Knowledge Objectification

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Agency as Inference: Toward a Critical Theory of Knowledge Objectification

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Abstract

This article evaluates the plausibility of synthesizing theory of knowledge objectification (Radford, 2003) with equity research on mathematics education. I suggest the cognitive phenomenon of mathematical inference as a promising locus for investigating the types of agency that equity-driven scholars often care for. In particular, I conceptualize students' appropriation of semiotic-cultural artifacts (e.g., algebraic symbols and forms) to objectify their pre-symbolic inferences as conditional on their agency to carefully and incrementally construct personal meaning for these artifacts. To empirically ground this emerging approach, this study focuses on algebraic generalization (as a type of mathematical inference) and applies Radford's framework to video data of two iterations of an instructional intervention conducted in a high school program for academically at-risk youth. I analyze and compare students' acts of appropriation/objectification during whole-class conversations centered on pattern-finding tasks, in relation to the instructional mode adopted for each of the iterations—"direct instruction" vs. "inquiry-based." The analysis shows that the implementation involving inquiry-based instruction enabled more equitable access to opportunities for agency-as-mathematical inference, whereas the implementation involving direct-instruction was ostensibly more productive. Implications for future equity research involving cognition-and-instruction analyses are discussed.

Keywords: algebraic reasoning, agency, equity, generalization, inference.

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La Agencia como Inferencia: Hacia una Teoría Crítica sobre la Objetivización del Conocimiento

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Resumen

Este artículo evalúa la verosimilitud de sintetizar la teoría del conocimiento objetivado (Radford, 2003) con investigación sobre equidad en didáctica de las matemáticas. Propongo el fenómeno cognitivo de inferencia matemática como un concepto prometedor para la investigación de los tipos de agencia de la equidad impulsadas por los estudiantes. Conceptualizo la apropiación de los estudiantes de los artefactos semióticos-culturales (p.e. símbolos y formas algebraicas) como medios para objetivar sus inferencias pre-simbólicas como condiciones de su agencia para construir cuidadosamente el significado de esos artefactos. A fin de basar empíricamente este enfoque emergente, este estudio se centra en la generalización algebraica (como un típo de inferencia matemática) y aplica el marco desarrollado por Radford a los datos de vídeo de dos iteraciones de una intervención educativa llevada a cabo en una escuela secundaria con jóvenes en riesgo. Se analizan y comparan las conversaciones de los estudiantes sobre la apropiación / obetivación, centradas en el patrón de enseñanza adoptado por cada una de las iteraciones ("instrucción directa" versus "basada en la investigación.") El análisis muestra que la ejecución que implica instrucción basada en la investigación permitió un acceso más equitativo a las oportunidades de inferencia agencia-como-matemática, mientras que la aplicación directa de la participación de la instrucción era aparentemente más productiva. Implicaciones para la investigación de acciones futuras que incluyan análisis de la cognición y la instrucción se discuten.

Palabras Clave: razonamiento algebraico, agencia, generalización, inferencia

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A thematic objective of mathematics education researchers focusing on algebra content is to develop theoretical frameworks that account for students' difficulty with problem solving. Yet while these frameworks are being developed, the national achievement gap persists, even amidcalls for equity in mathematics education (DiME, 2007; R. Gutiérrez, 2002) and, in particular, to improve the accessibility of algebra content for students from underrepresented minority groups and economically disadvantaged backgrounds (Moses & Cobb, 2001; Oakes, Joseph, & Muir, 2004). Notwithstanding, I propose that recent theoretical work on algebraic reasoning has the potential to illuminate new directions for broadening diverse passage through this "gatekeeper" content. The objective of this paper is to provide theoretical rationale and build upon some preliminary empirical data so as to illustrate what we may need to attend to as we pave these new directions.

Central to the theoretical argument of this article is Luis Radford's (2003, 2006, 2008) theory of *knowledge objectification* and, in particular, his *semiotic–cultural framework* for the study of students' algebraic reasoning. Taken together, this powerful approach views learning as an evolving process co-constrained by both cognitive and socio-cultural factors. Specifically, mathematics learning is conceptualized as constructing personal meaning for semiotic-cultural artifacts (e.g., algebraic symbols such as the variable *x*). In this article I attempt to synthesize Radford's approach with equity research on mathematics education. To support this synthesis, I suggest the phenomenon of mathematical inference, which is central to cognitivist analyses of learning, as a promising locus for investigating the types of agency that equity-driven scholars have deemed as vital for student identity and, in turn, participation and learning (Boaler & Greeno, 2000; Gresalfi, Martin, Hand, & Greeno, 2009; Wagner, 2004).

Equity studies on mathematics education are framed primarily in terms of access to *opportunities to learn* (see DiME, 2007). At the classroom or interactional level, for example, opportunities to learn are understood and analyzed in terms of access to mathematics content and discourse practices, as well as access to constructive mathematical identities that are congruous with students' sociocultural identities (Boaler, 2008;

Cobb & Hodge, 2002; Esmonde, 2009). Inherent to mathematical content/discourse and identity is the notion of *agency*, that is, "who is said to be making things happen" (Pickering, 1995; Wagner, 2004). In this article, I view the study of mathematical inference as revealing both of students' reasoning processes and, simultaneously, of their agency. In particular, I conceptualize students' appropriation of canonical mathematical artifacts as semiotic means of objectifying their budding (pre-symbolic) inferences as *conditional* on their agency to carefully and incrementally construct personal meaning for these cultural artifacts.

To support this claim, I first propose a qualification to theory of knowledge objectification that, in contrast to its extant formulation, does not assume classroom homogeneity in opportunities to appropriate mathematical semiotic artifacts. Next, to empirically ground this emerging approach, this study focuses on algebraic generalization —as a specific genre of mathematical inference— and applies Radford's framework to video data of a teacher's classroom orchestration around pattern-finding tasks.

The empirical context for this study is a participatory instructional design intervention conducted in a high school mathematics program for academically at-risk youth. The intervention was intended to implement a classroom participation structure that would facilitate a particular desirable interaction among students and ultimately give rise to authentic engagement and deep learning. The intervention was also an opportunity for the teacher to reflect critically on issues germane to equitable mathematics education, such that he continue to engage with these issues in his own practice. For this article I focus on two iterations of a whole-class problem solving activity, Group A and Group B, and conduct my data analysis through the semiotic-cultural perspective. The main questions that guide my analysis are the following:

- Are whole-class conversations that are ostensibly productive also necessarily equitable?
- How can we distinguish between "equity" and "productivity" in classroom mathematical discourse?

To address these questions, I analyzed and compared students' acts of appropriation/objectification during whole-class conversations centered on pattern-finding tasks, in relation to the instructional mode adopted for each of the iterations. Group A used "inquiry-based" instructional techniques, whereas Group B was implemented using mostly "direct instruction." Consequently, the analysis shows that Group A's implementation enabled more equitable access to opportunities for agency-as-mathematical inference, whereas Group B's implementation was ostensibly more productive.

Prior Research & Theoretical Background—Toward A Critical Semiotic–Cultural Perspective

It has been well documented that historically marginalized groups, such as African American, Latino/a, and economically disadvantaged students, are under-represented in higher education and, in particular, in the fields of science, technology, engineering, and mathematics (STEM) (NSB, 2008). The research further implicates high-school mathematics as the *de facto* "gatekeeper" into academic and technological communities of practice (RAND, 2003; Ladson-Billings, 1998). Namely, access to and completion of rigorous high-school mathematics courses has been shown to be among the strongest predictors of student success in higher education (Oakes, et al., 2004). Therefore, improving access to high-school mathematics education is an important goal toward bridging the academic achievement gap. A central focus of this effort should be on algebra content, because high-stakes exams define "success in school" directly in terms of success in algebra (Moses & Cobb, 2001).

In this section I review the literature on student algebraic reasoning. So doing, I introduce the cultural–semiotic perspective as a means of illuminating affective and not just cognitive factors that are critical for mathematics students from marginalized communities specifically.

Empirical Studies on Student Algebraic Reasoning

Modeling situations and manipulating symbols in Algebra problem solving.

Algebra, viewed as a human practice, can be characterized broadly as involving complex sense-making processes (Kaput, 2007; Schoenfeld, 2007). These processes are often further described as demanding two general capacities that both characterize algebraic problem-solving activity and constitute common goals and objectives of curricular design that inform classroom instruction:

1. *Modeling* – actions for making initial sense of a given problem situation, such as creating and expressing generalizations to model the source situation using increasingly formal representational forms (e.g., symbolic expressions, graphs, tables, verbal descriptions, or some combinations thereof).
2. *Solving* – reflecting and operating on those mathematical representations using conventional manipulation procedures to support reasoning about the source situation being modeled.

Distinguishing between these two core capacities has illuminated the challenges faced by my target population. Specifically, my previous studies suggest that discourse plays a more critical role in the development of (1) than (2) for struggling students from historically marginalized groups (Gutierrez, 2010). Before I unpack the details of this assertion, first it is necessary to review the literature on algebra learning challenges that are presumably faced by all learners. So doing will enable me to later leverage a critique of and propose a qualification to the predominant theoretical models pertaining to student algebraic reasoning.

Algebra learning challenges: focus on the semiotic–cultural perspective.

Learning algebra has historically been fraught with conceptual challenges (for a recent review, see Kieran, 2007). In particular, the literature has documented cognitive “gaps” that students must traverse as they transition from arithmetic to algebraic forms of reasoning. For example, Luis Radford (2003) applies semiotic analysis to implicate discontinuation in students’ spatial–temporal embodied mathematical experience, as they appropriate symbolic notation to express algebraic generalization of non-symbolic situations. This “rupture” designates a conceptually critical shift in the semiotic role of an inscription, such as *x*, from indexing a specific actual aspect of the problem space, such as the number of "toothpicks" in a geometric construction (see Figure 1, below) to meaning any element within the plurality or even infinity of imagined situated extensions of the problem. The *x*, in this case, has to be liberated, so to speak, from the grounding situation from which it emerged, so that the problem solver can manipulate the algebraic expressions unconstrained by a constant need to evoke the situated meaning of *x*. Consider the "toothpicks" problem (see Figure 1, below), a situation involving an initially unknown general principle governing the relation between a numerical and a geometric sequence.

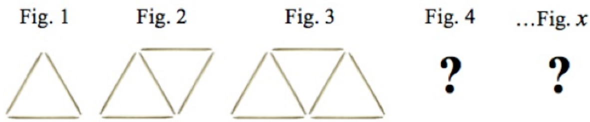


Figure 1. "Toothpicks"— a paradigmatic algebra generalization problem. The task objective is to express Ux the total number of toothpicks in the x^{th} figural extension. For example, "Fig. 1" consists of three toothpicks, "Fig. 2" consists of five, "Fig. 3" consists of seven, etc., so that "Fig. x" consists of $2x$+1 toothpicks.

Whereas Radford's rupture lives in the realm of (1) Modeling, researchers have also identified other gaps that live in the realm of (2) Solving. For instance, Filloy and Rojano (1989) identify a stark demarcation, which they call a didactic cut, between arithmetic and algebraic forms of reasoning in the context of solving first-degree equations with a single unknown. Equations such as $Ax+B=C$ can be solved using arithmetic means such as counting or inverse operations, whereas equations with unknowns on both sides of the equal sign, such as $Ax+B=Cx+D$, require "operations drawn from outside the domain of arithmetic —that is, operations on the unknown" (Filloy and Rojano, p. 19). These scholars conclude that focused instructional intervention is required at such didactic cut points. Note that whereas Filloy and Rojano characterize the arithmetic–algebraic gap in terms of specific mathematical forms ($Ax+B=C$ vs $Ax+B=Cx+D$) and strategies to deal appropriately with such forms, other researchers would characterize the same gap in more fundamental terms. In particular, Herscovics and Linchevski (1994) maintain that students' difficulty with equations involving double occurrence (e.g., *x+5=2x-1*) is not so much a didactical issue but rather suggests a deeper, underlying "*cognitive gap*" that can be characterized as a fundamental inability to operate *spontaneously* with or on unknown quantities.

Finally, research findings indicate that the capacities to model and solve algebraic problems do not necessarily develop at the same rate, and the research implicates traditional instruction as determining this developmental differential. Namely, curricular material and teacher practice tend to value symbol manipulation at the expense of creating opportunities for students to practice initially generating these symbols from problem situations (cf. Arcavi, 1994; see also *The Alegebra Problem* by Kaput, 2007). Thus the crux of algebra instruction is not only to support the development of both types of capacity but also to teach students to shift flexibly back and forth between them.

My earlier studies (Gutiérrez, 2010) support the implication of gaps inherent to algebraic problem solving as foci for productive research. I propose that discourse plays a greater role than has been theorized in explicating these gaps and how they may be forded. In particular, I submit, a critical examination of the role of discourse in algebraic learning reveals that these gaps present affective and not just cognitive challenges. Furthermore, for struggling students from historically marginalized groups, issues of discourse and identity may play a more critical and more nuanced role in the development of the core capacities than has been previously surmised and particularly more so in (1) modeling as compared to (2) solving. I elaborate on this last point in the next section, below.

A Critical Conceptual Analysis of the Semiotic-Cultural Approach

Building on Lev Vygotsky's cultural-historical psychology and Edmund Husserl's phenomenological philosophy, Luis Radford's (2003) semiotic–cultural approach views learning as an evolving process reflexively co-constrained by cognitive and socio-cultural factors. Specifically, mathematics learning is conceptualized as constructing personal meaning for canonical semiotic artifacts (e.g., algebraic symbols such as the variable *x*). Through consolidation and iteration of these constructions, students appropriate the mathematical semiotic artifacts and, reciprocally, build personal meaning for mathematical content as well as fluency with the disciplinary procedures.

Radford's approach takes into account a vast arsenal of personal and interpersonal resources that students bring to bear in solving mathematical situations, including linguistic devices and mathematical tools. A key construct in Radford's framework is *knowledge objectification*, which is defined as the process of making the objects of knowledge apparent (Radford, 2003). For example, a mathematics learner, in an attempt to convey a certain aspect of a concrete object, such as its shape or size, will make recourse to a variety of semiotic artifacts such as mathematical symbols and inscriptions, words, gestures, calculators, and so forth. In patterning activity, however, some of the objects of knowledge are general and therefore "cannot be fully exhibited in the concrete world" (Radford, 2008, p. 87). More broadly, knowledge objects in mathematics are not too cognitively accessible, because they do not exist in the world for empirical investigation (Duval, 2006), that is, these objects are never apparent to perception. Therefore, in order to instantiate (objectify) these ephemeral objects, students must resort instead to personally and culturally available forms such as linguistic, diagrammatic, symbolic, and substantive artifacts as well as the body, which Radford (2003, 2008) collectively terms *semiotic means of objectification* (see also Abrahamson, 2009).

The power of the semiotic–cultural approach is that critical steps within individual learning trajectories can be explained by noting subtle shifts in the subjective function and status of the semiotic artifacts (Duval, 2006; Sfard, 2007). In particular, mathematics learning in the context of algebraic generalizations can be monitored as subjective transitions along a desired chain of signification, from *factual*, to *contextual*, to *symbolic* modes of reasoning (Gutiérrez, 2010; Radford, 2010) (see Figure 2, below).

From this perspective, conceptual understanding is viewed as the capacity to flexibly shift across the three semiotic modes, which consequently requires that students assume agency in making these shifts so as to carefully and incrementally construct personal meaning for conventional semiotic artifacts (e.g., the variable x). Students' personal acts of generalization —which are a specific type of

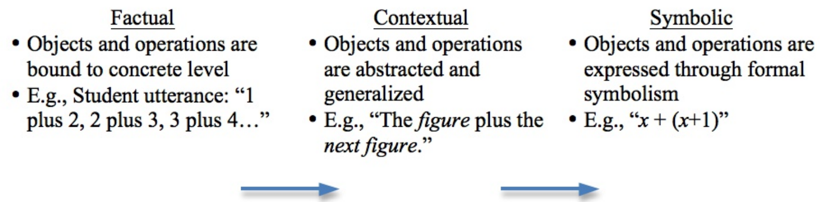


Figure 2. "F-C-S" framework (applied to the "Toothpicks" problem; see Figure 1, above): mathematics learning is conceptualized as shifts in semiotic modes.

mathematical inference— from one semiotic mode to the next mark both their conceptual understanding and their mathematical agency. That is, agency and conceptual understanding can be co-investigated by interrogating the process and content of students' mathematical inferences (generalizations) within and across the three semiotic modes. I conjecture that the development of agency-as-mathematical inference bears implications for students' nascent mathematical and social identities.

To operate in the symbolic mode is predicated on a tacit (if not explicit) alignment with the mainstream classroom discourse (Sfard, 2007). Many students may not experience tension due to shifts in discursive alignment, perhaps because their social identities remain intact and unthreatened by these public acts. However, for students whose mathematical understandings are not couched in the mainstream classroom discourse, these discursive shifts could threaten their social identities and loyalty to their communities, because they perceive the more "mathy" (symbolic) language as indexing the hegemonic cultural values and ideologies[1].

Furthermore, returning to Radford's construct of a rupture, note that he describes it as largely a sensuous–cognitive phenomenon. What I identify here is perhaps a different kind of rupture that is under-researched, a rupture that is still sensuous yet affective in nature and, through discourse, becomes imbricated with sociopolitical narratives of power, individual agency, and identity.

The theoretical work detailed above has enabled me to articulate a content-based definition of equity. Building on Esmonde's (2009) notion of "fair distribution of opportunities to learn," I define equity as the fair distribution of opportunities for agency-as-mathematical inference. I posit agency-as-mathematical inference is central in (1) developing both conceptual understanding and the institutionally sanctioned mathematical register, and (2) developing a constructive mathematical identity (cf. "dominant" versus "critical" mathematics, Gutiérrez, 2002; Veeragoudar-Harrell, 2009). This definition of equity implies that algebraic generalization activity is not merely to create opportunities for students to unreflectingly appropriate mathematical symbols and forms —to operate merely in the symbolic mode without having generalized to that mode (Gutiérrez, 2010) for the sake of classroom "productivity." Rather, generalization activity is to enable student agency to produce semiotically grounded inferences so as to progress along the desired chain of signification from the factual through to the symbolic mode.

Having presented a critical semiotic–cultural framework and a content-based definition of equity, I restate my research questions in light of both of these. Namely,

- What are the conditions that support equitable access to opportunities to produce semiotically grounded generalizations and progress along the F-C-S trajectory?
- How do instructional modes affect these classroom opportunities?

Next I describe the methods used to address these questions.

Methods

For this preliminary empirical study, I conducted a two-phase collaboration with a high-school mathematics teacher from the San Francisco Bay Area. In the first phase, I conducted an ethnography of the teacher's routine instruction, including videography, field-notes, and interviews. Based on this ethnographic data and the teacher's input, we co-designed a non-routine instructional intervention focusing on algebraic generalization; specifically, we implemented and videographed two iterations (Group A & Group B) of an instructional sequence using the "toothpicks" problem. The goals of this study were: (1) to empirically examine the challenges and opportunities that struggling students from diverse cultural and academic backgrounds, specifically, manifest with respect to reform and traditional algebra curricula; and then (2) to articulate appropriate responses to these challenges by identifying leverage points for effective pedagogy.

For this article I analyze and compare Group A with Group B, and conduct my data analysis from a critical semiotic–cultural perspective. To address my research questions, I examined students speech acts, gestures, and artifact production during whole-class collaborative engagement with algebraic generalization problems. I analyzed students' collective reasoning processes during whole-class conversations, in terms of whether and how their mathematical inferences were semiotically grounded across the three modes. Furthermore, I also analyzed students' reasoning processes in relation to the specific instructional mode adopted for each implementation. This cognition-and-instruction analysis reveals that specific design decisions backing the facilitation of each iteration resulted in differential access to opportunities for student agency-cum-mathematical inference.

Data Sources

The entire data corpus includes students' original work, a total of 10 hours of video footage, a total of two hours of audio recordings of conversations between the researcher and teacher, and a project wiki (online archive) that I used to store resources, document field-notes and meeting minutes, and upload ongoing reflections. However, for this article I focus on: a single 23-minute span of video footage from Group A, in which the "toothpicks" problem was implemented; a total of 14.5 minutes of video footage from Group B, in which an *x-y-table* exercise was implemented (Day 1), followed by the "toothpicks" problem (Day 2).

Analytic Techniques

I produced and analyzed transcriptions of Group A and Group B teaching episodes, which capture all verbal, gestural, inscriptional, and other semiotic actions that were clearly observable in the video. Similarly, I also produced and analyzed transcriptions of the interviews and design meetings conducted with the teacher, Amil (pseudonym). For this study, I focus only on student utterances involving mathematical propositions, for which two main analytic questions were asked pertaining to (1) its semiotic nature and (2) the instructional mode surrounding its manifestation:

1. Generalization Type (semiotically grounded versus ungrounded):
   1a. Is the proposition a mathematical inference based on a process of generalizing?
   1b. [If so,] Is the proposition an arithmetic (recursive) or algebraic (explicit) generalization?
   1c. [If so,] Is the proposition a factual, contextual, or symbolic generalization?
2. Instructional Mode:
   2a. Is the proposition —whether grounded or not— the result of a discernable feature of the instructional mode used to facilitate the activity?

Working with both the video/audio footage and the transcriptions, a first pass of the data involving Group A and Group B's implementations was done using analytic questions 1a-1c. I initially evaluated whether or not each utterance reflected a semiotically grounded mathematical generalization. This evaluation was based on a qualitative microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1993) of students' behaviors during their whole-class discussions. I determined whether the students engaged in authentic generalizing acts (i.e., producing inferences based on grasping and objectifying recurrent x-to-Ux relations and providing a direct expression for any term along the sequence) or resorted instead to other less-sophisticated strategies such as "guess and check" (see "generalizing" versus "naïve induction," Radford, 2008). Following this in-depth qualitative analysis, a second pass through the data was done using analytic question 2a, whereby students' mathematical propositions were analyzed vis-à-vis the active instructional mode. So doing, I traced students (un)grounded generalization acts to specific design decisions that were made prior to each implementation. Combined, questions 1 & 2 have enabled me to draw conclusions regarding the quality of learning underlying students discursive productions, as well as the equitable distribution of opportunities to learn in this local instructional context.

Results and Discussion

The goal of this article is to examine and compare two iterations of an instructional sequence involving the same algebraic generalization problem to understand how variations in instructional mode could affect equity in opportunities to gain deep conceptual understanding of algebra. In this section, I first present a brief overview of both implementations, including descriptions of the researcher and teacher's initial goals and objectives for instructional design, as well as descriptions of the students' behaviors during whole-class discussions. With this overview, it is my intention to help prepare the reader for a deeper analysis of the data that will be reported upon in the sections that follow.

Overview of Participatory Design Project

Researcher and teacher's initial goals and objectives for instructional design.

The goals of the project wherein the data for this study were collected were to occasionally observe Amil's classroom practice and provide him with ongoing feedback to foster critical reflection on learning issues that struggling mathematics students from historically marginalized communities face with respect to traditional and reform algebra curricula. In particular, Amil's routine teaching practice could be characterized as "teacher-centered" and we discussed the possibility of designing and implementing a student-centered instructional intervention involving algebra content and concepts.

Amil acknowledged that his practice was routinely teacher-centered and attributes the difficulty of implementing student-centered instruction to lack of resources and support at his school for facilitating such activities. Yet Amil recognized the benefits of student-centered inquiry-based instruction and was open to co-designing and implementing a non-routine mathematics activity.

We set out to design a student-centered inquiry-based activity for algebra that would be implemented in his two math classes. We engaged in a four-week long process of discussing and designing an activity involving a family of algebraic pattern-finding tasks. We reviewed relevant findings from recent mathematics educational research pertaining to algebraic generalization (e.g., Radford, 2003, 2008) and designed an instructional sequence based on the semiotic–cultural approach. We paid particular attention to scaffolding techniques; for example, we planned and rehearsed scripts for scaffolding students from the factual to the contextual mode, and from the contextual to the symbolic mode, and back again.

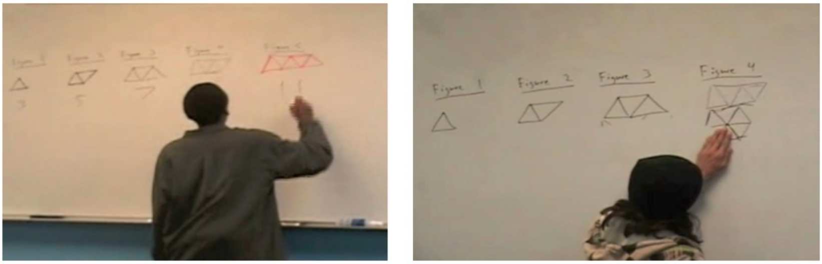
The final lesson plan for Group A, which emphasized "Modeling" (see section 2.1.1, above), involved three main components that would occur over the course of two days. The first half of Day 1 involved an introduction to patterning activity via a whole-class problem-solving session centered on the "toothpicks" problem (see Figure 1, above). The second half of Day 1 involved small-group work on worksheets of a set of similar pattern-finding tasks. Day 2 continued this small-group work and wrapped up with a final whole-class debrief.

For this study, I have elected to focus only on the teacher's classroom facilitation of the "toothpicks" problem (i.e., first half of Day 1). It's important to look at the very beginning of each of the implementations because these activities frame this genre of mathematical activity for the students for the first time. The students' encounter with the "toothpicks" problems sets up their expectations and elicits their resources for engaging with novel problem-situations, which offers a unique empirical context for classroom research.

Overview of Group A.

The Group A teaching episode begins when Amil drew the first three figural cues of the "toothpicks" sequence on the whiteboard. No specific instructions were provided; Amil simply used an open-ended prompt —"What comes next?"— to begin the problem-solving activity.

At the onset, some students immediately noticed that the figural sequence could be construed as a succession of accruing triangles. Whereas the students all agreed that the sequence could be extended by "adding another triangle," they disagreed, quite vehemently, over the type of growth the sequence was exhibiting. For example, some students argued that the sequence exhibits linear growth, whereby all figural cues are unique (e.g., Fig. 3 and *only* Fig. 3 consists of three triangles), extensions to the sequence are produced horizontally, and thus the sequence grows indefinitely (see Figure 3a, below). On the contrary, a student argued that the initial figural cues exhibit the growth of a single "hexagon" that terminates at Fig. 6 (see Figure 3b, below). Other students proposed that the sequence could be a repeating "hexagon" pattern, whereby Fig. 6 is a hexagon consisting of six triangles, Fig. 7 duplicates Fig. 1, Fig. 8 duplicates Fig. 2, and so forth.



a. b.

*Figure 3.*Student in the left image argues that the sequence of figural cues constitutes a linear progression and thus articulates a recursive relationship, whereas the student in the right image questions the apparent linearity of the sequence and instead considers a cyclic or repeating "hexagon" pattern.

For the first several minutes of the problem-solving activity, Group A students debated over the apparent linearity (or lack thereof) of the sequence. Realizing that the class had reached an impasse, Amil settled the argument by asserting that the sequence was linear; he then guided further exploration of the source situation with a series of questions (e.g., "How many toothpicks are in figure one? Figure two? Three?"). The students noticed that the number of toothpicks required to construct each consecutive figure always increases by a summand of two with respect to the previous figure; the students co-constructed an arithmetic generalization in the form of $U_{x+1}=U_x+2$. Furthermore, a key design feature for implementing the "toothpicks" problem was to substitute increasingly larger numbers (e.g., "Fig. 100") as a way to impress upon students that ultimately the arithmetic/recursive strategy is inefficient, thus motivating the need for more powerful tools and strategies such as algebraic generalizing and the use of explicit formulas.

For the remainder of the activity, most students were engaged in a process of authentic generalizing. For example, some students articulated an algebraic generalization in the form *(x+x)+1*, which directly calculates the number of toothpicks for a given figure. The students checked their formula with a few cases and, upon confirming its accuracy, successfully applied it to Fig. 100, concluding that it would consist of 201 toothpicks.

At the end of the activity, Amil reformulated the students' explicit formula, which was originally articulated in the *contextual* mode (i.e., St. utterance: "you add the figure number to itself, plus one"), as the *symbolic* expression $2x+1$. However, as my forthcoming analysis will demonstrate, despite the teacher having provided the symbolic version of the correct formula, by and large the instructional mode used during Group A's implementation enabled strong opportunities for student agency-as-mathematical inference.

In sum, Amil and I co-designed a facilitation strategy for Group A's implementation of the "toothpicks" problem, which utilized a student-centered inquiry-based instructional mode that, in turn, was generative of students' mathematical inference and spontaneous debate. Throughout, Group A students articulated and vigorously defended opposing arguments related to a complex mathematical topic (linear progression) and the entire class engaged in highly charged debates —mathematical discussions the likes of which I had not witnessed before in this particular classroom setting. These behaviors —proposing, questioning, and justifying mathematical inferences— are characteristic of expert mathematicians (see e.g., Rivera, 2008); Group B's implementation of similar pattern-finding problems, which used teacher-centered direct instruction, enabled much weaker opportunities for these same "expert" behaviors.

Overview of Group B.

Based on Amil's input after the first implementation, we modified the instructional sequence for Group B. Primarily, Amil had a deep concern for classroom efficiency and productivity, and he requested that Group B's implementation have greater continuity with mainstream curricular topics. In particular, he viewed the overall project as an opportunity for his students for review and enrichment of basic skills. As such, the intervention should foster the development of these basic skills as much as possible, which is something that Amil perceived was lacking in Group A's implementation. For example Amil expressly stated that he wanted students to develop fluency with basic procedural skills such as "going from a table to an equation." Moreover, he stated that he wanted to teach specific strategies for dealing with pattern-finding tasks, to instruct students *directly* on "how to recognize the equation in a procedural way." This "procedural way" was tantamount to drawing students' attention to possible relations between a figure's ordinal position (x) and a quantity related to its constituent elements (e.g., U_x) via direct instruction (as opposed to letting them "discover" this strategy on their own). The lesson plan for the second iteration thus resulted as an attempt to strike a balance between a radical constructivist approach and a more "traditional" approach that often relies too heavily on direct-instruction.

We modified the lesson plan such that it emphasized "Solving" over "Modeling" (see section 2.1.1, above). On Day 1, the lesson would begin with a short (5 min.) exercise involving a table of x and y values ($x = \{0, 1, 2, 3, 4, \dots, 50\}$, and $y = \{3, 5, 7, 9, 11, \dots, 103\}$), which can be modeled with the function $y=2x+3$. On Day 2, just before Amil introduced the "toothpicks" problem, he started off with a refresher of the table of values, its solution procedure and its symbolic reformulation, $2x+3$. The intent was for students to first familiarize themselves with the numerical values inherent to the "toothpicks" sequence and its algebraic solution, so that later they could retroactively appropriate this solution as means of accomplishing contextual goals (i.e., calculating U_x). Furthermore, Amil wanted to present this particular patterning task in a way that highlighted the *y-intercept* of an algebraic equation. We modified the figures' ordinal positions, so that sequences began with "Fig. 0" instead of "Fig. 1," thus explicitly linking the sequence of figural cues and its graphical representation (e.g., when x=0, *y-intercept*=3).

Next I briefly describe each of the two days of Group B's implementation.

Group B-day 1.

This first teaching episode begins when Amil presented an *x-y table* to the class as a "warm-up" exercise. Amil instructed the class to fill in the missing values by using the data provided in the table. Students noticed that the *x values* were increasing by a summand of 1 and the *y values* were increasing by a summand of 2; thus they were able to produce extensions to the table of values. Similar to Group A's implementation, a key design feature for presenting the table of values was to substitute increasingly larger numbers (e.g., "when *x* equals fifty, what does *y* equal?") as a way to render students' arithmetic strategies as insufficient and thus motivate them to search for explicit formulas. However, during this short exercise most students employed naïve induction and not generalizing as a means of dealing with the two numerical sequences. After about four minutes of whole-class exploration, and with no clear solution procedure yet articulated, Amil verbally explained the limitations of an arithmetic strategy, emphasizing the need to find an explicit formula for calculating *y*'s from large *x*'s without having to perform many iterations of repeated addition.

Amil also explained a strategy for "finding the rule" by drawing the students' attention away from the difference between consecutive *y values*, and having them focus instead on finding a relationship structure within *x-y* pairs. Upon this direct instruction, a student immediately articulated an explicit rule for the problem at hand: "*n* times two plus three."

Group B-day 2.

This teaching episode begun with a review of the previous day's table exercise, which then led to the introduction of the "toothpicks" problem.

Responding to the same open-ended prompt —"What comes next?"— the students immediately noticed that the figural sequence forms a succession of accruing triangles. They co-constructed an arithmetic generalization in the form of $U_{x+1}=U_x+2$. Using the same tactics as before, Amil extended the conceptual problem-space to include figural extensions that are further along the sequence, and wrote "Fig. 50" on the board. As expected, students immediately calculated that Fig. 50 would have 103 toothpicks. However, the evidence suggests that although students verbalized the correct solution for Fig. 50, their behaviors did not indicate generalizing as their main strategy. It is conceivable that students merely associated "Fig. 50" with "103" from their experience with the *x-y table* exercise, and thus assumed that it was the correct answer without actually verifying it.

In-Depth Look at Students' Mathematical Inferences Bearing Generalization

I here present qualitative data analysis of a series of selected transcript segments from both implementations. By conducting detailed and sequential analyses of students' contributions during both of the implementations, I aim to show that: (1) the instructional mode adopted for Sequence A enabled stronger opportunities for student agency-as-mathematical inference; whereas (2) the instructional mode adopted for Sequence B enabled greater classroom productivity from the perspective of "traditional" assessment. Specifically, I diagnose Group A's "F-C-S" trajectory as partially grounded and Group B's as ungrounded; yet I also determine that Group A was less productive than Group B, in terms of time spent on task and the amount of material covered.

Table 1
Contrasting profiles of two instructional sequences of the same "toothpicks" problem: "student-centered inquiry-based" (SCIB) versus "teacher-centered direct instruction" (TCDI).

Opprotunities to Learn	Classes		
Agency-as-Mathematical Inference	Group A (SCIB)	>	Group B (TCDI)
Productivity	Group A (SCIB)	<	Group B (TCDI)

Group A – Creating Opportunities for Students' Spontaneous Appropriation of Cultural Artifacts.

The transcript below begins just after Group A students articulate a recursive strategy, that "it goes up by two." Amil instructed them explicitly not to manually produce all the intermediate figural extensions between the initial set of cues (Fig.'s 1-6) and those that are further along the sequence (Fig. 100).

Amil: You know it's going up by two each time. Ok. What would—let's say—let's skip a little bit [writes "Figure 100" on the far right side of the board].
Sts.: One hundred! Figure one hundred! Oh my god! Why you skip so far?
Ami: Who thinks they can figure how many toothpicks go in figure one hundred?
St-T: Me! [jumps from his seat toward Amil, grabbing the marker from his hand]
Amil: How?
St-R: Don't draw it!
St-T: I'm gonna draw it!
Amil: The rule is, you can't draw it. You can't—you can't—you don't want to draw ninety-six figures!

Amil's edict of "you can't draw it" implicitly suggests to students that their recursive strategy is insufficient, and the instructional mode enabled them to explore other strategies for dealing with Fig.100. The students first express a solution procedure that relates the number of toothpicks to it's ordinal position, in the form U_x=x+2, but they soon realize that this strategy does not obtain for known cases. St-M then proposes a recursive strategy whereby two toothpicks are added to the last figural extension in order to produce the next one, that is, $U_{x+1}=U_x+2$. Although useful for producing extensions to the sequence, St-M articulates exactly why this strategy is insufficient as a closed-explicit formula and thus cannot be used to calculate the number of toothpicks in Fig. 100, stating: "No, because you don't know the figure before." (During the class discussions, St-M referred to U_x as "the figure," which Amil later rectifies, see below.)

Realizing the limitations of their recursive/additive strategies, St-M then spontaneously proposes the use of the variable x as a placeholder for the figure number.

St-M: You have to do *x* instead of a number
Amil: Ok, so you have to do *x* instead of a number. What do you mean?
St-M: Because if you use x then it could be any number.

Amil: Ok so you have to use *x*—we gotta use *x*. So what is *x* gonna be?
St-M: *x* is the amount of triangles.

Implicit in St-M's proposition is the understanding that *x* could serve as an indeterminate quantity, thus enabling operation on the figure numbers (ordinal positions) independent of the previous figures in the sequence. St-M guides her peers to look for patterned relations within and not just across the figures. So doing, St-M leads the class to a closed-explicit solution procedure in the form U_x=2x+1, as a contextual generalization (Radford, 2003, 2008):

St-M: I found something. Ok so if you add the figure number to itself plus one, it will equal the amount of toothpicks.

Amil then prompts St-M to verify the accuracy of her procedure by checking it on known cases; once they verify that it is correct, St-M then successfully applies her procedure to Fig. 100.

Amil: Ok figure number to itself. So [indicates Fig. 1] one plus one...plus one more? Equals three. So [indicates Fig. 2] two plus two—
St-M: Four, plus one is five.
Amil: Plus one equals five. [Indicates Fig. 3].
St-M: And then three plus three is six plus one, seven. Four plus four, eight, plus one, nine. Five plus five, ten, plus one, eleven.
Amil: Ok how about figure one hundred? How many [toothpicks] would it have?
St-M: A hundred plus a hundred, two hundred, two hundred minus—I mean two hundred plus one, two-o-one.
St-K: [inaudible] Two-o-one.

It is important to note that Amil did not explicitly instruct Group A to look for these within relationship structures. St-M, having realized the limitations of their previous recursive strategies, spontaneously searched for x-to-U_x relations. It appears that the instructional mode for Group A enabled an opportunity for St-M to spontaneously operate with unknown quantities, which in turn, constituted an opportunity for the teacher to assess this particular student as having mediated the "cognitive gap" (Herscovics & Linchevski, 1994).

St-M further elaborates on her (generalizing) search process:

St-M:	[addressing the class] The thing—see the thing that I did though, I was just looking for things that they all had in common. And they had the figure number plus another one.

Such a contextual generalization was missing from Group B's implementation of the same "toothpicks" problem (see below). Contextual generalization is vital for grounded appropriation of mathematical semiotic artifacts such as the variable x (Gutiérrez, 2010; Radford, 2003). However, accomplishing this necessary cognitive milestone enroute to a canonical symbolic reformulation does not guarantee one will actually arrive there. That is, contextual generalization is necessary but not sufficient for semiotically grounded F-C-S trajectories. Ultimately, based on my analysis of Group A's implementation, I diagnose student utterances as having generalized to the symbolic mode yet partially grounded, because it is Amil and not a student who verbalizes the final contextual generalization in symbolic form.

Amil:	Ok so you said the figure number...one...plus the figure number again, right? What's another way of saying that? Instead of saying the figure number plus the figure number again...
St-M:	I don't know.
Amil:	Ok Uhh let's see. [referring to Figure 2] Two plus two plus one. What's another way of saying two plus two? Or [indicating Fig. 3] three plus three? [No response from class] How about two times the figure number, plus one? Right?
St-M:	Yeah umm oh yeah.

Amil: Ok so that is... [writes "2n+1" on the board] so that's our—

St-M: So it could still be—it could still be *x*! So it'll be two *x* plus one.

Amil: Or two *x* plus one. You can put any letter there.

St-M: Ok.

Although St-M did not objectify the symbolic version herself, I maintain there was enough conceptual substrate —at the cognitive-semiotic level— for St-M to appropriate Amil's reformulation in a way that bore personal meaning. Taken together, these excerpts above suggest that St-M's articulation of the solution was partially grounded across the F-C-S trajectory.

Group B – Classroom Productivity at the Expense of Students' Grounded Appropriation of Cultural Artifacts.

On Day 1, Group B was also instructed explicitly not to draw or count between the initial set of figural cues and those that are further along the sequence. However, unlike Group A, Group B gave no indication that they recognized the limitations of their arithmetic strategies and/or fully appreciated the power of algebraic formulas. Instead, Amil flatly stated that their emerging strategies were insufficient.

For instance, during the table exercise, the students articulated a recursive functional relationship between the *x* and *y* table values, in the form $f(x+1)=f(x)+2$. The students employed naïve induction to guess *y-values* when x=50; Amil responds to their propositions with a mini lecture wherein he gives the strategy to look within *x-y* pair values and not just across the *y-values*.

Amil: Here's the deal, alright. So in order to figure out—it seems to me when you guys were figuring out what [figure] five and [figure] six were, you just were adding two to these [gestures across entries from the y-row], right? But it gets more tricky when you go further down the line [makes a sweeping gesture across the intermediate space where Fig.'s 6-49 would be] and say you want—when *x* equals fifty what does y equal? You can't just add two that many times, right? The thing is, the way to figure this out, there's a rule. Ok there's a relationship between this number [indicates x=0] and that number [indicates y=3]. Ok so you guys were just looking at these numbers [gestures across the *y-row*] just the bottom numbers. But there's a relationship between this number [x=0] and that number [y=0] and you have to figure that out.

Recall that Group A generated their own strategies and explored their utility in the context of a student-centered, inquiry-based whole-class discussion. In contrast, Group B was explicitly told what to do and what to look for, instead of providing an opportunity for them to discover strategies and the limitations and affordances of these strategies for themselves.

On Day 2, Group B articulates a recursive strategy for the "toothpicks" problem, $U_{x+1}=U_x+2$, and use it to produce extensions to the sequence. Fig. 50 is introduced into the problem space and the students immediately calculate its number of toothpicks.

Amil:	Five right? This one's got how many [indicates Fig. 2]?
St-L:	Seven.
St-P:	Nine [referring to Fig. 3].
St-L:	It goes up by two! It goes up by two.
St-P:	The other one is eleven [referring to Fig. 4].
St-B:	Twelve. I mean eleven [referring to Fig. 4]!
Amil:	[Writes "11" under Fig. 4] Ok so then [writes "Fig. 50" on the board].
St-N:	It's going to be one hundred and two!
Amil:	Figure fifty, how many toothpicks is it gonna be?
St-L:	A hundred and three!
Amil:	A hundred and three? Why a hundred and three? What rule are you using?
St-B:	It's the same thing! The same one as the other one [referring to the solution to the table exercise].

Although the students immediately and correctly calculated the number of toothpicks in Fig. 50, the evidence suggests that their mathematically correct propositions nevertheless constitute a conditional appropriation (of a previous solution procedure) and thus remained semiotically ungrounded. Right after St-B proclaims that it's the "same one as the other one," Amil asks for the specific rule governing the number of toothpicks for any figure, to which the St-B reproduces the symbolic formula as before, but was left confused as to its relevance to the actual figural cues.

Amil:	What rule was that?
St-B:	*x* times two plus three. But I didn't know you added three! I thought you only added two!
Amil:	Huh?
Sts.:	[Inaudible classroom chatter]
Amil:	Yes, that right. It's going to be one hundred and three. Remember, the key is figuring out the rule. The rule is going to tell you how to figure out how many toothpicks are in any number, in any number figure, right? [Erases all the work from the board, thus ending the problem-solving session for Day 2].

Based on St-B's behaviors —first correctly stating the number of toothpicks for Fig. 50, then articulating the symbolic version of the solution procedure, but then ultimately questioning the operations in that solution procedure and their relation to the growth of the actual figural sequence— I argue that the solution procedure was merely transferred from a previous context (the *x-y table*) to a new situation that involved similar quantities. That is, Group B objectified the symbolic formula during the table-of-values exercise on Day 1, and later appropriated this formula as the solution procedure to the toothpicks problem without grounding it in the actual problem space. Therefore, I diagnose Group B's symbolic formula as the final product of an ungrounded F-C-S trajectory. The formula they verbalized and successfully applied to the toothpicks was arguably semiotically grounded to the numerical quantities inherent to the toothpicks sequence but was derived and appropriated in isolation of the figural cues. That is, their solution procedure was semiotically grounded back in the *x-y* values of the table, not in actual constituent elements of the source situation.

Implications and Future Directions

The goals of this article were to consider two iterations of an instructional sequence involving patterning activity to explore how variations in instructional mode impact student learning. I proposed the cognitive phenomenon of mathematical inference as an analytic focus for synthesizing equity-driven and more "classical" research on mathematics education. Critical qualitative analysis of a teacher's classroom orchestration around a particular pattern-finding problem revealed differential —and therefore *inequitable*— opportunities for agency-as-mathematical inference across two instructional modes —"teacher-centered inquiry-based" versus "teacher-centered direct instruction." The iteration involving student-centered inquiry-based instruction provided stronger opportunities for students to assume agency during the patterning activity; thus, students were able to produce semiotically grounded mathematical inferences. In contrast, the iteration involving teacher-centered direct instruction created a participation structure that sanctioned mere participation in the symbolic mode at the cost of student agency, thus disenfranchising students from opportunities to build deep personal meaning for the content (generalizing). At the same time, however, the patterning activity facilitated using direct instruction was ostensibly more productive in terms of time spent on task and the amount of material covered.

Lastly, the emerging approach presented in this article relaxes tension in the "where's the math?" debate (Heid, 2010; Martin, Gholson, & Leonard, 2010). I maintain that by looking at agency-as-mathematical inference, researchers can contribute to both "classical" mathematics education research, which is typically framed and analyzed in terms of cognitive or conceptual challenges, and equity-driven research that attempts to theorize teaching and learning as socio-political acts and accounts for issues related to power, access, and identity (Ball, Battista, Harel, Thompson, & Confrey, 2010; Confrey, 2010). The study presented here represents first-steps in a longer research agenda that seeks to understand how organizational–hierarchical power structures shape local instructional contexts that, in turn, enable or constrain opportunities to learn.

Notes

[1] Broadly, from a sociological perspective, there are mainly two competing interpretations of students' apparent "oppositional" behavior in schools serving historically under-served communities. One perspective is that students are unequivocally rejecting schooling practices because these practices represent dominant, hegemonic cultural norms and values (e.g., Willis, 1977). Contrary to this perspective, Sánchez-Jankowski (2008) concludes that students' actions primarily affirm their own local culture, values, and knowledge and are not their effort to resist the conventional cultural norms of broader society. In this way, students' could experience tension in adopting a formal mathematical register not because they seek to flatly reject all things representing broader society, but because it is not immediately clear whether and how the new register is relevant to or affirms their local culture and norms (see also *Cultural Modeling Framework*, Lee, 2006).

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