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An Efficient **Liquidity Management For ATMs**

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Abstract

Bank liquidity management has become one of the main concerns of a bank during the financial crisis as liquidity shortages have put pressure on banks to diversity and improve their liquidity sources. Then, any programme of optimization of activities involving liquidity management are relevant issues as any significant improvement in cash management at the bank distribution channels may have positive effects in reducing liquidity tensions. Among these activities, there is cash management in ATMs. The purpose of this paper is to decide the optimum amount of money that will be placed in the ATM for minimizing opportunity costs as well as satisfying the customers uncertain demand. We propose a simple programme, easy to implement, which substitutes the actual method used by the branches (historical data).Our methodology is based on a simple dynamic model which describes and predicts the movements of the ATM cash flow, and whose equations have the capability of easily being changed if either the customers habits or the ATM rules of functioning change. The stochastic elements have been integrated in the model before applying suitable optimization programmes to all the costs involved. Some aspects of the Transaction Demand for the Cash are also present.

Keywords:

Liquidity management in (ATMs) banking, Dynamic mathematical model, Transactions demand for cash.

JEL classification:

C02, E50, O32

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Un manejo **eficiente del (dinero en) efectivo en ATMS**

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Resumen

El manejo del dinero en efectivo es una de las mayores preocupaciones de las entidades bancarias en esta crisis económica y financiera, puesto que la escasez de liquidez las obliga a diversificar y optimizar al máximo sus activos líquidos. Por tanto, todo programa de optimización de actividades que involucren manejo de activos líquidos se convierte en un tema de relevancia desde el momento en que cualquier mejora en los canales de uso y distribución de dinero, puede tener efectos positivos a la hora de reducir tensiones por la escasez de efectivo. Entre estas actividades está el manejo de efectivo en la red de cajeros automáticos de una entidad bancaria. Nuestro objetivo es optimizar la cantidad de efectivo que la entidad ha de depositar en sus cajeros a fin de minimizar costes al tiempo de satisfacer la demanda de efectivo de los usuarios. Así, el autor propone un programa simple y fácil de implementar, que sustituiría el actual método usado por las sucursales bancarias (histórico de datos).Nuestro método se basa en un modelo dinámico simple que describe (y puede predecir) los movimientos del flujo de efectivo del cajero, cuyas ecuaciones pueden ser fácilmente adaptadas a posibles cambios en los hábitos de los usuarios o en las reglas de funcionamiento de los cajeros, en caso de que dichos cambios se produjesen. Los elementos estocásticos se integran en el modelo antes de aplicar adecuados programas de optimización a los costes del proceso. Algunos aspectos de la Transacción de Demanda de Dinero están también presentes.

Palabras clave:

Manejo del efectivo en cajeros automáticos, modelo matemático dinámico, transacción de demanda de dinero.

1. Introduction

Along with risk, liquidity management represents the main reason for the existence of banks in the classical financial intermediation theory (see, for example, Allen and Gale, 2004; or Allen and Santomero, 1998). As far as liquidity is concerned, the basic management challenge in the standard framework is how to cover depositors' random consumption needs and how to set the subsequent deposit insurance mechanisms for these depositors.

Other studies have also dealt with liquidity management not just at the bank level but as a general feature of firm management. In this front, some papers have made use of stochastic and inventory theory to propose some models of firms' cash management. Among these approaches, Ferstl and Weissensteiner (2008) consider a cash management problem where a company with a given financial endowment and given future cash flows, minimizes the Conditional Value at Risk of final wealth using a lower bound for the expected terminal wealth written as a multi-stage stochastic linear program (SLP). In this framework, the company can choose between a riskless asset (cash), several default — and option — free bonds, an equity investment, and rebalances the portfolio at every stage. Other authors analyze the problem of cash management as an application of Operational Research by means of some stochastic programming models: Castro (2009) defines and solves two short-term and one mid-term models for ATMs.

Bank liquidity management has become one of the main concerns of a bank during the financial crisis as liquidity shortages have put pressure on banks to diversity and improve their liquidity sources. Thus, all those activities involving liquidity management are relevant issues as any significant improvement in cash management at the bank distribution channels may have positive effects in reducing liquidity tensions. Among the activities involving cash management in banking, there are cash management in ATMs and branches or compensation of credit card transactions. This paper focus on cash efficiency on ATMs.

Although the introduction of ATMs (Automatic Teller Machines) and other technological innovations has reduced cash management costs, there is still a pressing need to optimize resources, imposed by the high competition among bank institutions in the present scenario of economic and financial crisis. In this sense, this paper makes a proposal for the optimization of cash at ATM level. Actually, the 21st century has been recognized by the financial experts as the ATM age. This first ATM, known as DAC (De La Rue Automatic Cash System) was the starting point of a banking services revolution, making real the philosophy of the selfservice 24 hours per day. In addition it changed forever the relationship between the cash and billion of people around the world, who don't need anymore to keep great amounts of money at home in order to have it available. Forty years after this first DAC, a total of 1,3 millions of ATM terminals are distributed all world wide. In particular, Spain is the first country in number of ATM per habitant in Europe, and the second of the world after Japan.

From the TNS Consulting report, 46% of the customers prefers to operate through ATM facing 26% that rather go for a personal attention inside the office. Other sources point out that Spanish customers turn to ATM an average of 83 times per year, more that the double of the times they required for the branch personal attention. All these habits (customers behavior) has been amply addressed in several ATM reports. However, as far as we know, none of them models the customers habits into mathematical terms in order to predict their movements.

The crucial decision for the bank is what amount to daily maintain in that account given the overall sum to be charged. This problem is tried to be solved by the branches using historical data: basically the branch registers both the initial ATM quantity in some particular day (workable, weekend, holidays, etc) and the result obtained (i.e., success or failure). Thus, the branch imitates the ATM amounts that success.

This problem can be related with a classic issue: the transaction demand for the cash, which began with Baumol (1952) and Tobin (1956), and more recently with Álvarez and Lippi (2009). Basically, the transaction demand for the cash consists of managing an inventory of cash holdings: the decision maker holds two distinct types of assets, one asset which bears interest at a given rate, and a noninterest bearing asset where periodic receipts of income and expenditures are made. Transfers of funds between the two accounts are permissible but at a cost (transfer cost). They are involved other costs of different nature (opportunity costs) derived from the fact that funds into the noninterest bearing asset are loosing money while they are not into the interest bearing portfolio, as well as many others opportunity costs that includes the time spent waiting at a teller's window or in making decisions about purchases and sales, see Miller and Orr (1966).

One of the more illustrative example of the transaction demand for the cash is that of a family which has two different current accounts: one for the savings (bearing an interest) and a second one, dedicated to paying the periodic expenditures. Obviously, the second current account must be refilled from time to time to face the periodic payments. The present work can be included in this theoretical framework if we consider the branch of some bank having this "two-asset" setting: the ATM of the branch as the asset which is not bearing interest and where periodic expenditures are made, while the asset which bears interest at given rate would be conformed by the wide range of bank different products as Treasure bills, certificates of deposits, commercial paper and any other money bank instruments.

In this context, our objective is then to determine the quantity of money that must be placed into the ATM according to a future unknown demand. For this, we propose a simple programme of cash efficiency for AMT's, very easy to implement, which substitutes the actual method used by the branches based on historical data. The methodology we propose is based on the construction of a simple (dynamic) mathematical model which describes and predicts the movements of the ATM cash flow, whose equations have the capability of easily being changed if the customers habits change. The stochastic elements presented in the problem have been also captured and integrated in the model before applying suitable optimization programmes to all the costs involved in cash movements.

Apart from its simplicity, the more interesting point of this model is its capability of easy changing these equations if either the customers habits or the ATM rules of functioning change, by modifiying/adding more conditions as well as more significant variables if necessary.

The structure of this paper is as follows: section 2 starts by analyzing the real method the branch uses to provide enough service to the ATM customers along the day with regard to the determination of the initial quantity of money (x_0) which must be placed into the ATM to satisfy the customers unknown demand.

The second target of this section is to model mathematically our problem in order to make, in later sections, a deeper study of the cash management problem from the optimization point of view.

Section 3 is devoted to capturing the randomness of the problem by means of some stochastic processes whose presence in the mathematical model allows us to demonstrate some useful properties on the quantity x_0 as well as a formulae to calculate x_0 in real cases: the strength of this formulae lies on its extreme simplicity as well as on its easy implementation in practice. The above formulae drives us in a natural way to complete the constraints to the optimization problem which underlies the central problem of this paper: this optimization problem consists of minimizing the costs of the bank in all this process. Section 4 is devoted to considering and resolving this problem.

In section 5 we review the mathematical model made in section 2, which reproduced the process of customers making withdrawals. In a first stage, we resolve this model obtaining the same solution to that of the branch by its historical data. In a second step, we solve it after adding more real circumstances in the form of more conditions to the model.

Finally, in section 6, the model of former section is enforced by examining the different possibilities of making predictions. In the light of this, we obtain the most important formulaes to predict the initial amount of ATM money for optimizing its activity.

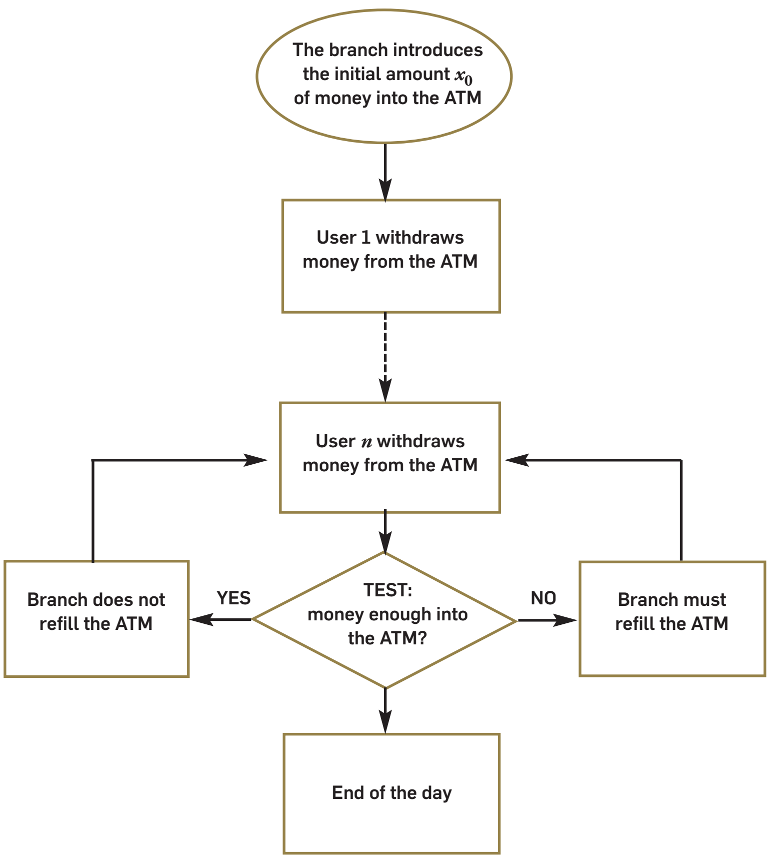
2. The Dynamic of the Model

We start this section by analyzing the real method the branch uses to provide enough ATM service to the customers along the day. The branch method consist of two basic points. The first one (and the key decision for the branch) is what amount x_0 to daily maintain in that account given the overall sum to be charged. The branch predicts x_0 from historical data of customers behavior in similar days to the referred one, and places x_0 into the ATM.

The second point concerns with the regulation of the ATM money for keeping it suitable (suitable means to fluctuate between some levels for giving enough service to the users): the branch staff check the remaining amount of ATM money at some key instant during the day in order to refill the ATM if necessary.

The branch method is reflected in the following picture:

Figure 1. **The Dynamic of the ATM Day**



As x_0 is the initial amount of money each day, let x_t be the amount of money in the ATM at the instant of time t. We will use discrete notation for x_t (which represents different amounts of money in different discrete points of time) instead of $x(t)$, since both the ATM money recharge by the branch as well as the ATM money extraction by the customers, occur in discrete instants of time. Note also that $x_t > 0$, for all t.

- How does the bank keep suitable the amount of money in the ATM? To offset the amounts of money that the customers withdraw from the ATM, the bank has to recharge it in order to keep it *suitable*. Let us write with mathematical terms the process of making withdrawals and of refilling the ATM.

  - Users withdraw money from an amount x_{t+a} , where $t+a$ represents a later moment of time to t. As the amount withdrawn is probably smaller than the total, this user's behavior can be written as

$$-Ax_{t+a}, \quad 0<A<1 \ .$$

  - The bank recharges the ATM, but never completing the initial total amount. This can be expressed as

$$+Bx_t, \quad 0<B<1 \, .$$

- Which is the process of customers making withdrawals from de ATM? Let us start from x_t, the amount of money at some instant t. After this instant of time t in which someone makes a withdrawal, the remaining amount in the ATM at the next instant $t+1$, is

$$x_{t+1} = x_t - A_1 x_t = (1-A_1)x_t, \quad 0<A_1<1 \, .$$

In a second instant of time $t+2$ in which someone makes a withdrawal, the remaining amount in the ATM is

$$x_{t+2} = x_{t+1} - A_2 x_{t+1} = (1-A_2)x_{t+1}, \quad 0<A_2<1 \, .$$

By substituting x_{t+1} in the former expression, we have

$$\begin{aligned} x_{t+2} &= x_{t+1} - A_2 x_{t+1} = (1-A_2)x_{t+1} = \\ &= (1-A_1)(1-A_2)x_t, \quad 0<A_1, A_2<1 \, . \end{aligned}$$

Consequently, after r instants of time, the remaining amount of money is

$$\begin{aligned} x_{t+r} &= \overbrace{(1-A_1)(1-A_2)\ldots(1-A_r)}^{K_r} x_t = \\ &= K_r x_t . \\ & \quad 0<A_1, A_2, \ldots, A_r<1, \\ & \quad \forall_r > 0 . \end{aligned} \tag{1}$$

Note that the counter of instants is a counter of ATM customers as well.

Let us simplify the above idea of users making withdrawals from the ATM, and also, analyze the elements that appear in the above formulae. Considering only two consecutive moments of time, the formulae is

$$x_n = (1-A_n)x_{n-1} \tag{2}$$

As x_n denotes the amount of money that remains at the ATM after the nth-user has made a withdrawal, and A_n represents the portion of x_{n-1} that the nth-user withdraws, this expression condenses both the customers behavior along this process as well as let explicit the ATM remaining money after each customer. Others consideration around this formulae are:

- The number of A_i's reflects the number of times the customers make withdrawals.

- As we said, each A_i's represents the part of money that has been taken out, so this is why each A_i's is $0 < A_i < 1$. On the other hand, given this portion A_i, the real amount of money (in euros) that has been withdrawn by the ith-customer is given by

$$A_i^* = A_i \cdot x_{i-1}.$$

Of course, all the referred information — the customers behavior — is known by the branch of the bank.

If we solve the simple difference equation (2), the solution does not clear the situation too much:

$$x_n = \Pi_{j=1}^{n-1}(1-A_{j+1}) \quad \text{or} \quad x_n = \Pi_{j=1}^{n}(1-A_i)x_0.$$

We operate then in the expression (2) in order to transform it as

$$\begin{aligned} x_n &= (1-A_n)x_{n-1} = \\ &= x_{n-1} - A_n x_{n-1} = \\ &= x_{n-1} - A_n^* \Rightarrow \end{aligned}$$

$$x_n = x_{n-1} - A_n^* \tag{3}$$

The solution to this difference equation is the logical one:

$$x_n = x_0 - \sum_{i=1}^{n} A_i^* .$$

That is, if the branch wants to know the remaining money after n users had withdrawn money (this is x_n), they logically have to subtract to x_0, the sum of all the quantities withdrawn during the day. Recall that the branch checks the remaining amount of ATM money at some key instant during the day in order to refill the ATM if necessary. Let x_z be the remaining money in the ATM at this checking instant.

Hence, the above expression is concerned with this second point of the branch method, that of the regulation of the ATM money after checking it at some instant of time. Indeed, since each x_n can be now determined by the above expression, also this can be done for the x_z at the checking point time. Thus, this could help the branch to readjust the ATM money by comparing the real amount at the check point with that obtained by this expression.

Let us now focus on the first point of the branch method: from the above expression, inverting the problem, if we want to know how much money the ATM must have at the beginning of the day in order to give service to the customers, this is

$$x_0 = x_k + \sum_i A_i^*$$

where x_k is the remaining of money the branch plans to leave at the ATM at the end of the day (for instance, x_k can be taken under the minimum amount one can extract of the ATM). As a result, if the branch plans to leave the ATM empty ($x_k=0$) and can estimate how many customers will make use of the ATM during the all day (let N be this number), x_0 is more accurate by being taken as

$$x_0 = \sum_{i=1}^{N} A_i^* \, . \tag{4}$$

From this expression, the initial amount of money, x_0 is a function on both variables N (total number of users during the day) and A_i^* (quantities of money the users withdraw).

3. The Two Poisson Processes

This section is devoted to capturing the randomness of our problem by means of some stochastic processes.

Through the analysis of our problem, some stochastic elements have been appearing: first of all, the number of ATM users during each day, that the branch of the bank deals with using its historical data.

In former sections, we called N to the number of customers which make use of the ATM during the all day. Let us recall that often, the arrival process of customers can be described by a Poisson process. Mathematically the process is described by the so called counting process (point process) $N(t)$ or N_t. The counter tells the number of arrivals that have occurred in the interval $(0,t)$, that is,

$$N_t = \text{number of ATM users in the interval } (0,t).$$

From this definition and considering a day as the unit of time, note that $N = N_1$.

For that reason, if N_t is a (Poisson) counter process of parameter, say, λ, some of their properties are

- the number of arrivals to the ATM in an interval of length t has a Poisson distribution with parameter $\lambda \cdot t$; that is, $P[N_t = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ measures the probability of n ATM customers in the time t.

- The mean and variance of N_t are

$$E[N_t] = \lambda \cdot t \quad \text{and} \quad var[N_t] = \lambda \cdot t\,.$$

Particularly, since $\lambda = E[N]$, it follows that the rate of the Poisson process N, λ, is the average of withdrawals made of ATM customers in a day.

Apart from the number of ATM customers per day, other random element of our problem is the amount of each withdrawal made by each customer. We capture all these quantities A_i^* by means of a compound Poisson process.

A compound Poisson process is a (random) stochastic process with jumps. The jumps arrive randomly according to a Poisson process and the size of the jumps is also random, with a specified probability distribution. In our context we define the withdrawal process, parameterized by certain rate λ, as the compound Poisson process given by

$$X_t = \sum_{i=1}^{Nt} A_i^*$$

where A_i^* denotes the amount of money (in euros) that has been withdrawn by the ith-customer, viewed as independent and identically distributed (i.i.d.) random variables. Then, X_t is the total amount that have been withdrawn by the N_t ATM users until the moment t of the day.

As proved in section 2,

$$x_n = x_0 - \sum_i^n A_i^*\,, \text{ for all } n\,.$$

Hence, the relationship of the new *variable* X_t with the former ones, is the following:

$$X_t = \sum_{i=1}^{Nt} A_i^* = x_0 - x_{Nt}.$$

Particularly, if we assume that the ATM gets empty at the end of the day (or it has a worthless quantity of money, which we approximate by zero), then x_0 coincides with the withdrawal process at $t=1$:

$$x_0 = \sum_{i=1}^{N} A_i^* = X_1.$$

At this point, recall that the mean of a compound Poisson process can be calculate by means of the mean of one of the i.i.d. variables, as

$$E[X_t] = \lambda \cdot t \cdot E[A_i^*],$$

where λ is the rate of the Poisson process N_t.

Consequently, for $t=1$, we have

$$\begin{aligned} E[X_1] &= \lambda \cdot E[A_i^*] = \\ &= \begin{array}{c}\text{Average of number of} \\ \text{withdrawals per day}\end{array} \cdot \begin{array}{l}\text{Average of quantity withdrawn} \\ \text{from ATM per day}\end{array}\end{aligned}$$

While $x_0 = X_1$, if we approximate X_1 by its mean, one mathematical proceed to calculate the quantity x_0 to be put into the ATM at the beginning of the day, is

$$x_0 = \begin{array}{c}\text{Average of number of} \\ \text{withdrawals per day}\end{array} \cdot \begin{array}{l}\text{Average of quantity withdrawn} \\ \text{from ATM per day}\end{array}$$

This provides a simple method of estimating the initial amount of money x_0 which could substitute the old method of the branches of the bank, based on success/failure of x_0 for similar days. Also this method is very easy to be implemented in branches, as (part of) a set of optimization instructions from the cash central to their branches.

4. The Underlying Optimization Problem

The main objective of this paper is to formulate a mathematical method which helps branches to decide the optimum amount of money that will be placed into the ATM for satisfying the daily customers demand.

However, along this paper, it has been reminded clear one key question, that should be considered on parallel form to the calculus of x_0: if it is crucial for this study that x_0 must be big enough to provide service to the ATM customers, it is important as well that x_0 must be small enough to avoid losses to the bank derived from the fact of holding too much cash in the ATM, while the surplus of x_0 should be generating money if it should be deposited into other bank products.

This idea suggests the necessity of optimizing the banks objective function, as well as still considering the main question of the paper (i.e., x_0 must give enough service to the customers). This leads us in a natural way to the following optimizing problem with constraints:

$$\begin{array}{rl} \text{Minimize:} & \text{banks objective function} \\ \text{s.a.} & x_0 \text{ must give enough service to the customers} \end{array}$$

Let us start with the banks objective function. For this, we follow the standard practice in inventory theory by assuming that the bank seeks to minimize the long-run average cost of managing the cash balance under some policy of simple form.

Specifically, we assume that the cash balance in ATM is allowed to fluctuate until it reaches either the lower bound, zero, or the upper bound, x_0. Let x_z be the checking point of the ATM cash level which indicates either branch must refill the ATM or not (see Figure 1.)

On these two variables, x_0 and x_z, the banks objective function $\varepsilon(x_0, x_z)$ is

$$\varepsilon(x_0, x_z) = \gamma \frac{A}{(x_0 - x_z)x_z} + \nu \frac{x_0 + x_z}{3},$$

This function is constructed with separated addends, one for transfers costs and the other for opportunity costs:

$$\varepsilon(x_0, x_z) = \underbrace{\gamma \frac{A}{(x_0 - x_z)x_z}}_{\textit{transfer costs}} + \underbrace{\nu \frac{x_0 + x_z}{3}}_{\textit{opportunity costs}},$$

where γ is the costs per transfer, and $\frac{A}{(x_0 - x_z)x_z}$ represents the total number of transfers, while ν is the daily rate of interest earned on the portfolio, and $\frac{x_0 + x_z}{3}$ represents the average daily cash balance on ATM, see Miller and Orr (1966).

As for the constraint of the optimization problem, i.e., x_0 must give enough service to the customers, this has been written into mathematical terms in the former section, simply considering that $x_0 = E[X_1]$.

Once both the bank objective function and the constraint are specified, the optimization problem turns out to be the following:

$$\text{Minimize}: \gamma \frac{A}{(x_0 - x_z)x_z} + \nu \frac{x_0 + x_z}{3}$$

$$\text{s.a.} \qquad x_0 = E[X_1]$$

To resolve this problem, we effect the following change of variables:

$$x = x_0 - x_z$$
$$y = x_z$$

Accordingly, the optimization problem turns out to be

$$\text{Minimize}: \gamma \frac{A}{xy} + \frac{\nu}{3}(x + 2y)$$

$$\text{s.a.} \qquad x + y = E[X_1]$$

which can be solved by substituting one of the new variables from the constraint into the objective function: if we do so with $y = E[X_1] - x$, the solutions to the problem are the roots of the following quartic equation:

$$x^4 - 2E[X_1]x^3 + E[X_1]^2x^2 - \frac{6\gamma A}{\nu}x + \frac{3\gamma AE[X_1]}{\nu},$$

which yields to the desired upper bound x_0, as well as the return point of the cash level, x_z.

Note that this return point, x_z, is in the reality the level of ATM money which is supervised by the branch staff after they close the branch at the middle of the day. Since usually branches do not open after this supervision, the ATM cash level must be high enough till next day. Consequently, it is crucial that the return point, x_z, has been adjusted properly.

5. The Mathematical Model Again. Resolution

In the former section, we interpreted the actual process the branch uses to determine x_0, by means of the equation that describes the dynamic of the model, (2), as the central equation. The conclusions from our model concord with that of the branch real process. In this section, we shall improve our model with equations that simulates the ATM real rules of working. Thus the model will be more accurate in their predictions.

The more interesting point of this model is its capability of changing these equations if either the customers habits or the ATM rules of functioning change, by modifiying/ adding more conditions as well as more significant variables.

For this, we modify the starting difference equation (3), to the equivalent one

$$x_{n-1} = x_n + A_n^* \,.$$

As a second equation which adds to the model the information concerning to the ATM working rules, we consider

$$A_n^* = c_{n-1} A_{n-1}^*$$

Why? Because it is impossible to predict the next amount withdrawn from the ATM, A_n^* from the former one, A_{n-1}^*. Indeed, A_n^* can be bigger, smaller o equal to A_{n-1}^* but in all the cases, these quantities must be multiple of 10 euros:

$$\left.\begin{array}{l} A_{n-1}^* = 10a_{n-1}^* \\ A_n^* \;\;\, = 10a_n^* \end{array}\right\} \Rightarrow A_n^* = \underbrace{\frac{a_n^*}{a_{n-1}^*}}_{c_{n-1}} A_{n-1}^*$$

Let us denote c_{n-1} to the quantity $\dfrac{a_n^*}{a_{n-1}^*}$, as indicated in the above formulae. The use of the notation c_n to represent the multiple between quantities A_n^* and A_{n-1}^* reveals that this multiple does depend on n. In fact, this number, c_n, can be consider as a discrete random variable, that is, one which may take on only a countable number of distinct positive values. Note also that each $c_n \in \mathbb{Q}$.

Hence, we arrive to the following homogeneous linear difference equations system, where the second one is stochastic (or that of variable coefficients at the variable A_n^*):

$$\left\{\begin{array}{l} x_n \;\; = x_{n-1} - A_n^* = x_{n-1} - c_{n-1} A_{n-1}^* \\ A_n^* = c_{n-1} A_{n-1}^*, \end{array}\right. \tag{5}$$

Thus, using the transition matrix

$$A(n) = \begin{pmatrix} 1 & -c_{n-1} \\ 0 & c_{n-1} \end{pmatrix} \tag{6}$$

the central system can be expressed as

$$\begin{pmatrix} x_n \\ A_n^* \end{pmatrix} = \begin{pmatrix} 1 & -c_{n-1} \\ 0 & c_{n-1} \end{pmatrix} \begin{pmatrix} x_{n-1} \\ A_{n-1}^* \end{pmatrix} \tag{7}$$

with significant variables both x_n and A_n^*. Through the matrix $A(n)$, it may also be written in the following equivalent matricial form

$$X_n = A(n)X_{n-1},$$

where X_n is $X_n = \begin{pmatrix} x_n \\ A_n^* \end{pmatrix}$ as usual.

Note that the variable coefficient c_n, viewed as discrete random variable could be replaced by a suitable RNG, random numbers generator if necessary.

5.1. Solving the Model

Considering the above system as a homogeneous system with variable coefficients, written as

$$X_n = A(n)X_{n-1}, \quad \text{where } X_n = \begin{pmatrix} x_n \\ A_n^* \end{pmatrix}.$$

Thus, the general solution is

$$\begin{aligned} X_k = \begin{pmatrix} x_k \\ A_k^* \end{pmatrix} &= \Pi_{j=r<k}^{k-1} A(j) \begin{pmatrix} x_r \\ A_r^* \end{pmatrix} \\ &= A(k-1)A(k-2)\dots A(r-1)A(r)\begin{pmatrix} x_r \\ A_r^* \end{pmatrix} \end{aligned}$$

due to, as the matrix product is non commutative, it is obliged to precise in which order we do the product. We shall take $r=2$ since $A(1)$, which involves the coefficient c_0, is meaningless. Hence, the solution to system (7) is

$$\begin{pmatrix} x_k \\ A_k^* \end{pmatrix} = A(k-1)A(k-2)\dots A(3)A(2)\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix}$$

To develop this matrix product, let us begin with the simplest case, namely

$$\begin{aligned} A(3)A(2) &= \begin{pmatrix} 1 & -c_2 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} 1 & -c_1 \\ 0 & c_1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & -\frac{a_3^*}{a_2^*} \\ 0 & \frac{a_3^*}{a_2^*} \end{pmatrix}\begin{pmatrix} 1 & -\frac{a_2^*}{a_1^*} \\ 0 & \frac{a_2^*}{a_1^*} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{a_2^*}{a_1^*} - \frac{a_3^*}{a_2^*}\frac{a_2^*}{a_1^*} \\ 0 & \frac{a_3^*}{a_2^*}\frac{a_2^*}{a_1^*} \end{pmatrix} \end{aligned}$$

Thus, the matrix product $A(3)A(2)$ turns out to be

$$A(3)A(2) = \begin{pmatrix} 1 & -\frac{a_2^*+a_3^*}{a_1^*} \\ 0 & \frac{a_3^*}{a_1^*} \end{pmatrix}$$

It is easy to check that next step $A(4)A(3)A(2)$ is

$$A(4)A(3)A(2) = \begin{pmatrix} 1 & -\frac{a_2^*+a_3^*+a_4^*}{a_1^*} \\ 0 & \frac{a_4^*}{a_1^*} \end{pmatrix}$$

and so forth

$$A(k-1)A(k-2)\ldots A(3)A(2) = \begin{pmatrix} 1 & -\frac{a_2^*+a_3^*+\ldots+a_{k-1}^*}{a_1^*} \\ 0 & \frac{a_{k-1}^*}{a_1^*} \end{pmatrix}$$

From this point on, the method follows by isolating each x_k and A_k^*. These are

$$\begin{cases} x_k = x_2 - A_2^* \dfrac{a_2^*+a_3^*+\ldots+a_{k-1}^*}{a_1^*} \\ A_k^* = A_2^* \dfrac{a_{k-1}^*}{a_1^*} \end{cases}$$

Apart from the information that gives second equation, from the first one, by putting x_2 as $x_0 - A_1^* - A_2^*$ and isolating x_0, we finally conclude that

$$x_0 = A_1^* + A_2^* + x_k + A_2^* \frac{a_2^*+a_3^*+\ldots+a_{k-1}^*}{a_1^*} \tag{8}$$

for $k = 0, 1, 2, \ldots$.

If N represents the instant of the day that the branch decides to get closed the ATM, then x_N denotes the remaining quantity of money which is left at the ATM at the end of this day. This quantity x_N may also be decided by the branch to be 0 or alternatively, other quantity (x_N can be taken under the minimum amount one can withdraw from the ATM): if they decide to get the ATM completely empty, x_N must be 0. Introducing this *branch decision* at the former equation (8), the initial amount x_0 turns out to be

$$x_0 = A_1^* + A_2^* + A_2^* \frac{a_2^*+a_3^*+\ldots+a_{N-1}^*}{a_1^*} \tag{9}$$

6. The Dynamic Model for the Real World: How to Determine A_n^*

If in the previous section we introduce a simple model to represent one ATM day, in this section we attempt to bring over the model to the reality. In regards of simplicity, we can say that the above formulae shows the initial quantity x_0 as a function of both A_n^* and N,

$$x_0 = f(A_n^*, N).$$

Even more, as N (the instant of time of the day that the ATM gets empty) is a branch decision (as well as how much money remains at the ATM after the last user), we can even assume that x_0 depends only on quantities A_n^* that users withdraw the ATM, i.e.,

$$x_0 = f(A_n^*).$$

Hence, to be closer to x_0 the point is how to calculate in a accurate form these quantities A_n^*.

For this, let us denote $E(A_n^*)$ the expected value of A_n^* at each instant n. The branch calculates these impacts by estimating them in view of its historical data, that is, they use the same items they know for a similar day (recall that *similar* means similar characteristic, i.e., working day versus bank holiday etc).

In this paragraph we analyze some different ways to calculate $E(A_n^*)$. In order to give a complete vision, we shall examine all the known expectations, even the non realistic ones. Let us note that this range of possibilities could be augmented by defining ad hoc other kind of expectations if necessary (ad hoc means in this context depending on the features of each branch, like size or location).

Then, the common way to procedure will be the direct substitution of items A_n^* by the expected ones, $E(A_n^*)$. Namely, we shall replace the original system (5)

$$\begin{cases} x_n &= x_{n-1} - A_n^* \\ A_n^* &= c_{n-1} A_{n-1}^*, \end{cases}$$

by the system

$$\begin{cases} x_n &= x_{n-1} - E(A_n^*) \\ E(A_n^*) &= c_{n-1}\, E(A_{n-1}^*), \end{cases} \tag{10}$$

6.1. Rational Expectations

One way to calculate the impacts $E(A_n^*)$ is by considering no possible estimation error, that is, taking them as equal to the real ones, A_n^*. This is the case of *rational expectations*, the forecast is equal to the actual impact:

$$E(A_n^*) = A_n^* ,$$

This case has been solved in the former section, bringing us the formulae obtained in (9).

6.2. Naïve Expectations

In the case of naïve expectations, the forecast is equal to the previous impact:

$$E(A_n^*) = A_{n-1}^* \ .$$

By direct substitution in the system

$$\begin{cases} x_n & = \ x_{n-1} - E(A_n^*) \\ E(A_n^*) & = \ c_{n-1}\, E(A_{n-1}^*), \end{cases}$$

we would get

$$\begin{cases} x_n & = \ x_{n-1} - A_{n-1}^* \\ A_{n-1}^* & = \ c_{n-1}\, A_{n-2}^* \end{cases}$$

However, to be faithful to the model, the second equation, which expresses the relationship between two consecutive quantities must be still the same. Hence, the key system at this paragraph is

$$\begin{cases} x_n & = \ x_{n-1} - A_{n-1}^* \\ A_n^* & = \ c_{n-1}\, A_{n-1}^* \end{cases}$$

Note also that, by shifting the time (or user number) subscripts by one unit, A_n^* denote now the quantity of money that n+1-user withdraws from the ATM. Due to this, A_0^* , which was meaningless, represents now the first quantity of money that is withdrawn from the ATM, corresponding to the first user in the day. Continuing this argument, A_1^* represents the second quantity corresponding to the 2-user in the day, and so on.

Let us solve the main homogeneous linear difference equations system:

$$\begin{cases} x_n & = \ x_{n-1} - A_{n-1}^* \\ A_n^* & = \ c_{n-1}\, A_{n-1}^* \end{cases} \qquad (11)$$

This may also be written in the following equivalent matricial form

$$X_n = A(n)X_{n-1}, \quad \text{where } A(n) \text{now is } A(n) = \begin{pmatrix} 1 & -1 \\ 0 & c_{n-1} \end{pmatrix}$$

To solve this system, we can proceed as for (5). Then, the solution is

$$\begin{pmatrix} x_k \\ A_k^* \end{pmatrix} = A(k-1)A(k-2)\ldots A(2)A(1) \begin{pmatrix} x_1 \\ A_1^* \end{pmatrix}$$

By developing the above matrix product, we find out that

$$A(k-1)A(k-2)\dots A(2)A(1) = \begin{pmatrix} 1 & -1-c_0-c_0c_1-c_0c_1c_2-\dots-c_0c_1c_2\dots c_{k-2} \\ 0 & \dfrac{a_k^*}{a_1^*} \end{pmatrix}$$

where it is easy to check that

$$-1-c_0-c_0c_1-c_0c_1c_2-\dots-c_0c_1c_2\dots c_{k-2} = -1-\frac{a_1^*+a_2^*+\dots+a_{k-1}^*}{a_0^*}$$

so the above matrix is

$$A(k-1)A(k-2)\dots A(2)A(1) = \begin{pmatrix} 1 & -1-c_0-c_0c_1-c_0c_1c_2-\dots-c_0c_1c_2\dots c_{k-2} \\ 0 & \dfrac{a_k^*}{a_1^*} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1-\dfrac{a_1^*+a_2^*+\dots+a_{k-1}^*}{a_0^*} \\ 0 & \dfrac{a_k^*}{a_1^*} \end{pmatrix}$$

From this point on, the procedure of isolating each x_k and A_k^* is the same we used before:

$$\begin{cases} x_k = x_1 - A_1^* - A_1^* \dfrac{a_2^*+a_3^*+\dots+a_{k-1}^*}{a_1^*} \\ A_k^* = A_1^* \dfrac{a_k^*}{a_1^*} \end{cases}$$

As $x_1 = x_0 - A_0^*$, isolating x_0, we finally conclude that

$$x_0 = A_0^* + A_1^* + x_k + A_1^* \frac{a_2^*+a_3^*+\dots+a_{k-1}^*}{a_1^*} \tag{12}$$

for $k = 0, 1, 2\dots$. Taking into account the numbered-user subscripts shifted, we obtain the same result as in the former paragraph, namely,

$$x_0 = A_0^* + A_1^* + A_1^* \frac{a_2^*+a_3^*+\dots+a_{N-1}^*}{a_1^*} \tag{13}$$

6.3. Adaptive Expectacions

A good deal of papers generalize naïve expectations into adaptative expectations. In the case of adaptative expectations, the forecast is equal to the convex combination of the previous impact and the previous forecast:

$$E(A_n^*) = \lambda A_{n-1}^* + (1-\lambda) E(A_{n-1}^*), \quad 0<\lambda\geq 1. \tag{14}$$

Then, the original system (5)

$$\begin{cases} x_n = x_{n-1} - c_{n-1} A_{n-1}^* \\ A_n^* = c_{n-1} A_{n-1}^* , \end{cases}$$

can now be enlarged with an additional equation:

$$\left\{\begin{array}{lcl} x_n & = & x_{n-1} - c_{n-1}A^*_{n-1} \\ A^*_n & = & c_{n-1}A^*_{n-1} \\ E(A^*_n) & = & \lambda A^*_{n-1} + (1-\lambda)E(A^*_{n-1}), \end{array}\right. \tag{15}$$

Hence, we have 3 difference equations as well as 3 variables: x_n , A^*_n and $E(A^*_n)$,

$$\left\{\begin{array}{lcl} x_n & = & x_{n-1} \; - c_{n-1}A^*_{n-1} \\ A^*_n & = & \qquad c_{n-1}A^*_{n-1} \\ E(A^*_n) & = & \qquad\quad \lambda A^*_{n-1} + \; (1-\lambda)E(A^*_{n-1}), \end{array}\right. \tag{16}$$

where the transition matrix is now the matrix

$$B(n) = \begin{pmatrix} 1 & -c_{n-1} & 0 \\ 0 & -c_{n-1} & 0 \\ 1 & \lambda & 1-\lambda \end{pmatrix} = B(n, \lambda)$$

i.e., the block partition matrix

$$B(n) = \left(\begin{array}{c|c} A(n) & 0 \\ \hline F(\lambda) & 1-\lambda \end{array}\right)$$

where the matrix $A(n)$ is the transition matrix (6),

$$A(n) = \begin{pmatrix} 1 & -c_{n-1} \\ 0 & c_{n-1} \end{pmatrix},$$

$F(\lambda)$ is the row matrix $F(\lambda) = (\; 0 \quad \lambda \;)$, 0 represents the column matrix $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $1-\lambda$ is an scalar.

The procedure to solve this system is the same as in former paragraph, namely:

$$X_k = \begin{pmatrix} x_k \\ A^*_k \\ E(A^*_k) \end{pmatrix} = \Pi^{k-1}_{j=r<k} B(j) \begin{pmatrix} x_r \\ A^*_r \\ E(A^*_r) \end{pmatrix}$$
$$= B(k-1)\,B(k-2)\ldots B(3)\,B(2) \begin{pmatrix} x_2 \\ A^*_2 \\ E(A^*_2) \end{pmatrix}$$

It is not difficult to check out that

$$B(3)\,B(2) = \left(\begin{array}{c|c} A(3)A(2) & 0 \\ \hline F(\lambda)A(2)+(1-\lambda)F(\lambda) & (1-\lambda)^2 \end{array}\right)$$

while

$$B(4)\,B(3)\,B(2) = \left(\begin{array}{c|c} A(4)A(3)A(2) & 0 \\ \hline \begin{array}{r} F(\lambda)A(3)A(2)+ \\ (1-\lambda)F(\lambda)A(2)+ \\ (1-\lambda)^2F(\lambda) \end{array} & (1-\lambda)^3 \end{array}\right)$$

and, in general,

$$B(k-1)\,B(k-2)\ldots B(3)\,B(2) = \left(\begin{array}{c|c} A(k-1)A(k-2)\ldots A(3)A(2) & 0 \\ \hline \begin{array}{c} F(\lambda)A(k-2)\ldots A(3)A(2)+ \\ (1-\lambda)F(\lambda)(k-3)\ldots A(3)A(2)+ \\ \vdots \\ +(1-\lambda)^{k-3}F(\lambda) \end{array} & (1-\lambda)^{k-2} \end{array}\right)$$

with the convention that the matrix $A(1)$ is the identity matrix. The left down block can also be expressed as

$$\sum_{j=0}^{k-3}(1-\lambda)^j F(\lambda)\,A(k-1-j-1)\,A(k-1-j-2)\ldots A(2).$$

Then, we solve the system in the usual way:

$$\begin{pmatrix} x_k \\ A_k^* \\ E(A_k^*) \end{pmatrix} = \left(\begin{array}{c|c} A(k-1)A(k-2)\ldots A(3)A(2) & 0 \\ \hline \begin{array}{c} F(\lambda)A(k-2)\ldots A(3)A(2)+ \\ (1-\lambda)F(\lambda)(k-3)\ldots A(3)A(2)+ \\ \vdots \\ +(1-\lambda)^{k-3}F(\lambda) \end{array} & (1-\lambda)^{k-2} \end{array}\right) \begin{pmatrix} x_2 \\ A_2^* \\ E(A_2^*) \end{pmatrix}$$

and due to the block partition, we obtain that

$$E(A_k^*) = \underbrace{\begin{array}{c} F(\lambda)A(k-2)\ldots A(3)A(2)+ \\ (1-\lambda)F(\lambda)(k-3)\ldots A(3)A(2)+ \\ \vdots \\ +(1-\lambda)^{k-3}F(\lambda) \end{array}}_{\textit{row matrix } 1\text{x}2} \begin{pmatrix} x_2 \\ A_2^* \end{pmatrix} + (1-\lambda)^{k-2}E(A_2^*)$$

Hence, this expression reveals that the expected value of the k-user depends on the previous ones, and in a very significant way, it depends on the first one, A_2^*. That point is highly interesting for the real implications it has since the branch could predict the amounts of money A_i^* almost only from A_2^*.

Let us analyze in detail the former expression:

***k* = 3** For $k=3$ the former expression derives into the known definition of adaptative expectations, which expresses the expected value to be equal to the convex combination of the previous impact and the previous forecast. Namely,

$$\begin{aligned} E(A_3^*) &= F(\lambda)\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix} + (1-\lambda)E(A_2^*) \\ &= (0\ \lambda)\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix} + (1-\lambda)E(A_2^*) \\ &= \lambda A_2^* + (1-\lambda)E(A_2^*). \end{aligned}$$

***k* = 4** For $k = 4$, we have

$$
\begin{aligned}
E(A_4^*) &= [F(\lambda)A(2)+(1-\lambda)F(\lambda)]\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix}+(1-\lambda)^2E(A_2^*) \\
&= \left[(0\ \lambda)\begin{pmatrix} 1 & -c_1 \\ 0 & c_1 \end{pmatrix}+(1-\lambda)(0\ \lambda)\right]\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix}+(1-\lambda)^2E(A_2^*) \\
&= \left[(0\ \lambda c_1)+(0\ \lambda(1-\lambda))\right]\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix}+(1-\lambda)^2E(A_2^*) \\
&= (0\ \lambda c_1+\lambda(1-\lambda))\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix}+(1-\lambda)^2E(A_2^*) \\
&= \lambda[c_1+(1-\lambda)]\,A_2^*+(1-\lambda)^2E(A_2^*).
\end{aligned}
$$

***k* = 5** For $k = 5$, we have

$$
E(A_5^*) = \begin{bmatrix} F(\lambda)A(3)A(2)+ \\ (1-\lambda)F(\lambda)A(2)+ \\ (1-\lambda)^2F(\lambda) \end{bmatrix}\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix}+(1-\lambda)^3E(A_2^*)
$$

$$
= \begin{bmatrix} (0\ \lambda)\begin{pmatrix} 1 & -c_1-c_1c_2 \\ 0 & c_1c_2 \end{pmatrix} \\ +(1-\lambda)(0\ \lambda)\begin{pmatrix} 1 & -c_1 \\ 0 & c_1 \end{pmatrix} \\ +(1-\lambda)^2(0\ \lambda) \end{bmatrix}\begin{pmatrix} x_2 \\ A_2^* \end{pmatrix}+(1-\lambda)^3E(A_2^*)
$$

$$
= \lambda[c_1c_2+(1-\lambda)\,c_1+(1-\lambda)^2]A_2^*+(1-\lambda)^3E(A_2^*).
$$

In general, the k-forecast is

$$
E(A_k^*)= \lambda\left[\sum_{j=0}^{k-3} c_1c_2\ldots c_{k-3-j}(1-\lambda)^j\right]A_2^*+(1-\lambda)^{k-2}E(A_2^*)
$$

This expression resembles the known formulae that relates the k-expected value for adaptative expectations with the backward previous forecasts, namely

$$
E(A_k^*)= \lambda\sum_{j=0}^{n}(1-\lambda)^jA_{j-1}^*+(1-\lambda)^{n+1}E(A_{k-1-n}^*).
$$

However, our formulae improves that one in the sense that the expected value of the k-user only depends on the first one, A_2^*. As we commented before, this point allows the branch to predict the amounts of money A_i^* only from A_2^* while solving our initial problem.

Conclusion

The liquidity management is one of the main concerns of banks, particularly in the actual situation of economic and financial crisis. In consequence, all studies for optimizing liquidity management are welcome in the present scenario, for all bank activities involved in banking cash management.

Our contribution to all these optimization efforts is done in the front of cash management for ATMs banks: we have designed some formulae which will help branches to decide the optimum amount of money that will be placed into the ATM for satisfying the customers daily demand as well as minimizing all costs.

The main obstacle of this modeling process is to control the stochastic ingredients involved in the problem, as number of daily ATM users or number of quantities of money they withdraw from ATM as well as the erratic behavior of ATM customers, where erratic means, for instance, that no conclusions on quantity of money withdrawn by an ATM user can be extracted for the next ATM user. Also, there are other impediments of different nature in modeling this situation: possible changes in the customers behavior as well as possible changes at the ATM rules of functioning could make obsolete the mathematical model that reproduces this situation unless the model anticipates these troubles and integrates them as part of itself.

The stochastic ingredients have been captured by using Poisson processes. They allow us to demonstrate some formulae, simple and easy to implement in practice, of calculating the optimum amount of money that will be placed in the ATM for minimizing opportunity costs as well as satisfying the customers uncertain demand. The way to implement this mathematical routine in branches could be as (part of) a set of optimization *instructions* from the cash central to their branches. To help to illustrate better the accuracy of the model, it would be interesting to contrast this with real data from branches of the different bank companies. Nowadays the major part of these data are protected and it is difficult to access to them for the policy of maintaining both the names of the persons and the banks confidential. This could be a matter for future research in any case.

The last handicap — changes in the customers behavior and/or new rules of functioning — has been intended to save by designing our model with simple equations (difference equations —accord with the discrete-in-time situation we model) which can be restructured easily. Even more, although in the later development of the model it has been considered the three principal kind of expectations (rational, naïve and adaptive), many other types of expectations can be implemented by substituting them into the model, as the form of having being constructed their equations allows new possibilities for future research. Actually, as we observed in section 6, the range of possibilities for expectations could be augmented by *defining* our own *ad hoc* expectations accord with the features of each branch, like size or location. The inclusion of these new exogenous variables (branch size or location) is let by the author as an open problem.

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