Composition Operators between $\mu$-Bloch Spaces

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Abstract: In this paper we study continuity, boundedness from below and compactness of composition operators between $\mu$-Bloch spaces for very general weights $\mu$.

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1. Introduction

Let $D$ be the unit disk of the complex plane $\mathbb{C}$ and let $H(D)$ be the space of all holomorphic functions on $D$ with the topology of uniform convergence on compact subsets of $D$. The Bloch space, $B$, consist of all functions $f \in H(D)$ such that

$$
\|f\|_B := \sup_{z \in D} |1 - |z|^2| |f'(z)| < \infty.
$$

It is known that $B$ is a Banach space with the norm $\|f\| := |f(0)| + \|f\|_B$ (see, e.g., [1]). In the last decade, many authors have studied different classes of Bloch type spaces, where the typical weight function, $v(z) = 1 - |z|^2$, ($z \in D$), is replaced by a bounded continuous positive function $\mu$ defined on $D$. More precisely, a function $f \in H(D)$ is called a $\mu$-Bloch function, denoted as $f \in B^\mu$, if

$$
\|f\|_\mu := \sup_{z \in D} \mu(z) |f'(z)| < \infty.
$$

If $\mu(z) = v(z)^{\alpha}$ with $\alpha > 0$, $B^\mu$ is just the $\alpha$-Bloch space (see [26]). It is readily seen that $B^\mu$ is a Banach space with the norm $\|f\|_{B^\mu} := |f(0)| + \|f\|_\mu$. The $B^\mu$ spaces appear in the literature in a natural way when one study properties of some operators in certain spaces of holomorphic functions; for instance,

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if $\mu_1(z) = v(z) \log \frac{r}{v(z)}$ with $z \in \mathbb{D}$, Attele in [2] proved that the Hankel operator induced by a function $f$ in the Bergman space is bounded if and only if $f \in B^{\mu_1}$. The space $B^{\mu_1}$ is also known as the Log-Bloch space or the weighted Bloch space. Quie recently Stević in [18] introduced, the so called, logarithmic Bloch type space with $\mu(z) = v(z)\alpha \ln \frac{e}{v(z)}$, $\alpha > 0$ and $\beta \geq 0$, where some properties of this spaces are studied. Another Bloch type space, using Young’s functions, have been recently introduced by Ramos-Fernández in [17].

There is a big interest in the investigation of Bloch type spaces and various concrete linear operators $L : X \to Y$, where at least one of the spaces $X$ and $Y$ is Bloch. For some other recent results in the area see, for example, [1]–[26] and a lot of references therein. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two linear subspaces of $H(\mathbb{D})$. If $\phi$ is a holomorphic self-map of $\mathbb{D}$, such that $f \circ \phi$ belongs to $\mathcal{H}_2$ for all $f \in \mathcal{H}_1$, then $\phi$ induces a linear operator $C_\phi : \mathcal{H}_1 \to \mathcal{H}_2$ defined as

$$C_\phi(f) := f \circ \phi,$$

called the composition operator with symbol $\phi$. Composition operators has been studied by numerous authors in many subspaces of $H(\mathbb{D})$ and in particular in Bloch type spaces.

In [14], Madigan and Matheson characterized continuity and compactness for composition operators on the classical Bloch space $\mathcal{B}$. In turn, their results have been extended by Xiao [22] to the $\alpha$-Bloch spaces and by Yoneda [24] to the Log-Bloch space. On the other hand, Gathage, Zheng and Zorboska [10] characterized closed range composition operators on the Bloch space. This result has been extended by Chen and Gauthier [6] to $\alpha$-Bloch spaces. Also, in [25], Zhang and Xiao have characterized boundedness and compactness of weighted composition operators that act between $\mu$-Bloch spaces on the unit ball of $\mathbb{C}^n$. In this case it is required that $\mu$ be a normal function. The results of Zhang and Xiao have been extended by Chen and Gauthier [7] to the $\mu$-Bloch spaces being $\mu$ a positive and non-decreasing continuous function such that $\mu(t) \to 0$ as $t \to 0$ and $\mu(t)/t^\delta$ is decreasing for small $t$ and for some $\delta > 0$. Recently, Chen, Stević and Zhou (see [8]) have studied composition operators between Bloch type spaces in the polydisc. While Giménez, Malavé and Ramos-Fernández [11] have extended those results to certain $\mu$-Bloch spaces, where the weight $\mu$ can be extended to non vanishing complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk $D(1, 1)$. Ramos-Fernández in [17] have extended all the results mentioned above to the Bloch-Orlicz spaces. In this paper we study properties
such as continuity, boundedness from below and compactness of composition operators acting between $\mu$-Bloch spaces, for very general weights $\mu$.

The essential norm of a continuous linear operator $T$ is the distance from $T$ to the compact operators, that is, $\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}$. Notice that $\|T\|_e = 0$ if and only if $T$ is compact, so that estimates on $\|T\|_e$ lead to conditions for $T$ to be compact. The essential norm of a composition operator on the Bloch space was calculated by Montes-Rodríguez in [15]. Similar results for the essential norms of weighted composition operators between weighted Banach spaces of analytic functions were obtained by Montes-Rodríguez in [16], and by Contreras and Hernández-Díaz in [9], in particular, formulas for the essential norm of weighted composition operators on the $\alpha$-Bloch spaces are obtained (see also the paper of McCluer and Zhao in [13]). Recently have appeared many extensions of the above results, for instance, we can mention the paper of Yang and Zhou in [23] and a lot of references therein.

Let us explain the organization of the paper. In Section 2, we summarize preliminaries on spaces $H^\infty_v$, associated weight and essential weights. Thus, in Section 3, inspired by the results in [5], we characterize continuity and compactness of the composition operator $C_\phi : B^{\mu_1} \to B^{\mu_2}$, in fact, if we denote by $\|\delta_\cdot\|$ the norm of the evaluation functional at $z$ acting on the weighted Banach space of analytic functions $H^\infty_{\mu_1}$, then we have the following results:

- The operator $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is continuous if and only if
  \[ \sup_{z \in \mathbb{D}} \mu_2(z) \|\delta_{\phi(z)}\| |\phi' (z)| < \infty. \]

- The composition operator $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is compact if and only if $\phi \in B^{\mu_2}$ and
  \[ \lim_{|\phi(z)| \to 1^-} \mu_2(z) \|\delta_{\phi(z)}\| |\phi' (z)| = 0. \]

Finally, in Section 4, we characterize composition operators $C_\phi : B^{\mu_1} \to B^{\mu_2}$ with closed range in term of certain sampling sets for the space $B^{\mu_1}$.

2. The associated weight

Let $v : \mathbb{D} \to \mathbb{R}_+$ be an arbitrary weight, that is, $v$ is a bounded, continuous and strictly positive function. A function $f \in H(\mathbb{D})$ belongs to the space $H^\infty_v$ if

\[ \|f\|_{H^\infty_v} := \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty. \] (2.1)
It is known (see [3]) that $H_v^\infty$ is a Banach space with the norm defined in (2.1). The space $H_v^\infty$ is connected with the study of growth conditions of analytic functions and were studied in detail in [3, 4]. When $v(z) = (1 - |z|^2)\alpha$ with $\alpha > 0$, we get the Korenblum spaces $A^{-\alpha}$. The relation between the $\mu$-Bloch space and the space $H_v^\infty$ is evident, in fact, $f \in B^\mu$ if and only if $f' \in H_\mu^\infty$ and

$$
\|f\|_\mu = \|f'\|_{H_\mu^\infty}.
$$

From the relation (2.1), it is clear that, for $z \in \mathbb{D}$ fixed, there exists a constant $K_z$, depending on $z$ and $v$, such that

$$
|f(z)| \leq K_z \|f\|_{H_v^\infty},
$$

for all $f \in H_v^\infty$. This means that the evaluation functional at $z$, denoted as $\delta_z$, is continuous on $H_v^\infty$ and we can define the associated weight with $v$, denoted as $\tilde{v}$, by

$$
\tilde{v}(z) = \frac{1}{\|\delta_z\|} = \frac{1}{\sup\{|f(z)| : \|f\|_v \leq 1\}},
$$

where $z \in \mathbb{D}$ and $\|\delta_z\|$ denotes the norm of the evaluation functional at $z$. In [3], it is shown that $\tilde{v}$ satisfies the following useful properties:

1. $\tilde{v}$ is a weight and $0 < v(z) < \tilde{v}(z)$ for all $z \in \mathbb{D}$,

2. for every $z \in \mathbb{D}$, there exists $f_z \in H_v^\infty$ such that $\|f\|_{H_v^\infty} \leq 1$ and $\tilde{v}(z)|f_z(z)| = 1$,

3. $H_v^\infty$ is isometrically equal to $H_v^\infty$ and $\|f\|_v = \|f\|_{\tilde{v}}$ for all $f \in H_v^\infty$.

A weight $v$ is called essential if there exists a constant $C > 0$ such that $\tilde{v}(z) \leq Cv(z)$ for all $z \in \mathbb{D}$. For instance, if $v(z) = 1/M(f, |z|)$ for some analytic function $f \in H(\mathbb{D})$, then $v = \tilde{v}$, where $M(f, r) = \sup_{|z|=r} |f(z)|$. The following are examples of essential weights (see [3] for a reasonable amount of examples of essential weights):

- $v_\alpha(z) = (1 - |z|)^\alpha$ with $\alpha > 0$,
- $v(z) = \exp(-1/(1 - |z|)^\alpha)$, with $\alpha > 0$,
- $v(z) = 1/\max\{1, -\log(1 - |z|)\}$.

In general, is not easy to calculate the associated weight $\tilde{v}$; however, in [3, Section 3] Bierstedt, Bonet and Taskinen give some estimations of $\tilde{v}$.

Finally, we like to comment that some of the properties of composition operators acting on $H_v^\infty$ spaces has been studied by Bonet, Domariki, Lindström and Taskinen in [5], Contreras and Hernández-Díaz in [9] and Wolf in [20].
In this section, we study continuity and compactness of composition operators between $\mu$-Bloch spaces. Throughout this section, $\phi$ is a holomorphic self-map of $D$ and $C_\phi$ denote its associated composition operator. $\mu_1$ and $\mu_2$ are weight functions defined on $D$ and $\tilde{\mu}_1$ is the associated weight of $\mu_1$; $B^{\mu_1}$ and $B^{\mu_2}$ are their respective $\mu$-Bloch spaces. Also, $\|\delta_z\|$ denotes the norm of the evaluation functional at $z$ on the weighted Banach space of analytic functions $H^{\infty}_{\mu_1}$. With this notations we have the following results.

3.1. Continuity. The following, generalize many results about continuity of composition operators acting on Bloch type spaces. A similar result was obtained recently by Wolf in [21], while this article was under review.

**Theorem 3.1.** The operator $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is continuous if and only if

$$\sup_{z \in D} \mu_2(z)\|\delta_{\phi(z)}\|\|\phi'(z)\| < \infty. \quad (3.1)$$

**Proof.** Suppose first that

$$L = \sup_{z \in D} \mu_2(z)\|\delta_{\phi(z)}\|\|\phi'(z)\| < \infty.$$  

Then, for each $f \in B^{\mu_1}$, since $\tilde{\mu}_1(s)\|\delta_s\| = 1$ for all $s \in D$, we have the following estimate

$$\|f \circ \phi\|_{\mu_2} = \sup_{z \in D} \mu_2(z)\|\delta_{\phi(z)}\|\|\phi'(z)\|\tilde{\mu}_1(\phi(z))|f'(\phi(z))|$$

$$\leq L\|f\|_{\tilde{\mu}_1} = L\|f\|_{\mu_1}.$$  

Also, since $\mu_1$ is continuous and positive on the compact set $[0, \phi(0)]$, there exists a constant $K_{\mu_1, \phi} > 0$, depending on $\mu_1$ and $\phi(0)$, such that

$$\int_0^{\phi(0)} \frac{|ds|}{\mu_1(s)} \leq K_{\mu_1, \phi}. \quad (3.2)$$

Hence, we have

$$|f(\phi(0))| \leq |f(0)| + \int_0^{\phi(0)} |f'(s)|ds \leq |f(0)| + K_{\mu_1, \phi}\|f\|_{\mu_1}.$$
We conclude that
\[ |f(\phi(0))| + \|f \circ \phi\|_{\mu_2} \leq |f(0)| + (L + K_{\mu_1,\phi})\|f\|_{\mu_1} \]
and the composition operator \( C_{\phi} : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2} \) is continuous.

Now, suppose that there exists a constant \( L > 0 \) such that \( \|f \circ \phi\|_{\mu_2} \leq L\|f\|_{\mu_1} \) for all function \( f \in \mathcal{B}^{\mu_1} \) with \( f(0) = 0 \) and let us fix \( z \in \mathbb{D} \). By definition of associated weight, for \( a = \phi(z) \in \mathbb{D} \), there exists a function \( f_a \in H(\mathbb{D}) \) such that \( \sup_{w \in \mathbb{D}} \mu_1(w)|f_a(w)| \leq 1 \) and \( \tilde{\mu}_1(a)|f_a(a)| = 1 \), hence the function \( g_a \) given by
\[ g_a(w) = \int_0^w f_a(s)ds, \]
with \( w \in \mathbb{D} \) belongs to \( \mathcal{B}_1^{\mu} \) and satisfies \( g_a(0) = 0 \). Thus, applying the hypothesis with \( f = g_a \), we have \( \|g_a \circ \phi\|_{\mu_2} \leq L \); that is,
\[ \sup_{w \in \mathbb{D}} \mu_2(w)|g'_a(\phi(w))|\|\phi'(w)\| \leq L. \]
This last, implies that
\[ \mu_2(z)||\delta_{\phi(z)}||\|\phi'(z)\| \leq L. \]
The proof of the theorem is complete.

As an immediate consequence of Theorem 3.1, we have the following result, which has been obtained by many authors for various type of weight \( \mu \).

**Corollary 3.2.** If \( \mu_1 \) is a essential weight, then the composition operator \( C_{\phi} : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2} \) is continuous if and only if
\[ \sup_{z \in \mathbb{D}} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| < \infty. \]

**Proof.** It follows from the fact that \( \tilde{\mu}_1(s)||\delta_s|| = 1 \) for all \( s \in \mathbb{D} \) and \( \tilde{\mu}_1 \sim \mu_1 \).

**Example 3.3.** If \( \mu_1(z) = (1 - |z|)^{\alpha} \) and \( \mu_2(z) = (1 - |z|)^{\beta} \) with \( \alpha, \beta > 0 \), we have the result of Xiao in [22]. If \( \mu_1(z) = \mu_2(z) = (1 - |z|) \log \left( \frac{2}{1-|z|} \right) \), we have the result of Yoneda in [24].
3.2. Compactness. Now, we are going to characterize compactness of composition operators that act between $\mu$-Bloch spaces. Our goal is to obtain genuine extensions of the results in [17, 11, 14]. In [19], Tjani showed the following result.

**Lemma 3.4.** Let $X, Y$ be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that

1. The point evaluation functionals on $X$ are continuous.
2. The closed unit ball of $X$ is a compact subset of $X$ in the topology of uniform convergence on compact sets.
3. $T : X \to Y$ is continuous when $X$ and $Y$ are given the topology of uniform convergence on compact sets.

Then, $T$ is a compact operator if and only if given a bounded sequence $\{f_n\}$ in $X$ such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of $Y$.

Observe that for $z \in \mathbb{D}$ fixed, since $\mu_1$ is positive and continuous on the compact set $[0, z]$, there exists a constant $K_{\mu_1, z} > 0$ such that

$$|f(z)| \leq |f(0)| + \int_0^z \frac{|ds|}{\mu_1(s)} \leq K_{\mu_1, z} \|f\|_{B^{\mu_1}}$$

and the point evaluation functionals on $B^{\mu_1}$ are continuous. Thus, as a consequence of Lemma 3.4, we have the following result.

**Lemma 3.5.** The composition operator $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is compact if and only if given a bounded sequence $\{f_n\}$ in $B^{\mu_1}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\phi(f_n)\|_{\mu_2} \to 0$ as $n \to \infty$.

Next we establish our criterion for the compactness of $C_\phi : B^{\mu_1} \to B^{\mu_2}$. It generalizes a result of Madigan and Matheson in [14].

**Theorem 3.6.** The composition operator $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is compact if and only if $\phi \in B^{\mu_2}$ and

$$\lim_{|\phi(z)| \to 1} \mu_2(z)\|\delta_{\phi(z)}||\phi'(z)| = 0.$$  \hspace{1cm} (3.3)
Proof. Let us suppose first that $\phi \in B^{v_2}$ and (3.3) holds. Let $\{f_n\}$ be a bounded sequence in $B^{v_1}$ converging to 0 uniformly on compact subsets of $\mathbb{D}$. Then, by Lemma 3.5, it is sufficient to show that $\|C_\phi(f_n)\|_{\mu_2} \to 0$ as $n \to \infty$.

To this end, we set $K = \sup_n \|f_n\|_{\mu_1} = \sup_n \|f_n\|\tilde{\mu}_1$. Then, for $\epsilon > 0$ we can find an $r \in (0, 1)$ such that

$$\mu_2(z)\|\delta_{\phi(z)}||\phi'(z)|| < \frac{\epsilon}{K},$$

for any $z \in \mathbb{D}$ satisfying $r < |\phi(z)| < 1$. Hence, we have

$$\mu_2(z)|f_n \circ \phi)'(z)| = \mu_2(z)\|\delta_{\phi(z)}||\phi'(z)||\tilde{\mu}_1(\phi(z))|f_n'(\phi(z))|$$

$$\leq \frac{\epsilon}{K}K = \epsilon$$

whenever $r < |\phi(z)| < 1$. Here, we have used the fact that $\|\delta_s\|\tilde{\mu}_1(s) = 1$ for all $s \in \mathbb{D}$.

On the other hand, since $\phi \in B^{v_2}$ and $\tilde{\mu}_1$ is continuous and positive on the compact set $\{w \in \mathbb{D} : |w| \leq r\}$, we can find a constant $C > 0$, depending only on $r$ and $\mu_1$, such that

$$\sup_{|\phi(z)| \leq r} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))}|\phi'(z)| \leq C\|\phi\|_{\mu_2}.$$ 

Thus, since $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ and $\tilde{\mu}_1$ is bounded on the compact set $|s| \leq r$, we have $\sup_{|s| \leq r} \tilde{\mu}_1(s)|f_n'(s)| \to 0$, as $n \to \infty$. Hence, for the given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sup_{|\phi(z)| \leq r} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))}|\phi'(z)| \leq C\|\phi\|_{\mu_2}\epsilon$$

whenever $n \geq N$. Finally, since $f_n \circ \phi(0) \to 0$ as $n \to \infty$, we conclude that

$$\|f_n \circ \phi\|_{\mu_2} = |f_n \circ \phi(0)| + \sup_{z \in \mathbb{D}} \mu_2(z)|f_n \circ \phi)'(z)| < (1 + C\|\phi\|_{\mu_2})\epsilon$$

whenever $n \geq N$, which means that $C_\phi : B^{v_1} \to B^{v_2}$ is a compact operator.

To prove the converse, suppose that there exists an $\epsilon_0 > 0$ such that

$$\sup_{|\phi(z)| \geq r} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))}|\phi'(z)| \geq \epsilon_0$$
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for any $r \in (0, 1)$. Then, given a sequence of real numbers $\{r_n\} \subset (0, 1)$ such that $r_n \to 1$ as $n \to \infty$, we can find a sequence $\{z_n\} \subset \mathbb{D}$ such that $|\phi(z_n)| > r_n$ and

$$\frac{\mu_2(z_n)}{\mu_1(w_n)} |\phi'(z_n)| \geq \frac{1}{2} \epsilon_0,$$

where $w_n = \phi(z_n)$. By taking a subsequence, if necessary, we may suppose that $w_n \to w_0 \in \partial \mathbb{D}$. Also, since $|w_n| \to 1$ as $n \to \infty$, we can find an increasing sequence $\{\alpha(n)\}$ of positive integers such that $\alpha(n) \to \infty$ as $n \to \infty$ and $|w_n|^{\alpha(n)} \geq \frac{1}{2}$ for all $n \in \mathbb{N}$.

Now, for $n \in \mathbb{N}$, we choose a function $f_n \in H(\mathbb{D})$ such that

$$\sup_{w \in \mathbb{D}} \mu_1(w) |f_n(w)| \leq 1$$

and $\mu_1(w_n)|f_n(w_n)| = 1$ and we set

$$g_n(z) = \int_0^z s^{\alpha(n)} f_n(s)\,ds,$$

with $z \in \mathbb{D}$. We can see that $\{g_n\}$ is a bounded sequence in $B^{\mu_1}$, in fact,

$$\|g_n\|_{\mu_1} = \sup_{z \in \mathbb{D}} \mu_1(z)|g_n'(z)| = \sup_{z \in \mathbb{D}} \mu_1(z)|z|^{\alpha(n)}|f_n(z)| \leq 1$$

for all $n \in \mathbb{N}$. Furthermore, because of the factor $z^{\alpha(n)}$, the sequences $\{g'_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, therefore, since

$$g_n(z) = \int_0^z g'_n(s)\,ds$$

for all $z \in \mathbb{D}$, we can see that $\{g_n\}$ is a sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$ and satisfying

$$\|C_\phi(g_n)\|_{\mu_2} \geq \mu_2(z_n)|g'_n(w_n)||\phi'(z_n)| = \mu_2(z_n)|w_n|^{\alpha(n)}|f_n(w_n)||\phi'(z_n)| = \frac{1}{2} \frac{\mu_2(z_n)}{\mu_1(w_n)} |\phi'(z_n)| > \frac{1}{4} \epsilon_0 > 0,$$

where, we have used the fact that $|f_n(w_n)| = 1/\mu_1(w_n)$. Therefore, $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is not a compact operator. This completes the proof of the theorem. 

As an immediate consequence, we have.
Corollary 3.7. If $\mu_1$ is an essential weight, then the composition operator $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ is compact if and only if $\phi \in \mathcal{B}^{\mu_2}$ and
\[
\lim_{|\phi(z)| \to 1^- \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| = 0.
\]

Example 3.8. If $\mu_1(z) = \mu_2(z) = 1 - |z|$, we get the result of Madigan and Matheson in [14]. When $\mu_1(z) = (1 - |z|^2)^\alpha$ and $\mu_2(z) = (1 - |z|^2)^\beta$ with $\alpha, \beta > 0$, we obtain a criterion for the compactness of composition operators between $\alpha$-Bloch type spaces, similar result was found by Montes-Rodríguez in [16], later, independently, by Xiao in [22] and by McCluer and Zhao in [13]. If $\mu_1(z) = \mu_2(z) = (1 - |z|) \log(2/(1 - |z|))$ we obtain the result of Yoneda in [24].

4. Composition operators with closed range between $\mu$-Bloch spaces

In this section, we characterize the composition operators $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ with closed range in terms of certain sampling sets for the $\mu$-Bloch space. The purpose here is to generalize the results in [6, 10, 11, 17] for the $\mu$-Bloch spaces. Recall (see [12]) that a subset $G$ of the unit disk $D$ is said to be a sampling set for the Korenblum space $\mathcal{A}^{-\alpha}$ if there exists a positive constant $L > 0$ such that
\[
\sup_{z \in G} (1 - |z|^2)^\alpha |f(z)| \geq L \|f\|_{\mathcal{A}^{-\alpha}}
\]
for all $f \in \mathcal{A}^{-\alpha}$. Observe that $v(z) = (1 - |z|^2)^\alpha = \tilde{v}(z)$ and for this reason, we introduce the following definition.

Definition 4.1. Let $v$ be a weight defined on $D$. A subset $G$ of the unit disk $D$ is said to be a sampling set for $\mathcal{B}^v$ if there exists a positive constant $L > 0$ such that
\[
\sup_{z \in G} \tilde{v}(z) |f'(z)| \geq L \|f\|_v,
\]
(4.1)
for all $f \in \mathcal{B}^v$.

Remark 4.2. When $v$ is an essential weight, we can replace, in the preceding definition, the weight $\tilde{v}$ by $v$.

Let $\mu_1$ and $\mu_2$ two weights. For $\varepsilon > 0$, let us denote
\[
\Omega_\varepsilon := \left\{ z \in D : \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| \geq \varepsilon \right\}.
\]
With this notation, we have the following result.

**Theorem 4.3.** Let $C_\phi : B^{\mu_1} \to B^{\mu_2}$ be a continuous composition operator. $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is bounded below if and only if there exists $\varepsilon > 0$ such that $G_\varepsilon = \phi(\Omega_\varepsilon)$ is a sampling set for $B^{\mu_1}$.

**Proof.** Let us suppose first that there exists $\varepsilon > 0$ such that $G_\varepsilon = \phi(\Omega_\varepsilon)$ is a sampling set for $B^{\mu_1}$. In this case, we can find a constant $L > 0$ such that

$$
\|f\|_{\mu_1} \leq L \sup_{z \in G_\varepsilon} \tilde{\mu}_1(z) |f'(z)|
$$

for all functions $f \in B^{\mu_1}$. Hence, we have that

$$
\|f\|_{\mu_1} \leq L \sup_{z \in \Omega_\varepsilon} \frac{\tilde{\mu}_1(\phi(z))}{\mu_2(z)} |f'(\phi(z))|
$$

$$
= L \sup_{z \in \Omega_\varepsilon} \frac{\tilde{\mu}_1(\phi(z))}{\mu_2(z)} |f(\phi(z))'|
$$

$$
\leq \frac{L}{\varepsilon} \|f \circ \phi\|_{\mu_2},
$$

and since

$$
|f(0)| \leq |f(\phi(0))| + K_{\mu_1, \phi} \|f\|_{\mu_1},
$$

where $K_{\mu_1, \phi}$ is the constant in (3.2), we conclude that

$$
|f(0)| + \|f\|_{\mu_1} \leq |f(\phi(0))| + (1 + K_{\mu_1, \phi}) \frac{L}{\varepsilon} \|f \circ \phi\|_{\mu_2}
$$

and the operator $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is bounded below.

To prove the converse, suppose that $C_\phi : B^{\mu_1} \to B^{\mu_2}$ is bounded below. For any non constant function $g \in B^{\mu_1}$, we set

$$
f(z) = \frac{1}{\|g\|_{\mu_1}} (g(z) - g(\phi(0)))
$$

and we have that $f(\phi(0)) = 0$ and $\|f\|_{\mu_1} = 1$. Hence, by hypothesis, there exists a constant $K > 0$ (not depending on $g$), such that $\|C_\phi(f)\|_{B^{\mu_2}} \geq K \|f\|_{B^{\mu_1}}$; this last, implies that

$$
\|C_\phi(f)\|_{\mu_2} = \sup_{z \in \mathbb{D}} \mu_2(z) |f'(\phi(z))| |\phi'(z)| \geq K.
$$
Thus, by definition of supremum, we can find $z_f \in \mathbb{D}$, such that
\[
\mu_2(z_f)|f'(\phi(z_f))||\phi'(z_f)| \geq \frac{K}{2},
\]
which, in turn, implies that
\[
\frac{\mu_2(z_f)}{\tilde{\mu}_1(\phi(z_f))} |\phi'(z_f)||\tilde{\mu}_1(\phi(z_f))|f'(\phi(z_f))| \geq \frac{K}{2}.
\] (4.2)

Thus, since $\tilde{\mu}_1(\phi(z_f))|f'(\phi(z_f))| \leq 1$, it must be
\[
\frac{\mu_2(z_f)}{\tilde{\mu}_1(\phi(z_f))} |\phi'(z_f)| \geq \frac{K}{2}.
\]
Therefore, putting $\varepsilon := \frac{K}{2}$, we have $z_f \in \Omega_\varepsilon$.

Now, since $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous, Theorem 3.1 implies that there is a constant $M > 0$, such that
\[
\frac{\mu_2(z_f)}{\tilde{\mu}_1(\phi(z_f))} |\phi'(z_f)| \leq M.
\]
From (4.2) we conclude that
\[
\tilde{\mu}_1(\phi(z_f))|f'(\phi(z_f))| \geq \frac{K}{2M}.
\]
Finally, since $\phi(z_f) \in G_\varepsilon$, it must be
\[
\sup_{z \in G_\varepsilon} \tilde{\mu}_1(z)|f'(z)| \geq \frac{K}{2M}.
\]
That is,
\[
\sup_{z \in G_\varepsilon} \tilde{\mu}_1(z) \frac{|f'(z)|}{\|g\|_{\mu_1}} \geq \frac{K}{2M}
\]
and therefore $G_\varepsilon$ is a sampling set for $\mathcal{B}^{\mu_1}$. The proof of the theorem is complete.

**Example 4.4.** If $\mu_1(z) = \mu_2(z) = 1 - |z|^2$, we have the result of Ghatage, Zheng and Zorboska in [10]. If $\mu_1(z) = (1 - |z|)^\alpha$ and $\mu_2(z) = (1 - |z|)^\beta$ with $\alpha, \beta > 0$, we obtain the result of Chen and Gauthier in [6].

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References


