NEW SOLUTIONS FOR THE KDV EQUATION BY THE EXP-FUNCTION METHOD

NUEVAS SOLUCIONES PARA LA ECUACIÓN KDV POR EL MÉTODO DE LA FUNCIÓN-EXP

Álvaro H. Salas S.[1]

Resumen

En este artículo se obtienen soluciones para la ecuación KdV. Estas soluciones son obtenidas a través del método de la function-Exp, con ayuda del computador.

Palabras clave

ecuación diferencial no lineal, ecuación diferencial parcial no lineal, ecuación de evolución de tercer orden. Ecuación KdV, soluciones solitónicas, onda viajera, soliton, método de la function-Exp, ecuación diferencial parcial, ecuación de evolución no lineal.

Abstract

In this paper we obtain some exact solutions for the KdV equation. These solutions are obtained via the Exp-function method with the aid of a computer.

Key words

Nonlinear differential equation, nonlinear partial differential equation, third order evolution equation, KdV equation, solitonic solution, traveling wave, soliton, Exp-function method, partial differential equation, nonlinear evolution equation.

1. INTRODUCTION

Nonlinear evolution and wave equations are partial differential equations (PDEs) involving first or second-order derivatives with respect to time. Such equations have been intensively studied for the past decades [1, 2], and several new methods to solve nonlinear PDEs, either numerically or analytically, are now available. When the dependent variable u in the PDE corresponds to a physical quantity (such as the surface height of a water wave, the magnitude of an electromagnetic wave, etc.), it is important to study the propagation or aggregation properties of u . This motivates the study of methods to analytically solve evolution or wave equations via symbolic methods. The goal is to find exact traveling wave solutions. If these solutions do not change their form during propagation, they are called solitary waves. Solitary waves that preserve their shape upon collision are called solitons [3]. Solitary-waves and solitons arise due to a critical balance between dispersion

1 Department of Mathematics, Universidad de Caldas, Universidad Nacional de Colombia – Manizales, Caldas. Correo electrónico: asalash2002@yahoo.com

and nonlinearity. Due to the complexity of the mathematics involved in finding exact solutions for these PDEs, the use of algorithmic techniques that can be implemented in the symbolic language of computer algebra systems becomes a necessity. Several computer algebra packages now exist to aid in the study of nonlinear PDEs [4, 5, 6]. For example, Painlev´e analysis offers an algorithm for testing whether or not a PDE is a good candidate to be completely integrable. In addition, the Painlev´e method allows one to construct solitary wave solutions in explicit form. A more powerful technique is Hirota's bilinear method [7] which allows one to find N-soliton solutions of large classes of completely integrable PDEs [8]. The story of the first observation of solitary waves is worth telling. In 1834, while riding horseback beside the narrow Union canal near Edinburgh in Scotland, J. Scott Russell noticed that a bow wave, rolling away from a large barge, traveled as a huge heap of water for quite a long distance before finally dispersing into smaller ripples. In order to study this intriguing phenomenon, Russell did extensive experiments in a large water tank. Further investigations of solitary waves were done by Airy, Stokes, Boussinesq, and Rayleigh in an attempt to understand the mechanism behind this remarkable phenomenon [9]. The latter two scientists derived approximate models to describe solitary waves. In order to obtain his result, Boussinesq derived a one-dimensional nonlinear wave equation which now bears his name. The issue was finally resolved (in 1895) by two Dutchmen, Korteweg and de Vries, when they derived a nonlinear evolution equation governing long, one-dimensional surface gravity waves (with small amplitude) propagating in shallow water:

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \zeta^2}\right), \quad \sigma = \frac{1}{3}h^3 - \frac{Th}{\rho g}, \tag{1}$$

where η is the surface elevation of the wave above the equilibrium level h, α is a small arbitrary constant related to the uniform motion of the liquid, g is the gravitational constant, T is the surface tension, and ρ is the density. The independent variables τ and ζ are scaled versions of the time and space coordinates. Equation (1), which is called the Korteweg-de Vries (KdV) equation, can be brought into a non-dimensional form via the change of variables

$$t = \frac{1}{2}\sqrt{\frac{g}{h\sigma}}\tau, \quad x = -\sigma^{-\frac{1}{2}}\zeta, \quad u = \frac{1}{2}\eta + \frac{1}{3}\alpha \tag{2}$$

Here subscripts denote partial derivatives, e.g. $u_{3x} = \frac{\partial^3 u}{\partial x^3}$. After some algebra, one obtains

$$u_t + 6uu_x + u_{3x} = 0. \tag{3}$$

Despite this early derivation of the KdV equation, it was not until 1960 that any new applications of the equation were discovered [9]. In 1960, while studying collision-free hydrodynamic waves, Gardner and Morikawa rediscovered the KdV equation [10]. Amazingly, the KdV equation started to show up in a number of other physical contexts such as the study of stratified internal waves, ion-acoustic waves in plasma physics, lattice dynamics, and so on (further details can be found in Jeffrey and Kakutani [11], Scott et al. [12], Miura [13], Ablowitz and Segur [14], Lamb [15], Calogero and Degasperis [16], Dodd et al. [17], and Novikov et al. [18]). Since the late 1960's, the study of the properties of solitons, and the search for solitonic equations and methods to solve them, has been an active and exciting area of research.

In this paper we give some new exact solutions of equation (4) by the exp-function method.

2. EXACT SOLUTIONS THE KDV EQUATION. THE EXP- FUNCTION METHOD

Using the transformation

$$u = v(\xi),\ \ \xi = \mu x + \lambda t, \tag{4}$$

where λ , u are constants, Eq. (4) becomes

$$v'''(\xi)\mu^3 + v(\xi)v'(\xi) + \lambda v'(\xi) = 0. \tag{5}$$

In view of the Exp-function method, we assume that the solution of Eq. (5) can be expressed in the form

$$v(\xi) = \frac{\sum_{n=c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)} = \frac{a_{-c}\exp(-c\xi) + \cdots + a_d \exp(d\xi)}{b_{-p}\exp(-p\xi) + \cdots + b_q \exp(q\xi)}, \tag{6}$$

where c , d , p and q are positive integers which are unknown to be determined later, a_n and b_m are unknown constants.

In order to determine values of c and p , we balance the linear term of highest order in Eq. (6) with the highest order nonlinear term, and the linear term of lowest order in Eq. (6) with the lowest order nonlinear term, respectively.

By simple calculation, we have

$$v'''(\xi) = \frac{k_1 \exp[(7p + c)\xi] + \cdots}{k_2 \exp[8p\xi] + \cdots} \tag{7}$$

and

$$v(\xi)v'(\xi)) = \frac{k_3 \exp[(p + 2c)\xi] + \cdots}{k_4 \exp[3p\xi] + \cdots} = \frac{k_3 \exp[2(3p + c)\xi] + \cdots}{k_4 \exp[8p\xi] + \cdots}, \tag{8}$$

where the k_i are some constants. Balancing highest order of Exp-function in Eqs. (8) and (9), we have $7p + c = 2(3p + c)$ so that $c = p$. Similarly, to determine values of d and q, we balance the linear term of lowest order in Eq. (6)

$$v'''(\xi) = \frac{\cdots + k_1{}'\exp[-(7q + d)\xi]}{\cdots + k_2{}'\exp[-8q\xi]} \tag{9}$$

and

$$v(\xi)v'(\xi) = \frac{\cdots + k_3{}'\exp[-(d + 2q)\xi]}{\cdots + k_4{}'\exp[-3q\xi]} = \frac{\cdots + k_3{}'\exp[-2(3q + d)\xi]}{\cdots + k_4{}'\exp[-8q\xi]}, \tag{10}$$

where the k_i are some constants. Balancing lowest order of Exp-function in Eqs. (10) and (11), we obtain $7q + d = 2(3q + d)$ so that $d = q$.

The considerations below say that any solution of the KdV equation (6) must have the form

$$v(\xi) = \frac{a_{-c}\exp(-c\xi) + \cdots + a_c \exp(d\xi)}{b_{-p}\exp(-p\xi) + \cdots + b_p \exp(q\xi)}.$$

We will consider two cases. In these cases we set $b_{-p} = 1$, that is, the trial solution has the form

$$v(\xi) = \frac{a_{-c}\exp(-c\xi) + \cdots + a_c \exp(d\xi)}{\exp(-p\xi) + \cdots + b_p \exp(q\xi)}. \tag{11}$$

2.1 Case 1: $p = c = 1$ and $d = q = 1$.

The trial solution Eq. (12) becomes

$$v(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{12}$$

Substituting Eq. (13) into (6) and equating to zero the coefficients of all powers of *exp (ξ)* yields a set of algebraic equations. Solving it with the aid of a computer, we obtain the following solutions :

I. $a_1 = a_1,\ b_0 = b_0,\ a_{-1} = \frac{1}{4}a_1 b_0^2,\ a_0 = a_1 b_0 + \mu^2 b_0,\ b_{-1} = \frac{b_0^2}{4},\ \lambda = -\mu^3 - 6\mu a_1,\ \mu = \mu.$

For $b0 = \pm 2$ the soliton solutions corresponding to these values are :

$$u_1(x,t) = a_1 + \frac{4\mu^2 b_0 e^{\mu(x+(\mu^2+6a_1)t)}}{(2e^{\mu x} + b_0 e^{\mu(\mu^2+6a_1)t})^2}. \tag{13}$$

For $b0 = \pm 2$ and real μ.

$$u_2(x,t) = a_1 + \frac{\mu^2}{1+\cosh\left(\mu(x-(\mu^2+6a_1)t)\right)}. \tag{14}$$

$$u_3(x,t) = a_1 + \frac{\mu^2}{1-\cosh\left(\mu(x-(\mu^2+6a_1)t)\right)}. \tag{15}$$

We obtain periodic solutions in the following cases :

$b_0 = 2$ and $\mu = \sqrt{-1}\,m$:

$$u_4(x,t) = a_1 - \frac{m^2}{1+\cos\left(m(x+(m^2-6a_1)t)\right)}. \tag{16}$$

$b_0 = 2$ and $\mu = \sqrt{-1}\,m$:

$$u_5(x,t) = a_1 - \frac{m^2}{1-\cos\left(m(x+(m^2-6a_1)t)\right)}. \tag{17}$$

2.2. Case 2: $p = c = 2$ and $d = q = 2$.

The trial solution Eq. (12) becomes

$$v(\xi) = \frac{a_2\exp(2\xi) + a_1\exp(\xi) + a_0 + a_{-1}\exp(-\xi) + a_{-2}\exp(-2\xi)}{\exp(2\xi) + b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi)}. \tag{18}$$

Substituting Eq. (16) into (6) and equating to zero the coefficients of all powers of *exp(ξ)* yields a set of algebraic equations. Solving it with the aid of a computer, we obtain many solutions. For space reasons, we only give some of them.

II. $a_2 = a_2,\ b_0 = b_0,\ a_1 = 0,\ b_1 = 0,\ a_{-2} = \frac{a_2 b_0^2}{4},\ a_{-1} = 0,\ a_0 = a_2 b_0 + 4\mu^2 b_0,\ b_{-1} = 0,\ b_{-2} = \frac{b_0^2}{4},\ \lambda = -6\mu a_2 - 4\mu^3.$

$$u_6(x,t) = a_2 + \frac{16 b_0 \mu^2 e^{-2\mu(x-2(2\mu^2+3a_2)t)}}{\left(2 + b_0 e^{-2\mu(x-2(2\mu^2+3a_2)t)}\right)^2}. \tag{19}$$

In the case when $b0 = \pm 2$ we obtain periodic solutions. More exactly,

$$u_7(x,t) = a_2 + 2\mu^2 \operatorname{sech}^2(\mu(x - 2(2\mu^2 + 3a_2)t)). \tag{20}$$

$$u_8(x,t) = a_2 - 2\mu^2 \operatorname{csch}^2(\mu(x - 2(2\mu^2 + 3a_2)t)). \tag{21}$$

For $b_0 = \pm 2$ and $\mu = \sqrt{-1}\, m$:

$$u_9(x,t) = a_2 - 2m^2 \sec^2\left(m(x + 2(2m^2 - 3a_2)t)\right). \tag{22}$$

$$u_{10}(x,t) = a_2 - 2m^2 \csc^2\left(m(x + 2(2m^2 - 3a_2)t)\right). \tag{23}$$

III. $a_1 = a_1$, $a_2 = a_2$, $b_1 = 0$, $a_{-2} = \dfrac{1}{16}\dfrac{a_2 a_1^4}{\mu^8}$, $a_{-1} = \dfrac{1}{4}\dfrac{a_1^3}{\mu^4}$, $a_0 = -\dfrac{1}{2}\dfrac{a_1^2(a_2 + 2\mu^2)}{\mu^4}$,

$b_0 = -\dfrac{1}{2}\dfrac{a_1^2}{\mu^4}$, $b_{-1} = 0, b_{-2} = \dfrac{1}{16}\dfrac{a_1^4}{\mu^8}$,

$\lambda = -6\mu a_2 - \mu^3$.

$$u_{11}(x,t) = \frac{\left(4\mu^4 e^{2\mu x} + a_1^2 e^{2\mu(\mu^2+6a_2)t}\right)a_2 + 4\mu^2 a_1 \left(a_2 + \mu^2\right) e^{\mu(x+(\mu^2+6a_2)t)}}{\left(2\mu^2 e^{\mu x} + a_1 e^{\mu(\mu^2+6a_2)t}\right)^2}. \tag{24}$$

IV. $a_1 = a_1$, $a_2 = a_2$, $b_0 = b_0$, $b_1 = b_1$, $\lambda = -\mu^3 - 6\mu a_2$,

$$a_{-2} = \frac{a_2}{16\mu^8}(3a_1^4 + 3a_2^4 b_1^4 + 4\mu^4 a_2^2 b_0 b_1^2 - 8\mu^4 a_1 a_2 b_0 b_1 + 12\mu^2 a_1^2 a_2 b_1^2 - 12\mu^2 a_1 a_2^2 b_1^3 + 18a_1^2 a_2^2 b_1^2 - 12a_1 a_2^3 b_1^3 - 12a_1^3 a_2 b_1 - 4\mu^2 a_1^3 b_1 + 4\mu^2 a_2^3 b_1^4 + 4\mu^4 a_1^2 b_0),$$

$$a_{-1} = -\frac{1}{4\mu^6}(12\mu^2 a_1^2 a_2 b_1 + 4\mu^4 a_1^2 b_1 - 15\mu^2 a_1 a_2^2 b_1^2 - 8\mu^4 a_1 a_2 b_1^2 - 3\mu^2 a_1^3 + 6\mu^2 a_2^3 b_1^3 + 4\mu^4 a_2^2 b_1^3 + 2a_2^4 b_1^3 - 6a_1 a_2^3 b_1^2 + 4\mu^4 a_2^2 b_0 b_1 + 6a_1^2 a_2^2 b_1 - 4\mu^4 a_1 a_2 b_0 - 2a_1^3 a_2 + 4\mu^6 a_2 b_0 b_1 - 4\mu^6 a_1 b_0),$$

$$a_0 = \frac{1}{\mu^2}\left(2a_1 a_2 b_1 + \mu^2 a_1 b_1 + \mu^2 a_2 b_0 - a_2^2 b_1^2 - \mu^2 a_2 b_1^2 - a_1^2\right),$$

$$b_{\ 2} = \frac{1}{16\mu^8}(3a_1^4 + 3a_2^4 b_1^4 + 4\mu^4 a_2^2 b_1^2 b_0 \quad 8\mu^4 a_1 a_2 b_0 b_1 + 12\mu^2 a_1^2 a_2 b_1^2 \quad 12\mu^2 a_1 a_2^2 b_1^3 + 18a_1^2 a_2^2 b_1^2 \quad 12a_1 a_2^3 b_1^3 \quad 12a_1^3 a_2 b_1 \quad 4\mu^2 a_1^3 b_1 + 4\mu^2 a_2^3 b_1^4 + 4\mu^4 a_1^2 b_0),$$

$$b_{-1} = -\frac{1}{4\mu^6}(2a_2^3 b_1^3 + 3\mu^2 a_2^2 b_1^3 - 6a_1 a_2^2 b_1^2 - 6\mu^2 a_1 a_2 b_1^2 + 4\mu^4 a_2 b_0 b_1 + 6a_1^2 a_2 + 3\mu^2 a_1^2 b_1 - 4\mu^4 a_1 b_0 - 2a_1^3).$$

$$u_{12}(x,t) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi) + a_{-2}\exp(-2\xi)}{\exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi)}, \tag{25}$$

Where $\xi = \mu(x - (\mu^2 - 6a_2)t)$.

V. $a_2 = a_{-2} = 0$, $a_1 = 2q\sqrt{-1}$,

$a_0 = -4q$, $a_{-1} = -2q\sqrt{-1}$, $b_{-2} = 1$,

$b_{-1} = b_1 = 0$, $\mu = \sqrt{q}\sqrt{-1}$,

$\lambda = q\sqrt{q}\sqrt{-1}$.

$$u_{13}(x,t) = -\frac{2q\left(1 + \sin\left(\sqrt{q}(x + q\,t)\right)\right)}{1 + \cos\left(2\sqrt{q}(x + q\,t)\right)} \quad (q > 0). \tag{26}$$

VI. $a_2 = a_{-2} = 0$, $a_1 = 2q\sqrt{-1}$,

$a_0 = -4q$, $a_{-1} = -2q\sqrt{-1}$, $b_{-2} = 1$,

$b_{-1} = b_1 = 0$, $\mu = -\sqrt{q}\sqrt{-1}$, $\lambda = -q\sqrt{q}\sqrt{-1}$.

$$u_{14}(x,t) = -\frac{2q\left(1 - \sin\left(\sqrt{q}(x + q\,t)\right)\right)}{1 + \cos\left(2\sqrt{q}(x + q\,t)\right)} \quad (q > 0). \tag{27}$$

VII. Other interesting periodic solutions are :

$$u_{15}(x,t) = -\frac{m^2}{2r}\left(4 + r\csc^2\left(\frac{m}{2r}(rx + m^2(r + 12)t)\right)\right). \tag{28}$$

$$u_{16}(x,t) = \frac{m^2}{2r}\left(4 - r\sec^2\left(\frac{m}{2r}(rx + m^2(r-12)t)\right)\right). \tag{29}$$

$$u_{17}(x,t) = \frac{m^2\left(r^2 - 24 + 8\cos\left(2m(x-2m^2t)\right) + 8\sqrt{16-r^2}\,\sin\left(m(x-2m^2t)\right)\right)}{2\left(r + 4\cos\left(m(x-2m^2t)\right)\right)^2}. \tag{30}$$

VIII. Finally, other interesting solitonic solutions are :

$$u_{18}(x,t) = \frac{\mu^2}{2r}\left(4 - r\,\mathrm{csch}^2\left(\frac{\mu}{2r}(rx - \mu^2(r+12)t)\right)\right). \tag{31}$$

$$u_{19}(x,t) = -\frac{\mu^2}{2r}\left(4 - r\,\mathrm{sech}^2\left(\frac{\mu}{2r}(rx - \mu^2(r-12)t)\right)\right). \tag{32}$$

$$u_{20}(x,t) = -\frac{r^2 + 24 + 8\sqrt{16+r^2}\,\cosh\left(x+2t\right) + 8\cosh\left(2(x+2t)\right)}{2\left(r - 4\sinh\left(x+2t\right)\right)^2}. \tag{33}$$

Figure 1 and Figure 2 illustrate graphically some solutions. The soliton solution $u_2(x,t)$ and the periodic solution $u_{14}(x,t)$.

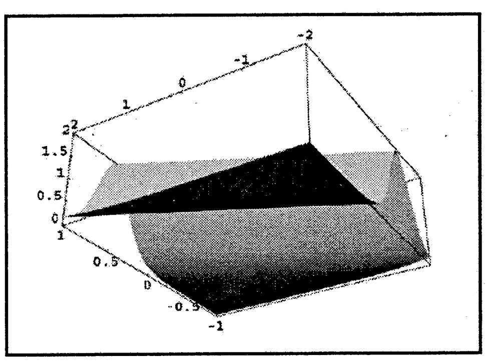

Figure 1 : The solution $u_2(x,t)$ for $m = 2$, $a_1 = 0$, $|t| \leq 1$ and $|x| \leq 2$.

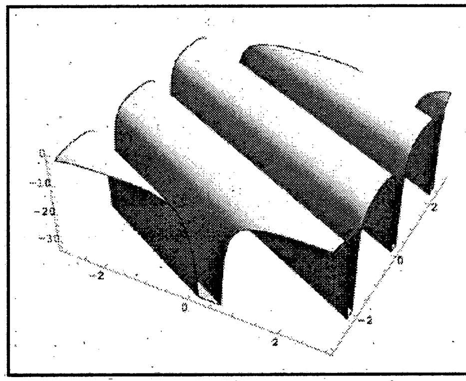

Figure 2: The function $u_{14}(x,t)$ for $q = 2$ and $t, x \in [-3,3]$.

4. CONCLUSIONS

In this paper, by using the exp-function method and the help of a computer, we obtained some exact solutions for the KdV equation (4). The method is direct and effective. We may apply this method to solve other partial and ordinary nonlinear differential equations. The Exp-function method is a promising and powerful new method for NLEEs arising in mathematical physics. Its applications are worth further studying.

REFERENCES

[1] P. G. Drazin, and R.S. Johnson, Solitons: an Introduction, Cambridge, London (1989).

[2] P. L. Sachdev, Nonlinear Diffusive Waves, Cambridge, London (1987).

[3] N. J. Zabusky, and M. D. Kruskal, Phys. Rev. Letters 15, 240-243 (1965).

[4] W. Hereman, and M. Takaoka, J. Phys. A: Math. Gen. 23, 4805-4822 (1990).

[5] A. K. Head, Comp. Phys. Comm., 77, 241-248 (1993).

[6] W. Hereman, and W. Zhuang, Acta Applicanda e Mathematica (1995).

[7] R. Hirota, Phys. Rev. Lett. 27, 1192-1194 (1971).

[8] Y. Matsuno, Bilinear Transformation Method, Academic Press, Orlando (1984).

[9] M. J. Ablowitz, and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scat¬tering, London Mathematical Society Lecture Note Series 149, Cambridge Univ. Press, London (1991).

[10] C. S. Gardner, and G. K. Marikawa, Courant Inst. Math. Sci. Res. Rep. NYO-9082, N.Y. University, New York (1960).

[11] A. Jeffrey, and T. Kakutani, SIAM Rev. 14, 582-643 (1972).

[12] A. C. Scott, F. Y. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443-1483 (1973).

[13] R. M. Miura, SIAM Rev. 18, 412-459 (1976).

[14] M. J. Ablowitz, and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia (1981).

[15] G. L. Lamb, Elements of Soliton Theory, John Wiley, New York (1980).

[16] F. Calogero, and A. Degasperis, Spectral Transforms and Solitons I, Amsterdam, Holland (1982).

[17] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, New York (1982).

[18] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, Theory of Solitons. The Inverse Scattering Method, Plenum, New York (1984).

[19] Zhao Xuequin and others, A new Riccati equation expansion method with symbolic computation to construct new traveling wave solution of nonlinear differential equations, Applied Mathematics and Computation, 172 (2006) 24-39.

[20] R. CONTE & M. MUSETTE , Link betwen solitary waves and projective Riccati equations, J. Phys. A Math. 25 (1992), 5609-5623.

[21] ALVARO H. SALAS S., Some solutions for a type of generalized Sawada-Kotera equation, Applied Mathematics and Computation, 196 (March, 2008), pages 812-817.

[22] ABDUL Wazwaz, Analytic study of the fifth order integrable nonlinear evolution equation by using the tanh method, Applied Mathematics and Computation, 174 (2006), 289-299.

[23] J.H. He, X.H. Wu, Chaos, Solitons & Fractals 30 (2006) 700.