

## On Sobolev trace inequality for fractional–order derivatives

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Se establece una desigualdad de traza de tipo Sobolev para derivadas fraccionales.

Palabras Claves: desigualdad de traza de tipo Sobolev, derivadas fraccionales.

We establish a Sobolev trace inequality for fractional derivatives.

Keywords: Sobolev trace inequality, fractional derivatives.

MSC: 46E35, 42C99.

### 1 Introduction

We denote the Laplacian operator  $\Delta$  in  $\mathbb{R}_+^{n+1}$  by

$$\Delta = \frac{\partial^2}{\partial y^2} + \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2},$$

and the corresponding gradient  $\nabla$  by

$$|\nabla u(x, y)|^2 = \left| \frac{\partial u}{\partial y} \right|^2 + |\nabla_x u(x, y)|^2,$$

where

$$|\nabla_x u(x, y)|^2 = \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2.$$

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In [2] and [3], Beckner and Escobar independently established the following Sobolev theorem:

**Theorem.** *Let  $f$  be a real-valued function, sufficiently smooth with fast enough decay at infinity,  $(x, t) \in \mathbb{R}^n \times (0, t) = \mathbb{R}_+^{n+1}$ ,  $n \geq 2$ . Let*

$$\varphi(x, t) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{\mathbb{R}^n} (|x-y|^2 + t^2)^{-\frac{n+1}{2}} t f(y) dy.$$

Then

$$\left( \int_{\mathbb{R}^n} |f(x)|^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \frac{1}{\sqrt{\pi}(n-1)} \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{1}{n}} \times \int_{\mathbb{R}^n} |\nabla \varphi(x, t)|^2 dx dt. \quad (1)$$

The equality in (1) occurs only if  $f$  has the form  $c \left( |x - x_0|^2 + t_0^2 \right)^{(n-1)/2}$  for  $c \in \mathbb{R}$  and  $(x_0, t_0) \in \mathbb{R}_+^{n+1}$ .

In order to prove this optimal estimate, both authors used the conformally equivalent model  $S^n$  of  $\mathbb{R}_+^{n+1}$  as a tool, but their methods are quite different: In [1] Beckner verified the result via certain cases of the sharp Hardy–Littlewood inequality on  $S^n$ ; while in [2] Escobar obtained the result by finding the minimizer of the Sobolev quotient in  $S^n$  subject to its associated Euler–Lagrange equation. In this paper we shall give an analogue of [1] for fractional derivatives.

## 2 Preliminaries

In this section we include several lemmas that will be used throughout the paper.

**Lemma 2.1.** Suppose  $u$  is a harmonic strictly positive function. Then

$$\Delta u^p = p(p-1) u^{p-2} |\nabla u|^2,$$

for  $1 < p < \infty$ .

*Proof.* Note that for  $1 < p < \infty$  we have

$$\sum_{i=1}^n \frac{\partial^2 u^p}{\partial x_i^2} = p(p-1) u^{p-2} \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i} \right) + p u^{p-1} \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2},$$

and

$$\frac{\partial^2 u^p}{\partial x_i^2} = p(p-1) u^{p-2} \left( \frac{\partial u}{\partial x_i} \right) + p u^{p-1} \frac{\partial^2 u}{\partial x_i^2}.$$

Thus

$$\begin{aligned} \Delta u^p &= \frac{\partial^2 u^p}{\partial y^2} + \sum_{i=1}^n \frac{\partial^2 u^p}{\partial x_i^2} \\ &= p(p-1) u^{p-2} \left[ \left| \frac{\partial u}{\partial y} \right|^2 + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 \right] + p u^{p-1} \Delta u. \end{aligned}$$

Since  $u$  is harmonic, we have

$$\Delta u^p = p(p-1) u^{p-2} |\nabla u|^2.$$

□

**Lemma 2.2.** Suppose  $F(x, y)$  is a continuous function of class  $C^2$  in  $\mathbb{R}_+^{n+1}$  which is sufficiently small at infinity. Then

$$\int_{\mathbb{R}_+^{n+1}} y \Delta F(x, y) dx dy = \int_{\mathbb{R}_+^{n+1}} F(x, 0) dx. \quad (2)$$

*Proof.* Let  $D = B_r \cap \mathbb{R}_+^{n+1}$  with  $B_r$  the ball of radius  $r$  in  $\mathbb{R}_+^{n+1}$  centered at the origin.

We take  $v = F$  and  $u = y$  in the Green formula. Then we will obtain our result (2) if

$$\int_D y \Delta F(x, y) dx dy \rightarrow \int_{\mathbb{R}_+^{n+1}} y \Delta F(x, y) dx dy,$$

and

$$\int_{\partial D_0} \left( y \frac{\partial F}{\partial v} - \frac{\partial y}{\partial v} F \right) d\sigma \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Here  $\partial D_0$  is the spherical part of the boundary of  $D$ . This will certainly be the case if, for instance,  $\Delta F \geq 0$ ,  $|F| \leq 0(|x| + |y|)$  and

$$|\nabla F| = 0(|x| + |y|)^{-n-1-\epsilon} \quad \text{as } |x| + |y| \rightarrow \infty,$$

for some  $\epsilon > 0$ . Then, using the Green formula

$$\int_D (u \Delta v - v \Delta u) dx dy = \int_{\partial D} \left[ u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right] d\sigma.$$

We obtain

$$\begin{aligned} \int_{\mathbb{R}_+^{n+1}} y \Delta F(x, y) dx dy &= \int_{\mathbb{R}_+^{n+1}} F(x, y) \Delta y dx dy \\ &= \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+} F(x, y) \Delta y dy dx \\ &= - \int_{\mathbb{R}^n} \int_0^\infty DF(x, y) \cdot Dy dy dx \\ &= \int_{\mathbb{R}^n} -[F(x, \infty) - F(x, 0)] dx \\ &= \int_{\mathbb{R}^n} F(x, 0) dx. \end{aligned}$$

□

**Lemma 2.3.** Let  $f$  be a function with compact support on  $B_r(0)$ . If  $u(x, y)$  is the Poisson integral of  $f$ , then

$$\sup_{y>0} |u(x, y)| \leq M f(x),$$

where

$$M f(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| dy.$$

*Proof.* Observe that

$$\begin{aligned} u(x, y) &= \int_{\mathbb{R}^n} F(x - z) P(z, y) dz \\ &= \int_{\mathbb{R}^n} F(z) P(x - z, y) dz, \end{aligned}$$

where

$$P(z, y) = \frac{y}{c_n \left( |z_n|^2 + y^2 \right)^{(n+1)/2}}.$$

Let  $r > 0$ . Then

$$\begin{aligned} |P(z, r)| &= \frac{r}{c_n \left( |z_n|^2 + r^2 \right)^{(n+1)/2}} \\ &\leq \frac{r}{c_n r^{n+1}} = \frac{1}{c_n r^n}. \end{aligned}$$

In this latter case, we have

$$\begin{aligned} |u(x, r)| &\leq \int_{B_r(0)} |f(z)| |P(x - z, r)| dz \\ &\leq \frac{1}{c_n r^n} \int_{B_r(0)} |f(z)| dz \\ &\leq \frac{1}{m B_r(0)} \int_{B_r(0)} (|f(z)|) dz. \end{aligned}$$

Therefore

$$\sup_{y>0} |u(x, y)| \leq M f(x).$$

□

**Lemma 2.4.** Let  $f \in L_p$  with  $1 < p \leq 2$  and

$$g(f)(x) = \left( \int_0^\infty y |\nabla u(x, y)|^2 dy \right)^{\frac{1}{2}}.$$

Then

$$\|g(f)\|_p \leq A_p \|f\|_p.$$

*Proof.*

$$\begin{aligned} [g(f)(x)]^2 &= \int_0^\infty y |\nabla u(x, y)|^2 dy \\ &= \frac{1}{p(p-1)} \int_0^\infty y u^{2-p} \Delta u^p dy \\ &\leq \frac{1}{p(p-1)} (M f(x))^{2-p} \int_0^\infty y \Delta u^p dy, \end{aligned}$$

$$g(f)(x) \leq C_p (M f(x))^{(2-p)/2} [I(x)]^{1/2},$$

where  $I(x) = \int_0^\infty y \Delta u^p(x, y) dy$ . However, by the Theorem 1(c) in [4, p. 62], we have

$$\begin{aligned} \int_{\mathbb{R}_+^n} I(x) dx &= \int_{\mathbb{R}_+^n} \int_0^\infty y \Delta u^p dy dx \\ &= \int_{\mathbb{R}_+^{n+1}} y \Delta u^p dy dx \\ &= \int_{\mathbb{R}^n} u^p(x, 0) dx \\ &= \lim_{y \rightarrow \mathbb{R}^n} \int_{\mathbb{R}^n} u^p(x, y) dx \\ &= \|f\|_p^r. \end{aligned}$$

Then, if  $p = 2$ , we have

$$\begin{aligned} [g(f)(x)]^2 &\leq C_2 I(x) \\ \int_{\mathbb{R}^n} |g(f)(x)|^2 dx &\leq C_2 \int_{\mathbb{R}^n} I(x) dx \\ \int_{\mathbb{R}^n} |g(f)(x)|^2 dx &\leq C_2 \|f\|_2^2 \\ \left( \int_{\mathbb{R}^n} |g(f)(x)|^2 dx \right)^{\frac{1}{2}} &\leq C_2 \|f\|_2^2 \\ \|g(f)\|_2 &\leq C_2 \|f\|_2. \end{aligned}$$

Suppose now  $1 < p < 2$ , then

$$(g(f)(x))^p \leq C_p (M f(x))^{p(2-p)/2} [I(x)]^{p/2}.$$

Take  $r = \frac{2}{2-p}$  and  $r' = \frac{2}{p}$ . Then, by the Hölder inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^n} [(g(f)(x))^p] dx &\leq C_p \int_{\mathbb{R}^n} [M f(x)]^{p(2-p)/2} [I(x)]^{p/2} dx \\ &\leq C'_p \left( \int_{\mathbb{R}^n} [M f(x)]^p dx \right)^{\frac{1}{r}} \left( \int_{\mathbb{R}^n} I(x) dx \right)^{\frac{1}{r'}} \\ &\leq C \|f\|_p^{p/r} \|f\|_p^{p/r'} \\ &= C \|f\|_p^p. \end{aligned}$$

Thus we have proved that

$$\|g(f)\|_p \leq C_p \|f\|_p \quad \text{for } 1 < p \leq 2,$$

whenever  $f$  is a positive function which is indefinitely differentiable and of compact support.

In general, for  $f \in L_p(\mathbb{R}^n)$  (which we assume for simplicity to be real valued) write  $f^+ - f^-$ , we only need to approximate in norm  $f^+$  and  $f^-$ , each by a sequences of positive indefinitely differentiable functions with compact support. We omit the routine details that are needed to complete the proof. □

**Remark.** *It is unfortunate that the just given elegant argument is not valid for  $p > 2$ .*

### 3 Main result

To state our main result, let us introduce some notation. For  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$  we define  $x \cdot y = \sum_{i=1}^n x_i y_i$  and

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y) \exp(-2\pi i x \cdot y) dy,$$

and  $f^\vee(x) = \hat{f}(-x)$  the Fourier transform and the inverse Fourier transform of an integrable function  $f$ , respectively. Furthermore, given  $\alpha \in$

$(0, 1)$ ,  $\dot{H}^\alpha(\mathbb{R}^n)$  is the homogeneous fractional order Sobolev space —the completion of all infinitely differential functions  $f$  with compact support in  $\mathbb{R}^n$  under the norm

$$\|f\|_{\dot{H}^\alpha(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f(x)|^2 dx \right)^{\frac{1}{2}} < \infty;$$

where

$$(-\Delta)^{\alpha/2} f(x) = \left( (2\pi|\bullet|)^\alpha \hat{f}(\bullet) \right)^\vee(x),$$

stands for the derivative of  $f$  of order  $\alpha$  at  $x \in \mathbb{R}^n$ .

**Theorem 3.1.** *Let  $n \geq 2$  and  $\alpha \in (0, 1)$ . If  $f \in \dot{H}^\alpha(\mathbb{R})$ , then*

$$\left( \int_{\mathbb{R}^n} |f(x)|^{\frac{2n}{n-2\alpha}} dx \right)^{\frac{n-2\alpha}{n}} \leq C(n, \alpha) \int_{\mathbb{R}_+^{n+1}} |\nabla \varphi(x, t)|^2 t^{1-2\alpha} dx dt, \quad (3)$$

where

$$C(n, \alpha) = \left( \frac{2^{1-4\alpha}}{\pi^\alpha \Gamma(2(1-\alpha))} \right) \left( \frac{\Gamma((n-2\alpha)/2)}{\Gamma((n+2\alpha)/2)} \right) \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2\alpha}{n}}.$$

The equality in (3) holds if and only if  $f(x) = c(|x - x_0|^2 + t_0^2)$  for  $c \in \mathbb{C}$  and  $(x_0, t_0) \in \mathbb{R}_+^{n+1}$ .

To prove the Theorem 3.1, we need the Lieb's sharp version (see [3]) of the Hardy–Littlewood–Sobolev inequality. For the norm of the Lebesgue space  $L_p(\mathbb{R}^n)$ ,  $p > 1$ , we use the simple notation

$$\|f\|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

**Lemma 3.2.** Let  $\lambda \in (0, n)$  and  $f, g \in L_{\frac{2n}{2n-\lambda}}(\mathbb{R}^n)$ . Then

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \right| \\ & \leq \frac{\pi^{\lambda/2} \Gamma((n-\lambda)/2)}{\Gamma(n-\lambda/2)} \left( \frac{\Gamma(n/2)}{\Gamma(n)} \right)^{\frac{\lambda-n}{2}} \|f\|_{\frac{2n}{2n-\lambda}} \|g\|_{\frac{2n}{2n-\lambda}}. \quad (4) \end{aligned}$$



where the equation (4) holds if and only if  $f$  and  $g$  can be written as  $c(|x - x_0|^2 + t_0)^{(\lambda-2n)/2}$  for  $c \in \mathbb{C}$  and  $(x_0, t_0) \in \mathbb{R}_+^{n+1}$

**Proof of Theorem 3.1.** First of all, note that

$$\varphi(x, t) = \int_{\mathbb{R}^n} \hat{f}(y) \exp(-2\pi(iy \cdot x + |y|t)) dy.$$

Thus, differentiating and integrating (see [4], p. 83) we have

$$\int_{\mathbb{R}^n} |\nabla\varphi(x, t)|^2 dx = 8\pi^2 \int_{\mathbb{R}^n} |x|^2 |\hat{f}(x)|^2 e^{-4\pi|x|t} dx,$$

and

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} |\nabla\varphi(x, t)|^2 t^{1-2\alpha} dx dt \\ = 8\pi^2 \int_0^\infty \int_{\mathbb{R}^n} |x|^2 |\hat{f}(x)|^2 t^{1-2\alpha} e^{-4\pi|x|t} dx dt. \end{aligned}$$

By the Fubini theorem

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} |\nabla\varphi(x, t)|^2 t^{1-2\alpha} dx dt \\ = 8\pi^2 \int_{\mathbb{R}^n} \left( \int_0^\infty t^{1-2\alpha} e^{-4\pi|x|t} dt \right) |x|^2 |\hat{f}(x)|^2 dx \\ = 8\pi^2 \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \frac{y}{4\pi|x|} \right)^{1-2\alpha} e^{-y} \frac{dy}{4\pi|x|} \right) |x|^2 |\hat{f}(x)|^2 dx \\ = \frac{8\pi^2}{(4\pi)^{2(1-\alpha)}} \int_{\mathbb{R}^n} \left( \int_0^\infty y^{1-2\alpha} e^{-y} dy \right) |x|^{2\alpha} |\hat{f}(x)|^2 dx \\ = \frac{8\pi^2}{(4\pi)^{2(1-\alpha)}} \Gamma(1-2\alpha) \int_{\mathbb{R}^n} |x|^{2\alpha} |\hat{f}(x)|^2 dx, \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma(1-2\alpha)} \int_{\mathbb{R}^{n+1}} |\nabla\varphi(x, t)|^2 t^{1-2\alpha} dx dt \\ = \frac{8\pi^2}{(4\pi)^{2(1-\alpha)}} \int_{\mathbb{R}^n} |x|^{2\alpha} |\hat{f}(x)|^2 dx. \end{aligned} \tag{5}$$

This identity together with  $(-\widehat{\Delta})^{\alpha/2} f(x) = (2\pi|x|)^{\alpha} \hat{f}(x)$ , more or less explains why we refer to (3) as a Sobolev trace inequality for fractional derivatives. Next, writing

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \bar{g}(x) dx,$$

for the dual product between two functions  $f$  and  $g$  on  $\mathbb{R}^n$ , we employ the Parseval formula and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} |\langle f, g \rangle| &= |\langle \hat{f}, \hat{g} \rangle| \\ &\leq \left( \int_{\mathbb{R}^n} |x|^{2\alpha} |\hat{f}(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x|^{-2\alpha} |\hat{g}(x)|^2 dx \right)^{\frac{1}{2}}. \end{aligned} \quad (6)$$

Let us now observe that

$$\begin{aligned} &\int_{\mathbb{R}^n} |x|^{-2\alpha} |\hat{g}(x)|^2 dx \\ &= \pi^{2\alpha-n/2} \frac{\Gamma((n-2\alpha)/2)}{\Gamma(\alpha)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{g(x) \bar{g}(y)}{|x-y|^{n-2\alpha}} dx dy; \end{aligned} \quad (7)$$

see [3], Corollary 6.10. Thus, by (4) (where  $\lambda = n - 2\alpha$ ), (6) and (7) we conclude that

$$\begin{aligned} |\langle f, g \rangle| &\leq \left( \int_{\mathbb{R}^n} |x|^{2\alpha} |\hat{f}(x)|^2 dx \right)^{\frac{1}{2}} \pi^{\alpha/2} \left( \frac{\Gamma((n-2\alpha)/2)}{\Gamma((n+2\alpha)/2)} \right)^{\frac{1}{2}} \\ &\quad \times \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{\alpha}{n}} \|g\|_{\frac{2n}{n+2\alpha}}. \end{aligned} \quad (8)$$

This equation together with  $g = f|f|^{\frac{4\alpha}{(n-2\alpha)}}$ ,  $f \in \dot{H}^{\alpha}(\mathbb{R}^n)$  and (5) implies

$$\begin{aligned} \|f\|_{\frac{2n}{n-2\alpha}} &\leq \pi^{\alpha/2} \left( \frac{\Gamma((n-2\alpha)/2)}{\Gamma((n+2\alpha)/2)} \right) \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{\alpha}{n}} \\ &\quad \times \left( \int_{\mathbb{R}^n} |x|^{2\alpha} |\hat{f}(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \left( \frac{2^{1-4\alpha}}{\pi^\alpha \Gamma(2(1-\alpha))} \right)^{\frac{1}{2}} \left( \frac{\Gamma((n-2\alpha)/2)}{\Gamma((n+2\alpha)/2)} \right)^{\frac{1}{2}} \\ &\quad \times \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{\alpha}{n}} \left( \int_{\mathbb{R}^{n+1}} |\nabla\varphi(x, t)|^2 t^{1-2\alpha} dx dt \right)^{\frac{1}{2}}. \quad (9) \end{aligned}$$

Finally, from (6)-(9) and Lemma 3.2 (with  $\lambda = n - 2\alpha$ ) we can see that if the equality in (3) holds then

$$|f(x)|^{\frac{n+2\alpha}{n-2\alpha}} = C_0 \left( |x - x_0|^2 + t_0 \right)^{-\frac{n+2\alpha}{2}},$$

for  $C_0 \geq 0$  and  $(x, t_0) \in \mathbb{R}_+^{n+1}$ . This is just the desired function. On the other hand, a change of variables implies that if  $\phi(x) = \lambda x + x_0$  for  $\lambda > 0$  and  $x_0 \in \mathbb{R}^n$  then

$$\left( \int_{\mathbb{R}^n} |f(\phi(x))|^{\frac{2n}{n-2\alpha}} dx \right)^{\frac{n-2\alpha}{n}} = \left( \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |f(x)|^{\frac{2n}{n-2\alpha}} dx \right)^{\frac{n-2\alpha}{n}}, \quad (10)$$

and

$$\int_{\mathbb{R}^{n+1}} |\nabla\varphi(\phi(x, t))|^2 t^{1-2\alpha} dx dt = \lambda^{2\alpha-n} \int_{\mathbb{R}^{n+1}} |\nabla\varphi(x, t)|^2 t^{1-2\alpha} dx dt, \quad (11)$$

since a straightforward computation with (5) at both sides of (3) yields that the equality in (5) is valid for  $f(x) = (1 + |x|^2)^{(2\alpha-n)/2}$ . We conclude from (10) and (11) that the equality in (5) is also true for the general functions described in Theorem 3.1. Now, the proof is complete.

Finally, let us distinguish the final cases:

**1.**  $\alpha = 0$ . Let us observe that

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} |\nabla\varphi(x, t)|^2 t \, dx \, dt &= \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} t \Delta f^2(x, t) \, dx \, dt \\
&= \frac{1}{2} \int_{\mathbb{R}^n} f^2(x, 0) \, dx \\
&= \frac{1}{2} \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} f^2(x, t) \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} |f(x)|^2 \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} |\hat{f}(x)|^2 \, dx.
\end{aligned}$$

$2\alpha = 1$  (which forces  $n > 2$ ). Since

$$\begin{aligned}
\lim_{\alpha \rightarrow 1} \frac{\int_{\mathbb{R}_+^{n+1}} |\varphi(x, t)|^2 t^{1-2\alpha} \, dx \, dt}{2\Gamma(1-\alpha)} &= 4\pi^2 \int_{\mathbb{R}^n} \left( |x| |\hat{f}(x)| \right)^2 \, dx \\
&= \int_{\mathbb{R}^n} |\nabla\varphi(x)|^2 \, dx.
\end{aligned}$$

The Theorem 3.1 naturally reduces to the Sobolev inequality

$$\left( \int_{\mathbb{R}^n} |f(x)|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \leq \left( \frac{1}{\pi n(n-2)} \right) \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2}{n}} \int_{\mathbb{R}^n} |\nabla\varphi(x)|^2 \, dx, \tag{12}$$

where the equality in (12) holds if and only if  $f(x) = c(|x - x_0|^2 + t_0)^{-(n-2)/2}$  for  $c \in \mathbb{C}$  and  $(x_0, t_0) \in \mathbb{R}_+^{n+1}$ .

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