Lecturas Matemáticas
Volumen **33** *(2) (2012), páginas 95–106*
ISSN 0120–1980

A uniparametric family of modifications for Chebyshev's method

Una familia uniparamétrica de modificaciones del método de Chebyshev

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Abstract. In this paper, we propose a new uniparametric family of modifications for Chebychev's method, free from second derivatives, to solve non-linear equations. It is proved that each method in this family is cubically convergent. Every iteration of the family requires one evaluation of the function and two of the first derivative. Hence, the efficiency index of each method is $3^{1/3} = 1{,}442$ that is better than that of Newton's method. Several numerical examples are, also, given to illustrate the performance of the presented method.

Key words and phrases. Chebyshev's method; Newton's method; Non-linear equations; Root-finding; Iterative method.

Resumen. En este artículo proponemos una nueva familia de modificaciones del método de Chebychev, independiente de las segundas derivadas para resolver ecuaciones no lineales. Se demuestra que cada método de esta familia converge cúbicamente. Cada iteraciónde la familia requiere una evaluación de la función y dos de la primera derivada. Luego el índice de eficiencia de cada método es $3^{1/3} = 1{,}442$ el cual es mejor que el del método de Newton, Sea dan varios ejemplos numéricos para ilustrar el comportamiento de los métodos.

Palabras y frases claves. Método de Chebyshev, método de Newton, ecuaciones no lineales, búsqueda de raíces, método iterativo.
2010 AMS Mathematics Subject Classification. 65H05, 65B99

1. Introduction

One of the most widely studied problems in Mathematics is to find a solution for non-linear equations. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$, where $f : D \subset R \mapsto R$, for an open interval D, is a scalar function. There are a number of situations in different scientific disciplines in which this problem appears. Although $f(x) = 0$ is a relatively easy problem to state, it is also well known that for many particular choices of the function f, it is very difficult or even impossible to find an exact solution. So, since centuries ago, different iterative techniques have been developed in order to approximate a solution of the aforementioned equation.

Undoubtedly, Newton's method is the most widely studied and used method to solve this problem. For a real-valued function f, with an initial value x_0, Newton's method is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\,. \tag{1}$$

Under the appropriate conditions for the function f and the initial value x_0, Newton's method (1) generates a sequence that converges to a solution of the equation $f(x) = 0$. One of the most interesting features of Newton's method is its balance between computational cost and speed of convergence, considered as an important and basic method which converges quadratically [20].

There are many other methods to approximate the solution of non-linear equations, some of which are variants of Newton's method that seek either to reduce the computational cost or to increase the speed of convergence. In the latter, Chebyshev's method plays an important role. For a real-valued function f, with an initial value x_0, Chebyshev's method is defined by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\,L_f(x_n)\right)\frac{f(x_n)}{f'(x_n)}\,,$$

in which

$$L_f(x_n) = \frac{f(x_n)f''(x_n)}{f'(x_n)}. \tag{2}$$

It is well known that Chebyshev's method is cubically convergent [1,22].

Traub [22] credits this method to Euler but in the Russian literature it is attributed to Chebyshev [2,3]. Although the very method is also known as Euler's method or Euler-Chebyshev method, throughout this paper we refer to it as Chebyshev's method. Chebyshev's method together with its variations and improvements has drawn the attention of many researchers. For instance, from the historical point of view, we can find an equivalent way to write Chebyshev's method known as Schröder's formula [21]. As a sample of more recently published papers about this topic, we can cite [13] and the references cited therein. Chebyshev's method can be deduced by different ways. For instance, it can be obtained by quadratic interpolation of the inverse function of f, in order to approximate $f^{-1}(0)$ [22]. It also assumes a geometric derivation, in terms of an osculating parabola $aw(x)^2 + w(x) + bx + c = 0$ that satisfies the tangency conditions $w(x_n) = f(x_n)$, $w'(x_n) = f'(x_n)$, and $w''(x_n) = f''(x_n)$ (see [1,19] for more details). In [13] Chebyshev's method is obtained from Obreshkov's techniques.

It is observed that Chebyshev's method depends on the second derivatives in computing process, making its practical utility rigorously restricted so that Newton's method is frequently used as an alternative in solving non-linear equations. To remove the second derivative of (2), recently, some variants of Chebyshev's method have been obtained [6,7,8,9,10,14,15,16,17,23].

Hernández [14] developed a second-derivative-free variant of Chebyshev's method while approximating the second derivative $f''(x_n)$ by a finite difference between first derivatives

$$f''(x_n) \approx \frac{f'(x_n) - f'(\frac{1}{2}\,(x_n + y_n))}{\frac{1}{2}\,(x_n - y_n)}\,, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}\,.$$

So,

$$L_f(x_n) \approx \frac{f'(x_n) - f'\left(x_n - \frac{1}{2}\,\frac{f(x_n)}{f'(x_n)}\right)}{\frac{1}{2}\,f'(x_n)}\,. \tag{3}$$

In [10,11] a second-derivative-free variant class of Chebyshev's method is obtained through approximating the second derivative $f''(x_n)$ as follows:

$$f''(x_n) \approx \frac{f'(x_n) - f'(y_n)}{x_n - y_n}\,, \quad y_n = x_n - \theta\frac{f(x_n)}{f'(x_n)}\,, \quad \theta \in (0, 1],$$

that results in

$$L_f(x_n) \approx \frac{f'(x_n) - f'(y_n)}{\theta f'(x_n)}\,. \tag{4}$$

Another class of approximations of $L_f(x_n)$ is obtained in [16], where authors consider a new finite difference approximation of $f''(x_n)$ as

$$f''(x_n) \approx \frac{f'(y_n) - f'(x_n)}{y_n - x_n}\,, \quad y_n = x_n + \theta\,\frac{f(x_n)}{f'(x_n)}\,, \quad \theta \in (0, 1]$$

to derive the approximation

$$L_f(x_n) \approx \frac{f'(x_n) - f'(y_n)}{\theta f'(x_n)}\,. \tag{5}$$

Chun [8] used

$$f''(x_n) \approx \frac{f'(y_n) - f'(x_n)}{y_n - x_n}\,, \quad y_n = x_n - \theta\frac{f(x_n)}{f'(x_n)}\,, \quad \theta \neq 0$$

and obtained a class of approximations as follows:

$$L_f(x_n) \approx \frac{f'(x_n) - f'(y_n)}{\theta f'(x_n)}\,. \tag{6}$$

In [17] authors consider Taylor expansion of $f(y_n)$ about x_n, where

$$y_n = x_n - \theta \frac{f(x_n)}{f'(x_n)}\,, \quad \theta \neq 0$$

and get the class of approximations

$$L_f(x_n) \approx 2\,\frac{f(y_n) + (\theta - 1) f(x_n)}{\theta^2 f(x_n)}\,. \tag{7}$$

Zhou [23] considers approximating the equation $f(x) = 0$ around the point $(x_n, f(x_n))$ by a hyperbola of the form $axw + w + bx + c = 0$ and imposes the tangency conditions $w(x_n) = f(x_n)$, $w'(x_n) = f'(x_n)$, $w(y_n) = f(y_n)$, to obtain

$$f''(x_n) \approx w''(x_n) = \frac{2f'^2(x_n) f(y_n)}{f^2(x_n) - f(x_n) f(y_n)}\,, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}\,.$$

Therefore,

$$L_f(x_n) \approx \frac{2f(y_n)}{f(x_n) - f(y_n)}\,. \tag{8}$$

To derive an approximation of $f''(x_n)$ in (2),Chun [6] considered the approximation

$$f(x) \approx h(x) := ax^3 + bx^2 + cx + d$$

which satisfied the conditions $f'(x_n) = h'(x_n)$ and $f'(y_n) = h'(y_n)$, in which

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\,.$$

He, then, derived the approximation

$$f''(x_n) \approx h''(x_n) = \frac{f'(y_n) - f'(x_n)}{y_n - x_n} - \lambda(y_n - x_n)\,, \quad \lambda = 3a,$$

that resulted in

$$L_f(x_n) \approx 1 - \frac{f'(y_n)}{f'(x_n)} + \lambda \frac{f^2(x_n)}{f'^3(x_n)}\,. \tag{9}$$

Also, Chun [7] considers approximating the equation $f(x) = 0$ around the point $(x_n, f(x_n))$ by the quadratic equation in x and w of the form $x^2 + aw^2 + bx + cw + d = 0$. He imposes the tangency conditions $w(x_n) = f(x_n)$, $w'(x_n) = f'(x_n)$, $w(y_n) = f(y_n)$, in which

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\,,$$

to derive the approximation

$$f''(x_n) \approx w''(x_n) = \frac{2f(y_n)f'^2(x_n)(1+af'^2(x_n))}{f^2(x_n)+af'^2(x_n)[f(y_n)-f(x_n)]^2}\,.$$

Hence,

$$L_f(x_n) \approx \frac{2f(y_n)f(x_n)(1+af'^2(x_n))}{f^2(x_n)+af'^2(x_n)[f(y_n)-f(x_n)]^2}\,. \tag{10}$$

It is proved that all of the above modifications for Chebyshev's method are cubically convergent.

In this paper, we will consider a new technique which is suitable for Chebyshev's method, to construct a finite difference between the first derivatives to replace the second derivative. So, a new family of modifications for Chebyshev's method, free from second derivatives, is obtained. The third order of convergence of these new methods is proved and their best efficiency, in terms of function evaluations, is provided, too. Some examples are given to show the efficiency and superiority of the new methods.

2. A family of modifications for Chebyshev's method free from second derivative

In this section, we consider a new finite difference approximation to $f''(x_n)$ as follows:

$$f''(x_n) \approx \frac{f'(x_n+\beta f(x_n))-f'(x_n)}{\beta f(x_n)}$$

in which $\beta \neq 0$ is a parameter. Using the above approximation, we can obtain

$$L_f(x_n) \approx \frac{f'(x_n+\beta f(x_n))-f'(x_n)}{\beta f'^2(x_n)} \tag{11}$$

that results in the following modifications for Chebyshev's method:

$$x_{n+1} = x_n - \left(1+\frac{1}{2}\,\frac{f'(x_n+\beta f(x_n))-f'(x_n)}{\beta f'^2(x_n)}\right)\frac{f(x_n)}{f'(x_n)}\,, \quad \beta \neq 0. \tag{12}$$

Considering different values of the parameter β in (12), we can obtain a family of Chebyshev-type methods that include, as particular cases, the following:

1. As a limit case, when $\beta \longrightarrow 0$, the classical Chebyshev's method is obtained.
2. If we allow the parameter β to vary in each iteration of (12), then the choice $\beta = -0{,}5/f'(x_n)$ gives the method (3), the choice $\beta = -\theta/f'(x_n)$, $\theta \in (0,1]$, gives the method (4), the choice $\beta = \theta/f'(x_n)$, $\theta \in (0,1]$, gives the method (5), and the choice $\beta = -\theta/f'(x_n)$, $\theta \neq 0$, gives the method (6).

In the sequel, we prove that the methods (12) are cubically convergent for any choice of the constant parameter β. To this end, we need the following facts.

Definition 1. Let $f(x)$ be a real function with a simple root α and $\{x_n\}_{n\geq 0}$ be a sequence of real numbers, converging towards α. Then, we say that the order of convergence of the sequence is p, if there exists a real $p \geq 1$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = C$$

for some $C \neq 0$. C is known as the asymptotic error constant.

If $p = 1, 2$, or 3, the sequence is said to have linear convergence, quadratic convergence or cubic convergence, respectively.

Definition 2. Let $e_n = x_n - \alpha$ be the error in the n-th iteration. We call the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1})$$

as the error equation.

If we obtain the error equation for any iterative method, then the value of p is its order of convergence.

Definition 3. Let r be the number of new pieces of information required by a method. A "piece of informationïs typically any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index [12] and is defined by

$$\rho = p^{1/r}$$

where p is the order of the method.

It is noticed that each iteration of methods defined by (12) requires one evaluation of the function and two of its first derivative. Therefore, according to the definition 3, the methods defined by (12) have the efficiency indexes equal to $\sqrt[3]{3} \approx 1{,}442$, which are better than that of Newton's method $\sqrt{2} \approx 1{,}414$. Thus, these new methods are preferable if the computational cost of the first derivative is not greater than that of the function itself.

As we know, to solve the non-linear equation $f(x) = 0$ most methods have fixed point style: they transform the equation $f(x) = 0$ to the $x = \varphi(x)$ in such a way that α is a fixed point of φ, namely $\alpha = \varphi(\alpha)$. With an initial approximation x_0 to the α, they generate the sequence $\{x_n\}$, in which $x_{n+1} = \varphi(x_n)$, $n \geq 0$. It is obvious that if φ is continuous and the sequence $\{x_n\}$ is convergent, then $x_n \longrightarrow \alpha$. Using the p-th order Taylor series of $x_{n+1} - \alpha = \varphi(x_n) - \varphi(\alpha)$ about α, we can easily prove the following theorem.

Theorem 4. *Let sequence $x_{n+1} = \varphi(x_n)$, $n \geq 0$, be convergent to the fixed point α of φ. If*

$$\varphi'(\alpha) = \varphi''(\alpha) = \cdots = \varphi^{(p-1)}(\alpha) = 0, \qquad \varphi^{(p)}(\alpha) \neq 0,$$

then the sequence $\{x_n\}$ is convergent of order p with asymptotic error constant $C = |\varphi^{(p)}(\alpha)|/p!$.

Theorem 5. *Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \mapsto R$ for an open interval I. If x_0 is sufficiently close to α, then the class of methods defined by (12) has third-order convergence for any choice of the parameter $\beta \neq 0$, satisfying the error equation*

$$e_{n+1} = \left(2c_2^2 - (1 + 1{,}5\beta f'(\alpha))c_3\right) e_n^3 + O(e_n^4),$$

where $e_n = x_n - \alpha$ and $c_k = \dfrac{f^{(k)}(\alpha)}{k! f'(\alpha)}$.

Proof. Using Taylor's expansion and taking $f(\alpha) = 0$ into account, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)\right) \tag{13}$$

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)\right). \tag{14}$$

Dividing (13) by (14) gives

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{15}$$

If $z_n = x_n + \beta f(x_n)$, then $z_n - \alpha = e_n + \beta f'(\alpha)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)\right)$ so that the Taylor's expansion of $f'(z_n)$ about α can be read as

$$\begin{aligned} f'(z_n) &= f'(\alpha)\left(1 + 2c_2(z_n - \alpha) + 3c_3(z_n - \alpha)^2 + 4c_4(z_n - \alpha)^3 + O(e_n^4)\right) \\ &= f'(\alpha)\left(1 + 2c_2(1 + \beta f'(\alpha))e_n + k e_n^2 + m e_n^3 + O(e_n^4)\right), \end{aligned}$$

in which

$$\begin{aligned} k &= 2c_2^2 \beta f'(\alpha) + 3c_3(1 + \beta f'(\alpha))^2, \\ m &= 2c_2 c_3 \beta f'(\alpha)(4 + 3\beta f'(\alpha)) + 4c_4(1 + \beta f'(\alpha))^3. \end{aligned}$$

This together with (14) implies that

$$f'(z_n) - f'(x_n) = f'(\alpha)\left[2c_2 \beta f'(\alpha) e_n + (k - 3c_3)e_n^2 + (m - 4c_4)e_n^3 + O(e_n^4)\right]. \tag{16}$$

On the other hand,

$$f'^2(x_n) = f'^2(\alpha)\left(1 + 4c_2 e_n + (4c_2^2 + 6c_3)e_n^2 + (12c_2c_3 + 8c_4)e_n^4 + O(e_n^4)\right)$$

and hence,

$$\begin{aligned} \frac{1}{2\beta f'^2(x_n)} = \frac{1}{2\beta f'^2(\alpha)} &\left[1 - 4c_2 e_n + (12c_2^2 - 6c_3)e_n^2 \right. \\ &\left. + (36c_2c_3 - 32c_2^3 - 8c_4)e_n^3 + O(e_n^4)\right]. \end{aligned} \tag{17}$$

Now, using (16) and (17), we have that

$$\frac{f'(z_n)-f'(x_n)}{2\beta f'^2(x_n)} = \frac{1}{2\beta f'(\alpha)}\left[2c_2\beta f'(\alpha)e_n + \left(k-3c_3-8c_2^2\beta f'(\alpha)\right)e_n^2 \right.$$
$$\left. + \left(m-4c_4-4kc_2-12c_2c_3-12c_2c_3\beta+24\beta c_2^3\right)e_n^3 + O(e_n^4)\right]. \quad (18)$$

From relations (15) and (18), one can get

$$l\left(1+\frac{1}{2}\,\frac{f'(z_n)-f'(x_n)}{\beta f'^2(x_n)}\right)\frac{f(x_n)}{f'(x_n)} =$$
$$e_n + \left(2(c_2^2-c_3)+\frac{1}{2\beta f'(\alpha)}\left(k-3c_3-10c_2^2\beta f'(\alpha)\right)\right)e_n^3 +$$
$$O(e_n^4). \quad (19)$$

Therefore,

$$e_{n+1} = \left(2c_2^2-(1+1{,}5\beta f'(\alpha))c_3\right)e_n^3 + O(e_n^4). \quad ☑$$

It has been shown that the Maple package can be successfully employed to rederive error equations of iterative methods, or, to find their order of convergence (see [4,5] for details). The class of methods (12) in this case is found to be third-order convergent as shown in the following theorem.

Let α be a simple zero of f. The iteration function F, corresponding to (12), is defined by

$$F(x) = x - \left(1+\frac{1}{2}\,\frac{f'(x+\beta f(x))-f'(x)}{\beta f'^2(x)}\right)\frac{f(x)}{f'(x)}\,.$$

According to the theorem 4, it is sufficient to show that

$$F(\alpha)=\alpha, \quad F'(\alpha)=0, \quad F''(\alpha)=0,$$

$$F'''(\alpha) = 3\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 - \left(\frac{1}{f'(\alpha)}+\frac{3\beta}{2}\right)f'''(\alpha).$$

The computations of the above derivatives can be performed using mathematical software package Maple, one of the computer algebra systems. To do that, we run the following Maple statements consecutively:

>L1:=x− >(D(f)(x+β*f(x))−D(f)(x))/(β*D(f)(x)^2);

$$L1 := x- > \frac{D(f)(x+\beta f(x)) - D(f)(x)}{\beta D(f)(x)^2}$$

>F:=x− >x−(1+0.5*L1(x))*f(x)/D(f)(x);

$$F := x- > x - (1 + 0{,}5L1(x))\frac{f(x)}{D(f)(x)}$$

>algsubs(f(α)=0, F(α));

$$\alpha$$

>algsubs(f(α)=0, D(F)(α));

$$0.$$

>algsubs(f(α)=0, (D@@2)(F)(α));

$$0.$$

>algsubs(f(α)=0, (D@@3)(F)(α));

$$-\frac{-3{,}0(D^{(2)})(f)(\alpha)^2 + 1{,}0(D^{(3)})(f)(\alpha)D(f)(\alpha) + 1{,}5(D^{(3)}(f)(\alpha)\beta D(f)(\alpha)^2}{D(f)(\alpha)^2}$$

Thus, it can be read

$$F(\alpha) = \alpha, \quad F'(\alpha) = 0, \quad F''(\alpha) = 0,$$

$$F'''(\alpha) = 3\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 - \left(\frac{1}{f'(\alpha)} + \frac{3\beta}{2}\right) f'''(\alpha).$$

Hence,

$$e_{n+1} = F(x_n) - F(\alpha) = F'(\alpha)e_n + \frac{F''(\alpha)}{2!}\, e_n^2 + \frac{F'''(\alpha)}{3!}\, e_n^3 + O(e_n^4) = Ce_n^3 + O(e_n^4),$$

where $C = \dfrac{F'''(\alpha)}{3!} = 2c_2^2 - (1 + 1{,}5\beta f'(\alpha))c_3$.

3. Numerical examples

Now, we employ some new modifications of Chebyshev's method (NMCH), Eq. (12), with $\beta = 0{,}2$, obtained in this paper to solve some non-linear equations to be compared with Newton's method (NM), Chebyshev's method (CHM), the method of Hernández (HM), Eq. (3), the method of Kou, Li, Wang version 1 (KLWM1), Eq. (5) with $\theta = 1/2$, the method of Kou, Li, Wang version 2 (KLW2), Eq. (7), with $\theta = -1/2$, Zhou's method (ZM), Eq. (8), Chun's method version 1 (CM1), Eq. (9) with $\lambda = 0$, and Chun's method version 2 (CM2), Eq. (10) with $a = 1$.

All computations were done using MATLAB software with format of long floating point arithmetics. We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. We use the following stopping criterion for computer programs: $|x_{n+1} - x_n| < \varepsilon$. So, when the stopping criterion is satisfied, $x^* := x_{n+1}$ is taken as the exactly computed root α. For numerical illustrations in this section, we used the fixed stopping criterion $\varepsilon = 10^{-15}$.

We used the test functions and obtained the approximate zeros x^* up to the 16 digits.

$$f_1(x) = x^3 + 4x^2 - 10, \qquad f_2(x) = \sin^2 x - x^2 + 1,$$
$$f_3(x) = x^2 - e^x - 3x + 2, \qquad f_4(x) = \cos x - x,$$
$$f_5(x) = (x + 2)e^x - 1,$$

As convergence criterion, it is required that the distance of two consecutive approximations for the zero be less than 10^{-15}. Also, displayed is the number of iterations to approximate the zero (IT) and the value of $|f(x^*)|$.

The test results in Table 1 show that for most of the functions we tested, our method has better performance compared to other third-order methods, and can also compete with Newton's method.

4. Conclusions

In this paper, we proposed a new uniparametric family of modifications for Chebychev's method, free from second derivatives, to solve non-linear equations. It is proved that each method in this family is cubically convergent. Every iteration of the family requires one evaluation of the function and two of the first derivative. Hence, the efficiency index of each method is $3^{1/3} = 1{,}442$ that is better than that of Newton's method. Numerical experiments shown that our family is comparable to other third-order convergent methods in terms of iteration number.

Table 1. Comparison of various cubically convergent methods and Newton's method

	Method	x^*	IT	$\lvert f(x^*)\rvert$
$f_1,\ x_0 = 1$				
	(NMCH)	1.365230013414097	4	0
	(NM)	1.365230013414097	6	0
	(CHM)	1.365230013414097	5	0
	(HM)	1.365230013414097	5	0
	(KLWM1)	1.365230013414097	5	0
	(KLWM2)	1.365230013414096	4	1.07e-14
	(ZM)	1.365230013414097	5	0
	(CM1)	1.365230013414097	5	0
	(CM2)	1.365230013414097	5	0
$f_2,\ x_0 = 2$				
	(NMCH)	1.404491648215341	5	3.33e-16
	(NM)	1.404491648215341	6	3.33e-16
	(CHM)	1.404491648215341	5	3.33e-16
	(HM)	1.404491648215341	5	3.33e-16
	(KLWM1)	1.404491648215341	6	4.44e-16
	(KLWM2)	1.404491648215341	5	9.99e-16
	(ZM)	1.404491648215341	5	4.44e-16
	(CM1)	1.404491648215341	6	4.44e-16
	(CM2)	1.404491648215341	5	3.33e-16
$f_3,\ x_0 = -1{,}0$				
	(NMCH)	0.2575302854398608	4	0
	(NM)	0.2575302854398608	6	0
	(CHM)	0.2575302854398608	4	0
	(HM)	0.2575302854398608	4	0
	(KLWM1)	0.2575302854398607	5	0
	(KLWM2)	0.2575302854398608	4	0
	(ZM)	0.2575302854398607	5	0
	(CM1)	0.2575302854398607	6	0
	(CM2)	0.2575302854398607	6	0
$f_4,\ x_0 = 1{,}5$				
	(NMCH)	0.7390851332151607	4	0
	(NM)	0.7390851332151606	5	1.11e-16
	(CHM)	0.7390851332151607	4	0
	(HM)	0.7390851332151607	4	0
	(KLWM1)	0.7390851332151607	5	0
	(KLWM2)	0.7390851332151607	4	0
	(ZM)	0.7390851332151607	6	0
	(CM1)	0.7390851332151607	6	0
	(CM2)	0.7390851332151607	5	0
$f_5,\ x_0 = 1$				
	(NMCH)	-0.4428544010023886	5	0
	(NM)	-0.4428544010023885	8	0
	(CHM)	-0.4428544010023887	6	0
	(HM)	-0.4428544010023886	6	0
	(KLWM1)	-0.4428544010023886	7	0
	(KLWM2)	-0.4428544010023888	5	2.2e-16
	(ZM)	-0.4428544010023886	5	0
	(CM1)	-0.4428544010023886	5	0
	(CM2)	-0.4428544010023886	6	0

References

[1] S. Amat, S. Busquier, J. M. Gutiérrez, *Geometric constructions of iterative functions to solve non-linear equations*, J. Comput. Appl. Math. **157** (2003), 197–205.

[2] P. Butzer, F. Jongmans, *"P.L. Chebyshev": A guide to his life and work*, J. Approx. Theory **96** (1999). 111–138.

[3] P. L. Chebyshev, *Complete Collected Works*, Izdatel̓stvo Akad. Nauk SSR, vol. V, 1951.
[4] C. Chun, *Iterative methods improving Newton's method by the decomposition method*, Comput. Math. Appl. **50** (2005) 1559–1568.
[5] C. Chun, *A new iterative method for solving non-linear equations*, Appl. Math. Comput. **178** (2006), 415–422.
[6] C. Chun, *Some variants of Chebyshev-Halley methods free from second derivative*, Appl. Math. Comput. **191** (2007), 193–198.
[7] C. Chun, *Some second-derivative-free variants of Chebyshev-Halley methods*, Appl. Math. Comput. **191** (2007), 410–414.
[8] C. Chun, *Some third-order families of iterative methods for solving non-linear equations*, Appl. Math. Comput. **188** (2007), 924–933.
[9] J. A. Ezquerro, *A modification of the Chebyshev method*, IMA J. Numerical Analysis **17** (1997), 511–525.
[10] J. A. Ezquerro, M. A. Hernández, *A uniparametric Halley-type iteration with free second derivative*, Int. J. Pure Appl. Math. **6** (2003) 103–114.
[11] J. A. Ezquerro, M. A. Hernández, *On Halley-type iterations with free second derivative*, J. Comput. Appl. Math. **170** (2004), 455–459.
[12] W. Gautschi, *Numerical Analysis: An Introduction*, Birkhäuser, Boston, MA, 1997.
[13] M. Grau–Sánchez, J. M. Gutiérrez, *Zero-finder methods derived from Obreshkov's techniques*, Appl. Math. Comput. **215** (2009), 2992–3001.
[14] M. A. Hernández, *Second-derivative-free variant of the Chebyshev method for non-linear equations*, J. Optim. Theory Appl. **104** (2000), 501–515.
[15] J.–S. Kou, Y.–T. Li, X.–H. Wang, *On a family of second-derivative-free variants of Chebyshev's method*, Appl. Math. Comput. **181** (2006), 982–987.
[16] J.–S. Kou, Y.–T. Li, X.–H. Wang, *A uniparametric Chebyshev-type method free from second derivatives*, Appl. Math. Comput. **179** (2006), 296–300.
[17] J.–S. Kou, Y.–T. Li, X.–H. Wang, *Modified Halley's method free from second derivative*, Appl. Math. Comput. **183** (2006), 704–708.
[18] J.–S. Kou, Y.–T. Li, *Modified Chebyshev's method free from second derivative for non-linear equations*, Appl. Math. Comput. **187** (2007), 1027–1032.
[19] A. Melman, *Geometry and convergence of Euler's and Halley's methods*, SIAM Rev. **39** (1997), 728–735.
[20] A. M. Ostrowski, *Solution of Equations in Eucilidean and Banach Space*, third ed., Academic Press, New York, 1973.
[21] E. Schröder, *Über unendlich viele Algorithmen zur Auflösung der Gleichungen*, Math. Annal. **2** (1870), 317-365.
[22] J. F. Traub, *Iterative Methods for Solution of Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
[23] X. Zhou, *Modified Chebyshev-Halley methods free from second derivative*, Appl. Math. Comput. **203** (2008), 824–827.

(Recibido en abril de 2012. Aceptado para publicación en agosto de 2012)

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