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Hrushovski constructions in non–elementary classes

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*Esta es una revisión breve acerca de algunos resultados sobre las construcciones de Hrushovski y clases elementales abstractas y algunos resultados de las construcciones de Hrushovski como clases elementales abstractas dada por Villaveces y Zambrano (*2009*).*

Palabras Claves: Construcciones de Hrushovski,
clases elementales abstractas.

*This is a brief survey about some results on Hrushovski constructions and abstract elementary classes and some results of Hrushovski constructions as an abstract elementary class given by Villaveces and Zambrano (*2009*).*

Keywords: Hrushovski constructions, abstract elementary classes.

MSC: 03C48, 03C95, 03C52.

1 Introduction

In this survey, we exhibit some results concerning particular examples of Hrushovski constructions as Abstract Elementary Classes (for short, AECs).

In the second section, we present part of the history of the development of ideas related to Hrushovski constructions, since when Zilber established in the 80's his conjecture about the tricotomy of strongly minimal $\aleph_1$–categorial structures, until recent works of Baudisch, Martin–Pizarro, Ziegler, Hasson, Hils *et al.*

In the third section, we present a general background about AECs, clarifying why tame AECs are important in this setting (assuming tameness, it is possible to prove a categoricity transfer theorem, see [14]; and a stability transfer theorem, see [6]).

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In the fourth section, for the sake of completeness, we present some basic definitions about Hrushovski fusions, following the terminology given by Holland in [21].

In the fifth section, we mention some results of the class of Hrushovski fusions as an AEC (see [34]).

In the sixth section, we mention a couple of open problems towards using the techniques suggested in [34] for proving general results without using algebraic arguments.

2 Some history of Hrushovski constructions

In the 80's, Zilber conjectured that strongly minimal $\aleph_1$–categorical structures are bi–interpretable with either a set without structure, or with a linear space, or with an algebraic closed field of a fixed characteristic. However, Hrushovski gave a counterexample to this conjecture. He constructed a new strongly minimal structure which is not bi–interpretable with any of the kind of structures given above (see [24]), using a technique generalizing Fraïssé limits (the structure obtained in this way is called *generic structure*), which allows to construct a countable model which is strongly minimal, saturated and homogeneous which has infinite Morley rank. After that, this structure is collapsed for obtaining a structure with finite Morley rank.

These examples carry a pre–dimension which is defined on finite subsets of structures in the same language, and include only the models such that this pre–dimension is a non–negative function. This is the *Schanuel condition*, because it is similar to the statement of the *Schanuel conjecture* in complex numbers:

Conjecture. (Schanuel) For every $x_1, \cdots, x_n \in \mathbb{C}$, if $\{x_1, \cdots, x_n\}$ are linearly independent over $\mathbb{Q}$ then we have that

$$trdeg_{\mathbb{Q}}\{x_1, \cdots, x_n; exp(x_1), \cdots, exp(x_n)\} \geq n\,.$$

The notion of self–sufficiency (a key notion in this setting) is strongly based on this pre–dimension.

In [23], Hrushovski made a variation to his construction given in [24] and proved that there exists a strongly minimal set which is bi–interpretable with two algebraic closed fields of distinct characteristics respectively, refuting in this way Zilber's conjecture.

Later, Poizat studied in [28] another example of this type of construction, which was called *bicolored fields*, where he constructed a generic

ω–stable structure of Morley rank $\omega \times 2$. This kind of structures consists of a field F with a distinguished subset N (whose elements are called *black points*). The pre–dimension involves the trascendence degree and the cardinality of certain subset of black points. When the set N corresponds to a divisible torsion–free subgroup of the multiplicative group $(F^X, \cdot)$, this construction is called a *green field* and the points inside N are called *green points* (see [29]). The collapse of this construction is called a *bad field*.

Baldwin and Holland generalized this type of constructions in [22, 4, 5]. Holland proved in [22] that under suitable conditions, the theory of the generic model is model–complete. Baldwin and Holland constructed in [BaHo00] a generic model in the setting of bi–colored fields which is ω–stable of Morley rank $\omega \times k$ $(k < \omega)$ and another one of Morley rank 2. Also, they studied in [5] a generic model in the setting of bi–colored fields, which is ω–stable and has Morley rank k.

Baudisch, Martén–Pizarro and Ziegler gave in [7] a simplified version of the construction of a bicolored field of Morley rank p with a predicate of rank $p-1$, giving also an explicit axiomatization of this class of models.

Hasson and Hils gave in [18] another generalization of the work of Hrushovski, similar to [23] but considering non–disjoint languages. In particular, they proved that if the intersection of the fusioned theories corresponds to the theory of infinite linear spaces over a finite field then the theory of the generic model is ω–stable of Morley rank ω.

Baudisch, Martén–Pizarro and Ziegler studied in [8] the case where the intersection of the involved theories corresponds to the theory of infinite linear spaces over a finite field, following the ideas of Hasson and Hils.

Because of Hrushovski's result in [24], Zilber reformulated his conjecture, saying that the other possibility for this kind of structures is an algebraic closed field of characteristic 0 which carries a pseudo–exponential which satisfies a suitable version of the Schanuel conjecture. In this setting, Zilber proved —under some Diophantine hypotheses— that the theory of the generic model is model–complete and that its completion is superstable ([Zi03]). Some members of the Oxford Logic Group are studying some variants of the Zilber's examples (see [10, 9, 25, 26, 40, 36, 37, 38, 39]).

Recent results by Hasson [17] and Hils [19] explore further connections of Hrushovski constructions to geometric stability theory (standard systems of geometries and an analysis of ranks in the supersimple case).

3 Abstract Elementary Classes

The notion of *abstract elementary class* (for short, AEC) corresponds to a generalization of the notion of *first order elementary class* (class of models of a certain first order theory), given by Jónsson and Shelah ([Jo56, Jo60, Sh88, Sh300]).

Definition 3.1. Let $\mathcal{K}$ be a class of L–structures, where L is a first order language, and $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$. We say that $(\mathcal{K}, \prec_{\mathcal{K}})$ is an *abstract elementary class* if and only if

1. $\prec_{\mathcal{K}}$ partially orders $\mathcal{K}$.
2. If $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
3. (Tarski–Vaught–like axiom[2]) If $M_0, M_1, M_2 \in \mathcal{K}$ are such that $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$ and $M_0 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$.
4. (Isomorphism (1)) Whenever $M \in \mathcal{K}$ and $M \cong N$ then $N \in \mathcal{K}$.
5. (Isomorphism (2)) If M_i and N_i are structures in $\mathcal{K}$ with $M_1 \subseteq M_2$ and $N_1 \prec_{\mathcal{K}} N_2$, and $f_i : M_i \overset{\cong}{\to} N_i$ $(i = 1, 2)$ are isomorphisms such that $f_1 \subseteq f_2$, then $M_1 \prec_{\mathcal{K}} M_2$.
6. (Łoś–Tarski unions of chains (1)) If $\{M_i \mid i < \lambda\} \subset \mathcal{K}$ is a $\prec_{\mathcal{K}}$–increasing and continuous chain, then $\bigcup_{i<\lambda} M_i \in \mathcal{K}$ and $M_k \prec_{\mathcal{K}} \bigcup_{i<\lambda} M_i$ for every $k < \lambda$.
7. (Łoś–Tarski unions of chains (2)) If $\{M_i \mid i < \lambda\} \subset \mathcal{K}$ is a $\prec_{\mathcal{K}}$–increasing and continuous chain and $N \in \mathcal{K}$ is such that $M_k \prec_{\mathcal{K}} N$ for every $k < \lambda$, then $\bigcup_{i<\lambda} M_i \prec_{\mathcal{K}} N$.
8. (Downward Léwenheim–Skolem) There exists a cardinal $LS(\mathcal{K})$ such that for every $M \in \mathcal{K}$ and $X \subseteq |M|$, there exists $N \in \mathcal{K}$ such that $X \subseteq N \prec_{\mathcal{K}} M$, where $\|N\| \leq |X| + LS(\mathcal{K}) + \aleph_0$

Examples 3.2.

1. $(Mod(T), \prec)$, where T is a first order theory and $\prec$ corresponds to the *elementary substructure* relation.
2. $Mod(\psi)$, where $\psi \in L_{\omega_1,\omega}$ (see [30, 31]).

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[2] Also called 'Coherence Axiom' or 'Triangle Axiom'.

For basic facts about AECs, see [3, 11].

Through this section, let $\mathcal{K}$ be an AEC.

Definition 3.3. We say that $\mathcal{K}$ is λ–categorical iff for every $M, N \in \mathcal{K}$ of size λ are isomorphic.

Example 3.4. Let $T := \{\forall x(x = x)\}$, where $L = \{=\}$. Notice that $M \cong N$ iff $\|M\| = \|N\|$. So, T is λ–categorical for every cardinality λ.

Two key points in the study of AECs are the stability (see definition 3.10) and categoricity spectrum, *i.e.*: we want to know the cardinalities where $\mathcal{K}$ is stable and categorical. In general, these are very difficult questions, but we know some partial results in this setting (see [6, 14]).

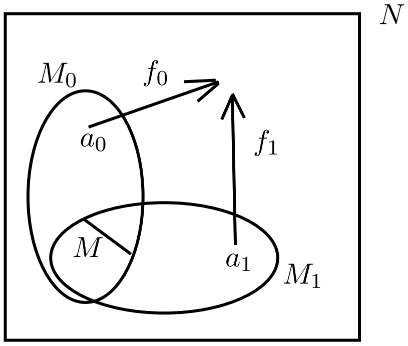
Shelah conjectured that there exists a cardinality $\mu(\kappa)$ such that for every AEC $\mathcal{K}$ with $LS(\mathcal{K}) \leq \kappa$, if $\mathcal{K}$ is λ–categorical for some $\lambda > \mu(\kappa)$ then $\mathcal{K}$ is μ–categorical for every $\mu > \mu(\kappa)$. This conjecture is still open. However, there exist some partial answers to this conjecture. One of them corresponds to the result given by Grossberg and VanDieren in the setting of *tame* AECs (see [13, 14]).

Definition 3.5. (Amalgamation property) We say that $\mathcal{K}$ satisfies the *amalgamation property* iff for every $M_i, M \in \mathcal{K}$ $(j = 0, 1)$ such that $M \prec_{\mathcal{K}} M_j$, there are $N \in \mathcal{K}$ and $\prec_{\mathcal{K}}$–embeddings $f_j : M_j \to N$ $(j = 0, 1)$ such that $f_0 \upharpoonright M = f_1 \upharpoonright M$

$$\begin{tikzcd}
M_0 \arrow[r, dotted, "f_0"] & N \\
M \arrow[u, "id"] \arrow[r, "id"'] & M_1 \arrow[u, dotted, "f_1"']
\end{tikzcd}$$

Definition 3.6. Let M_i, M be L–structures in $\mathcal{K}$ $(j = 0, 1)$ such that $M \prec_{\mathcal{K}} M_j$ and $\overline{a}_j \in M_j$ $(j = 0, 1)$ are tuples of the same length. Define the relation E by $(\overline{a}_0, M, M_0)E(\overline{a}_1, M, M_1)$ iff there are $N \in \mathcal{K}$ and $\prec_{\mathcal{K}}$–embeddings $f_j : M_j \to N$ $(j = 0, 1)$ such that $f_0(\overline{a}_0) = f_1(\overline{a}_1)$ and $f_0 \upharpoonright M = f_1 \upharpoonright M = id_M$

Remark 3.7. If $\mathcal{K}$ has the *amalgamation property*, then E is an equivalence relation.



Definition 3.8. Let $M, N \in \mathcal{K}$ (where $\mathcal{K}$ has the *amalgamation property*) and $\overline{a} \in N$, we define the *Galois–type* of $\overline{a}$ over M in N (which we denote by ga-tp$(\overline{a}/M, N)$) as the equivalence class $(\overline{a}, M, N)/E$. Additionally, if $\alpha > 0$ is an ordinal, we define ga-S$^{\alpha}(M) := \{$ga-tp$(\overline{a}/M, N) \mid M \prec_{\mathcal{K}} N$ and $\overline{a} \in N^{\alpha}\}$. We can drop the index α if it is clear,

Definition 3.9. Let $N, M_0, M_1 \in \mathcal{K}$ be such that $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} N$. If $p :=$ ga-tp$(\overline{a}/M_1, N)$, define $p \restriction M_0 :=$ ga-tp$(\overline{a}/M_0, N)$.

Definition 3.10. Let $\kappa \geq LS(\mathcal{K})$. We say that $\mathcal{K}$ is κ–stable iff for every $M \in \mathcal{K}$ of size κ we have that ga-S$(M) \leq \kappa$.

In first order logic, we have that if two syntactic types (over the same set of parameters) are different, so that difference can be codified by a countable countable subset (in fact, finite) of parameters. The following definition intends to generalize that behavior.

Definition 3.11. Let $\kappa \geq LS(\mathcal{K})$. We say that $\mathcal{K}$ is κ–tame iff for every $M \in \mathcal{K}$ of size $> \kappa$ and $p, q \in$ ga-S(M), if $p \neq q$ then there exists $N \prec_{\mathcal{K}} M$ of size κ such that $p \restriction N \neq q \restriction N$.

Example 3.12.

1. Let $\mathcal{K} := Mod(T)$, where T is a first order theory, where $L(T)$ is a countable language. Then $\mathcal{K}$ is $\aleph_0$–tame.
2. Excellent classes are tame (see [12]).

We have then the following results for the stability and categoricity spectrum in tame AECs.

Definition 3.13. We say that $\mathcal{K}$ is ω–local iff for every $\prec_{\mathcal{K}}$–increasing and continuous chain $\langle M_i : i < \omega\rangle$ and a sequence of Galois–types $\langle p_i : i < \omega\rangle$ such that $p_i \in$ ga-S(M_i) and $p_i = p_{i+1} \restriction M_i$ for every $i < \omega$, there exists a unique $p \in$ ga-S$(\bigcup_{i<\omega} M_i)$ such that $p_i = p \restriction M_i$ for every $i < \omega$.

Theorem 3.14. (Baldwin–Kueker–VanDieren [6]) *Let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) = \aleph_0$ that is ω–local and $\aleph_0$–tame. If $\mathcal{K}$ is $\aleph_0$–stable then $\mathcal{K}$ is stable in all cardinalities.*

In the setting of *metric abstract elementary classes* (MAECs, for short; see [20]) we have a similar result, but just for cardinalities κ which satisfy $\kappa = \kappa^{\aleph_0}$ (see [35]).

Definition 3.15. We say that $\mathcal{K}$ satisfies the *joint embedding property* (for short, JEP) iff for every $M_0, M_1 \in \mathcal{K}$ there exist $N \in \mathcal{K}$ and $\prec_{\mathcal{K}}$–embeddings $f_j : M_j \to N$ $(j = 0, 1)$.

$$\begin{tikzcd}
M_0 \arrow[r, dotted, "f_0"] & N \\
& M_1 \arrow[u, dotted, "f_1"']
\end{tikzcd}$$

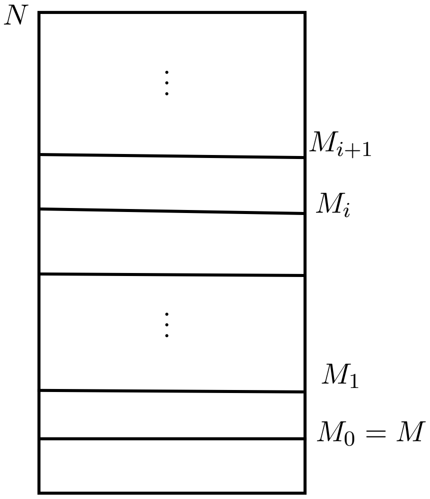
Theorem 3.16. *(Grossberg–VanDieren, see [15]) Suppose $\mathcal{K}$ is a χ–tame AEC satisfying the amalgamation and joint embedding properties. Let $\mu_0 := Hanf(\mathcal{K})$. If $\chi \leq \beth_{(2^{\mu_0})^+}$ and $\mathcal{K}$ is categorical in some $\lambda^+ > \beth_{(2^{\mu_0})^+}$, then $\mathcal{K}$ is μ–categorical for all $\mu > \beth_{(2^{\mu_0})^+}$.*

In this setting, uniqueness of *limit models* plays a very important role —similar to the role of saturated models in the classical Morley's theorem— in the proof given by Grossberg and VanDieren of their version of the categoricity transfer theorem in tame AECs. Under some assumptions of superstability in AECs —which are implied by the assumptions of the Grossberg–VanDieren result—, Grossberg, VanDieren and Villaveces proved in [16] that limit models are unique (up to isomorphisms).

Definition 3.17. Let $M, N \in \mathcal{K}$ be such that $M \prec_{\mathcal{K}} N$. We say that N is *μ–universal over M* iff for every $M' \succ_{\mathcal{K}} M$ of size μ we have that there exists a $\mathcal{K}$–embedding $f : M' \to N$ which fixes pointwise M. We say that N is universal over M iff N is $|M|$–universal over M.

$$\begin{tikzcd}
M \arrow[r, "id"] \arrow[d, "id"'] & N \\
M' \arrow[ur, dotted, "f"']
\end{tikzcd}$$

Definition 3.18. Let $M, N \in \mathcal{K}$ be such that $M \prec_{\mathcal{K}} N$, where $\|M\| = \mu$. We say that N is (μ, θ)–limit over M iff there exists an increasing and continuous $\prec_{\mathcal{K}}$–chain $(M_i : i < \theta)$ such that $M_0 = M$, $\bigcup_{i<\theta} M_i = N$, $\|M_i\| = \mu$ for every $i < \theta$ and also M_{i+1} is μ–universal over M_i.



Definition 3.19. (μ–Disjoint amalgamation property) We say that $\mathcal{K}$ satisfies the *μ–disjoint amalgamation property* (for short, μ–DAP) iff for every $M_j, M \in \mathcal{K}$ $(j = 0, 1)$ of size μ such that $M \prec_{\mathcal{K}} M_j$, there are $N \succ_{\mathcal{K}} M_1$ of size μ and a $\prec_{\mathcal{K}}$–embedding $f : M_0 \to N$ which fixes pointwise M such that $f(M_0) \cap M_1 = M$

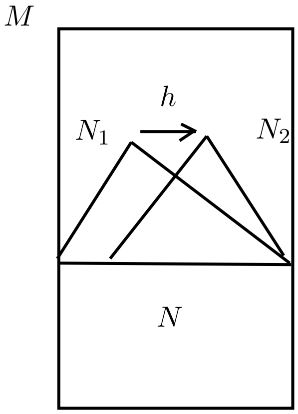
$$\begin{tikzcd}
M_0 \arrow[r, dotted, "f"] & N \\
M \arrow[u, "id"] \arrow[r, "id"'] & M_1 \arrow[u, dotted, "id"']
\end{tikzcd}$$

Example 3.20. If T is a complete first–order theory then $(Mod(T), \prec)$ has the λ–DAP for all $\lambda = |L(T)| + \aleph_0$.

In this setting, we do not work with syntactic types. However, we have a notion of independence which under stability assumptions satisfies nice properties such as locality, or existence and uniqueness of extensions over universal models.

Definition 3.21. A type $p \in ga - S(M)$ μ–splits over $N \in \mathcal{K}$ (of size $\leq \mu$) if and only if $N \prec_{\mathcal{K}} M$ and there there exist $N_1, N_2 \in \mathcal{K}$ of size

μ and a $\prec_{\mathcal{K}}$–embedding h such that $N \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} M$ for $l = 1, 2$ and $h : N_1 \cong N_2$ with $h \restriction N = id_N$ and $p \restriction N_2 \neq h(p \restriction N_1)$.



Definition 3.22. Let $\mathcal{K}$ be an AEC with the μ–DAP and JE. We say that non–μ–splitting satisfies the locality (also called continuity) and existence property (respectively) iff for all infinite α for every sequence $(Mi : i < \alpha)$ of limit models of cardinality μ and for every $p \in ga{-}S(M_\alpha)$ we have that

1. (locality) If for every $i < \alpha$ the type $p \restriction M_i$ does not μ–split over M_0, then p does not μ–split over M_0.
2. (existence) There exists $i < \alpha$ such that p does not μ–split over M_i.

Fact 3.23. *[Grossberg–VanDieren–Villaveces, see [16]] Let $\mathcal{K}$ be an AEC without maximal models, and $\mu > LS(\mathcal{K})$. Suppose $\mathcal{K}$ satisfies the μ–DAP. If $\mathcal{K}$ is μ–stable, and satisfies locality and existence of non–μ–splitting, then any two (μ, σ_l)–limits over M (for $l \in \{1, 2\}$) are isomorphic over M.*

4 Hrushovski fusions over disjoint languages.

We follow the setting given by Holland in [21]. In this section, for the sake of completeness, we give some of the most important results.

We are not considering here the more general fusions over non–disjoint languages, studied by Hasson and Hils in [18].

4.1 Pregeometries

Notation 4.1. Let A, B be sets. As usual, we denote the union $A \cup B$ by AB. If a is some element, we denote the union $A \cup \{a\}$ by Aa.

Notation 4.2. Given a set X, we denote $[X]^{<\omega} := \{B \in \mathcal{P}(X) \mid |B| < \aleph_0\}$. Aditionally, $A \subseteq_{finite} X$ means $A \in [X]^{<\omega}$

Definition 4.3. Let L be a first order language, M an L–structure and $A \subseteq |M|$. Then $a \in acl(A)$ if and only if there exist an L–formula $\varphi(x, \bar{y})$, $\bar{b} \in A$ and $n < \omega$ such that $M \models \varphi(a; \bar{b}) \wedge \exists^{\leq n} x \varphi(x; \bar{b})$. $acl(A)$ is called the *algebraic closure* of A.

Definition 4.4. For T a theory in a first order language L, we say that T is *strongly minimal* if and only if for every model $M \models T$ every definable set inside M is finite or cofinite.

The following basic remark is crucial for the treatment of fusions:

Remark 4.5. For T a strongly minimal theory, and $\mathfrak{A} \models T$, if $B \subseteq |\mathfrak{A}|$ and $a \in |\mathfrak{A}|$, the fact

$$a \notin acl(B)$$

is type–definable.

Definition 4.6. Let X be a non–empty set and $cl : \mathcal{P}(X) \to \mathcal{P}(X)$. We say that (X, cl) is a *pregeometry* iff for every $A, B \in \mathcal{P}(X)$:

1. $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$.
2. (Finite character) If $a \in cl(A)$ then there exists $B \subseteq_{finite} A$ such that $a \in cl(B)$.
3. (Monotonicity) $A \subseteq B$ implies $cl(A) \subseteq cl(B)$
4. (Exchange) If $a \in cl(Ab) \setminus cl(A)$ then $b \in cl(Aa)$

Examples 4.7.

1. (X, id) is a pregeometry. It is called the *trivial* pregeometry.
2. Let T be a strongly minimal theory, $M \models T$. Then, (M, acl) (where acl is the algebraic closure) is a pregeometry.
3. Let V a linear space. $(V, spam(\cdot))$ is a pregeometry.

Definition 4.8. Given a pregeometry (G, cl) and $X \subseteq G$, we say that X is *closed* if $X = cl(X)$.

Example 4.9. Let (X, cl) be a pregeometry. Notice that $cl(A)$ is a closed set (by definition of pregeometry).

Definition 4.10. Let (G, cl) be a pregeometry and $X \subseteq G$ be closed. $Y \subseteq X$ is a *base* for X if it is minimal such that $cl(Y) = X$. We say that $Y \subseteq G$ is *independent* if it is a base for $cl(Y)$.

Example 4.11. Let V a linear space and $(V, spam(\cdot))$ the pregeometry associated to V. $X \subseteq V$ is independent in the sense of pregeometry iff it is independent in the sense of linear spaces.

Proposition 4.12. *For every pregeometry (G, cl) and closed $X \subseteq G$, $Y \subseteq X$ is a base for X if and only if it is independent and $cl(Y) = X$.*

Fact 4.13. *Let (G, cl) be a pregeometry and $X \subseteq G$ a closed set. $Y \subseteq X$ is a base for X if and only if it is maximal among independent subsets of X.*

Fact 4.14. *Let (G, cl) be a pregeometry, $X \subseteq G$ a closed set and $Y \subseteq G$ such that $cl(Y) = X$. Then there exists $W \subseteq Y$, a base for X.*

Notation 4.15. By exchange property, if A and B are bases for $cl(X)$ then $|A| = |B|$. We call that cardinal number the *cl–dimension* of X and denote it by $d(X)$. The prefix *cl* may be omitted when obvious from context.

The following fact is well known.

Fact 4.16. *Let (G, cl) be a pregeometry, $X, Y \subseteq G$ and d the corresponding cl–dimension. Then d satisfies:*

1. $d(X) \leq |X|$
2. *(submodularity)* $d(XY) + d(X \cap Y) \leq d(X) + d(Y)$.
3. *(monotonicity) If* $X \subseteq Y$ *then* $d(X) \leq d(Y)$.

Definition 4.17. If (G, cl) is a pre–geometry and $Y, W \subseteq G$, we say that Y is *cl–independent* over W if and only if $d(Y'W') = |Y'| + d(W')$ for every $Y' \in [Y]^{<\omega}$ and every $W' \in [W]^{<\omega}$. A *base* for X over W is a set $Y \subseteq X$, which is maximal independent over W.

Notation 4.18. If $Y_1, Y_2 \subseteq X$ are bases for X over W, then $|Y_1| = |Y_2|$. Therefore, it makes sense to define the *cl*–dimension of X over W as the cardinality of a base of X over W; we denote this cardinal by $d(X/W)$.

When clear from context, we omit the prefix *cl* from the previous definition.

Lemma 4.19. *Let (G, cl) be a pregeometry, $W \subseteq G$ and $a \in G \setminus cl(W)$. Then $d(Wa) = d(W) + 1$*

Proposition 4.20. *Let (G, cl) be a pregeometry and $Y, W \subseteq G$. Then Y is independent over W if and only if for every $a \in Y$ we have that $a \notin cl(W(Y \setminus \{a\}))$.*

Proof. Let $a \in Y$ and $B \subseteq_{finite} W(Y \setminus \{a\})$. If $a \in cl(B)$, then $cl(B) = cl(Ba)$, so $d(B) = d(Ba)$. Let $B_1 := B \cap (Y \setminus \{a\})$ and $B_2 := B \cap W$. Since $B_1, B_1 a \in [Y]^{<\omega}$, $B_2 \in [W]^{<\omega}$, then $|B_1| + d(B_2) = d(B_1 B_2) = d(B) = d(Ba) = d((B_1 a)B_2) = |B_1 a| + d(B_2)$, hence $|B_1| = |B_1 a|$ (impossible, since $a \notin B_1$ and B_1 is finite). Therefore $a \notin cl(B)$ and by the finite character of *cl* we have $a \notin cl(W(Y \setminus \{a\}))$

Conversely, assume that for every $a \in Y$ we have that $a \notin cl(W(Y \setminus \{a\}))$. Let $Y' := \{a_1, \cdots, a_n\} \in [Y]^{<\omega}$ and $W' \in [W]^{<\omega}$. As $a_1 \notin cl(W')$ (since otherwise $a_1 \in cl(W(Y \setminus \{a_1\}))$) then $d(W' a_1) = d(W) + 1$, by lemma 4.19. Following a similar reasoning, we get that $a_i \notin cl(W' \cup \{a_1, \cdots, a_{i-1}\})$ and therefore $d(W' \cup \{a_1, \cdots, a_i\}) = d(W') + i$ $(i = 2, \cdots, n)$. So, $d(W'Y') = d(W') + |Y'|$.

□

Proposition 4.21. *If $X \subseteq cl(W)$ then $d(X/W) = 0$.*

Proof. Let $X \subseteq cl(W)$ and $Y \subseteq X$ be independent over W. If $Y \neq \emptyset$, there exists $a \in Y$, and since $Y \subseteq X \subseteq cl(W) \subseteq cl(W(Y \setminus \{a\}))$ then $a \in cl(W(Y \setminus \{a\}))$. (contradicts proposition 4.20). Then $Y = \emptyset$, so $d(X/W) = 0$.

□

4.2 Fusions over disjoint languages

Through this subsection, let T_1, T_2 be complete first order, strongly minimal and model–complete theories, in languages L_1 and L_2 respectively, where $L_1 \cap L_2 = \{=\}$. Also consider the corresponding dimension function d_i based on algebraic closures in the language L_i $(i = 1, 2)$.

Definition 4.22. Let $X \subseteq_{finite} |M|$, where $M \models T_1 \cup T_2$. Then we define $d_0 : [|M|]^{<\omega} \to \mathbb{Z}$ by

$$d_0(X) := d_1(X) + d_2(X) - |X|.$$

If $M \models T_1 \cup T_2$ and $d_0(X) \geq 0$ for every $X \in [|M|]^{<\omega}$ then we say that M is a *fusion* over L_1 and L_2.

Fact 4.23. *If T is a strongly minimal theory, $M \models T$ and $\{a_1, \cdots, a_n\} \subseteq |M|$ is such that $d(\{a_1, \cdots, a_n\}) = k$, then there exists an $L(T)$–formula $\varphi(x_1, \cdots, x_n)$ such that*

1. *$M \models \varphi[a_1, \cdots, a_n]$ and*
2. *$M \models \varphi[b_1, \cdots, b_n]$ iff $d(\{b_1, \cdots, b_n\}) \leq k$.*

The class of fusions over T_1 and T_2 is axiomatizable:

Fact 4.24 (Holland)**.** *The class of fusions over T_1 and T_2 is elementary, with the axiomatization $T_1 \cup T_2$ plus axioms of the form*

$$\forall \overline{x} \left((\varphi_1(\overline{x}) \wedge \varphi_2(\overline{x})) \to \bigvee_{i \neq j} x_i = x_j \right)$$

where for k_1, $k_2 \in \mathbb{N}$ such that $k_1 + k_2 < |\overline{x}|$ we have that φ_i is a L_i–formula such that if φ_i occurs in a model of T_i then $d_i(\overline{x}) \leq k_i$ ($i = 1, 2$).

Notation 4.25. T_{fus} denotes the previous axiomatization.

Here are some properties of the function d_0 we defined above.

Fact 4.26. *For $M \models T_1 \cup T_2$, d_0 the previously defined function and $X, Y \in [|M|]^{<\omega}$, we define $d_0(X/Y) := d_0(XY) - d_0(Y)$. Then for every $X, Y \in [|M|]^{<\omega}$ we have:*

1. *$-|X| \leq d_0(X/Y)$*
2. *$d_0(X) \leq |X|$.*
3. *(submodularity) $d_0(XY) + d_0(X \cap Y) \leq d_0(X) + d_0(Y)$*

Definition 4.27. A function $\delta : \mathcal{K} \to \mathbb{Z}$ (where $\mathcal{K}$ is a class of finite subsets of structures in the same language) is said to be a *predimension* if it satisfies properties (2) and (3) of fact 4.26

Remark 4.28. Notice that $d_0(X/Y) = d_0(XY) - d_0(Y) = d_0(X \setminus Y/Y)$ if X, Y are finite.

Definition 4.29. For $X, Y \subseteq M$, where $M \models T_1 \cup T_2$ and X is finite, we define $d_0(X/Y) := \min\{d_0(X/Y') \mid X \cap Y \subseteq Y' \subseteq_{finite} Y\}$.

Remark 4.30. Definition 4.29 extends the case Y finite: if we write $d_0'(X/Y) := \min\{d_0(X/Y') \mid X \cap Y \subseteq Y' \subseteq_{finite} Y\}$ then $d_0'(X/Y) \leq d_0(X/Y)$ (as $X \cap Y \subseteq Y \subseteq_{finite} Y$); and as $X \cap Y \subseteq W \subseteq_{finite} Y$ is such that $d_0'(X/Y) = d_0(X/W)$, since $X \cap Y = X \cap W$ and $W \subseteq Y$ by Remark 4.28 we have $d_0(X/Y) \leq d_0(X/W) = d_0'(X/Y)$.

Fact 4.31. *Let $X, Y \subseteq M$, where $M \models T_1 \cup T_2$ and X is finite (and Y is possibly infinite). We have that $d_0(X/Y) = d_1(X/Y) + d_2(X/Y) - |X \setminus Y|$.*

Definition 4.32. Let $M \models T_1 \cup T_2$ be a fusion over T_1 and T_2, $U \subseteq |M|$ and $X \in [U]^{<\omega}$. We define $d(X;U) := \min\{d_0(X') \mid X \subseteq X' \subseteq_{finite} U\}$.

It is crucial to ask here that M be a fusion so that $d(X;U)$ exists —in that case it is the minimum of a nonempty set of natural numbers.

Remark 4.33. It is relatively easy to show that $d(\cdot) := d(\cdot;U)$ (where $U \subseteq |M|$ is fixed and M is a fusion) satisfies:

1. $d(X) \leq |X|$
2. (submodularity) $d(XY) + d(X \cap Y) \leq d(X) + d(Y)$
3. (monotonicity) If $X \subseteq Y$ then $d(X) \leq d(Y)$

Because of that, there exists a natural pregeometry on U defined in the following way: $a \in cl(X)$ if and only if there exists $Y \in [X]^{<\omega}$ such that $d(Ya) = d(Y)$ (intuitively, closure in Ya works just as in Y).

For the remainder of this section, we assume that A and B are subsets of a fusion. The following fact is very important, as the notion of being a *self–sufficient subset* depends on it.

Fact 4.34 (Holland)**.** *For every $A \subseteq B$, the following statements are equivalent:*

1. *For every $X \in [A]^{<\omega}$, $d(X;A) = d(X;B)$.*
2. *For $X \in [A]^{<\omega}$ there exists $X \subseteq Y \subseteq_{finite} A$ such that $d_0(Y) = d(X;B)$.*

3. *For every* $Y \in [B]^{<\omega}$ $d_0(Y/Y \cap A) \geq 0$.

Moreover, if A *is finite, then* **1**, **2** *and* **3** *are equivalent to*

4. $d_0(A) = d(A; B)$.

Proof.

1. (1) ⇔ (2). It is straightforward.
2. (1) ⇒ (3). Suppose that for some $Y \in [B]^{<\omega}$ we have that $d_0(Y/Y \cap A) < 0$. Let $Y \cap A \subset Z \subset_{finite} A$ such that $d_0(Z) = d(Y \cap A/A)$. Notice that $d_0(Z) = d(Z; A)$. Therefore

$$\begin{aligned} d_0(Y/Z) &= d_0(YZ) - d_0(Z) \\ &\leq d_0(Y) - d_0(Y \cap Z) \text{ (by submodularity)} \\ &= d_0(Y(Y \cap A)) - d_0(Y \cap A) \text{ (since } Y \cap A = Y \cap Z) \\ &= d_0(Y/Y \cap A) \\ &< 0 \end{aligned}$$

   Therefore, $d(Z; B) \leq d_0(YZ) < d_0(Z) = d(Z; A)$, so (1) fails (contradiction).
3. (3) ⇒ (2). Let $X \subset A$. Let $Z \subseteq B$ be such that $X \subseteq Z$ and $d_0(Z) = d(X; B)$. By (3), we have that $d(X; B) = d_0(Z) \geq d_0(Z \cap A)$. Since $X \subseteq Z \cap A \subseteq B$, then $d(X; B) \leq d_0(Z \cap A)$. Therefore $d_0(Z \cap A) = d(X; B)$. Take $Y := Z \cap A$.
4. (1) → (4). If A is finite, notice that $d_0(A) = d(A; A)$. Therefore by (1) we have that $d_0(A) = d(A; A) = d(A; B)$.
5. (4) ⇒ (3). Suppose that $d_0(A) := d(A; B)$, and let $Y \subseteq_{finite} B$. Since $d(A; B) \leq d_0(YA)$, then $0 \leq d_0(YA) - d_0(A) \leq d_0(Y) - d_0(Y \cap A) = d_0(Y/Y \cap A)$ (by submodularity and definitionof d_0, see 4.29).

□

Definition 4.35. Assume that $A \subseteq B$. We say that A is *self–sufficient* in B (denoted $A \leq B$) if and only if any of the conditions of 4.34 holds[3].

If M, N are fusions such that $M \subseteq N$, we say that M is *self–sufficient* in N (denoted $M \leq N$) if and only if $|M| \leq |N|$

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[3] Other authors use ‘strong’ instead of ‘self–sufficient’.

Proposition 4.36. *If* $A \subseteq B$*,* $d_0(X/A) \geq 0$ *for each* $X \in [B]^{<\omega}$ *iff* $A \leq B$*.*

Proof. Since $d_0(X/A) := \min\{d_0(X/Y) \mid X \cap A \subseteq Y \subseteq_{finite} A\}$ and $X \cap A \subseteq X \cap A \subseteq_{finite} A$ then $d_0(X/A) \leq d_0(X/X \cap A)$. By hipothesis, $0 \leq d_0(X/A)$, so $0 \leq d_0(X/X \cap A)$. Therefore, from Fact 4.34 (3) we may conclude $A \leq B$.

Conversely, let $X \cap A \subseteq Y \subseteq_{finite} A$ be such that $d_0(X/Y) = d_0(X/A)$. Take $X' := XY \in [B]^{<\omega}$. Since $A \leq B$,

$$\begin{aligned} 0 &\leq d_0(X'/X' \cap A) \\ &= d_0(XY/(XY) \cap A) \\ &= d_0(XY/(X \cap A)(Y \cap A)) \\ &= d_0(XY/Y) \\ &= d_0(XY) - d_0(Y) \\ &= d_0(X/Y) \\ &= d_0(X/A) \end{aligned}$$

□

Corollary 4.37. *If* $A \subseteq B$ *and* $d_0(X/A) \geq 0$ *for every* $X \in [B \setminus A]^{<\omega}$*, then* $A \leq B$*.*

Proof. Use proposition 4.36 and remark 4.28.

□

Proposition 4.38 (Holland)**.** *Let* $i \in \{1, 2\}$ *and* $j = 3-i$*; if* $acl_i(W) \setminus W$ *is* j*–independent over* W *then* $W \leq acl_i(W)$*.*

Proof. Let $X \subseteq_{finite} acl_i(W) \setminus W$. X is j–independent over W (otherwise, there would be some $X' \subseteq_{finite} X$ and some $W' \subseteq_{finite} W$ such that $d_j(X'W') \neq |X'| + d_j(W')$, and since $X \subseteq acl_i(W) \setminus W$ they would contradict the j–independence of $acl_i(W) \setminus W$ over W). Since $X \subseteq acl_i(W)$, we have $d_i(X/W) = 0$, by Proposition 4.21. Therefore, $d_0(X/W) = d_j(X/W) - |X|$. Since $d_j(X/W) = |X|$, we have $d_0(X/W) \geq 0$. So, $W \leq acl_i(W)$, using Fact 4.37.

□

5 Hrushovski fusions as an AEC

Villaveces and the author studied in [34] the class of Hrushovski fusions together with the self–sufficient relation $\leq$. In this work, we do not consider the theory of the generic model.

Definition 5.1. Let L be a first order language and $L' \supset L$. Let δ be a predimension function (see definition 4.27) defined on every finite subset of every structure in a fixed class $\mathcal{K}$ of L'–structures. We say that a complete L–type p is *δ–locally Schanuel for $\mathcal{K}$* if for every realization of p which is inside of a model in $\mathcal{K}$, say $\overline{e} \models p(\overline{x})$, every finite subtuple $\overline{e}' \lhd \overline{e}$ satisfies $\delta(\overline{e}') \geq 0$

Proposition 5.2. *Let $p_1(\overline{x})$ and $p_2(\overline{x})$ be two complete d_0–locally Schanuel for $\mathcal{K}_{fus}$ types over $\emptyset$ in L_1, L_2 respectively, where these types have different realizations in T_1 and T_2 respectively. Then there exists a fusion N and a realization $\overline{b}$ of $p_1(\overline{x}) \cup p_2(\overline{x})$ in N such that $\overline{b} \leq N$.*

Proof. Let $\overline{x}^0 := \overline{x}$, $p_i^0 := p_i$ $(i = 1, 2)$ and $\overline{m}^0$ be a realization of $p_1(\overline{x})$ in a model of T_1. Extend $\overline{m}^0$ to some enumeration $\overline{m}^1$ of $acl_1(\overline{m}^0)$ in that model, taking $p_1^1(\overline{x}^1) := tp_{L_1}(\overline{m}^1/\emptyset)$. Extend $p_2^0(\overline{x})$ to a complete L_2–type $p_2^1(\overline{x}^1)$ making sure the new variables in $\overline{x}^1$ are 2–independent over $\overline{x}^0$. Alternanting the roles of L_1 and T_1 along this process with those of L_2 and T_2, we obtain two chains $p_i^0(\overline{x}^0) \subseteq p_i^1(\overline{x}^1) \subseteq p_i^2(\overline{x}^2) \subseteq \cdots$ of complete L_i–types $(i = 1, 2)$, taking $q_i := \bigcup_{n<\omega} p_i^n$ $(i = 1, 2)$. Since $L_1 \cap L_2 = \{=\}$, by Robinson's Consistency Theorem we conclude that $q_1 \cup q_2$ is consistent. If $\overline{a}$ realizes $q_1 \cup q_2$, then we have $acl_i(\overline{a}) = \overline{a}$ $(i = 1, 2)$ (if we take $a' \subseteq_{finite} \overline{a}$ this subtuple has been considered in a step of the construction of q_1 and q_2; call this step $n < \omega$ and $\overline{b}^n$ the subtuple of $\overline{a}$ which realizes the types $p_j^n(\overline{x}^n)$ $(j = 1, 2)$, and since $\overline{b}^{n+1} = acl_k(\overline{b}^n)$ for some $k \in \{1, 2\}$ (by the construction of the types $p_j(\overline{x}^{n+1})$ $j = 1, 2)$ then $acl_i(a') \subseteq acl_i(acl_k(\overline{b}^n)) = acl_i(\overline{b}^{n+1}) \subseteq \overline{a}$ (if $k = i$ $acl_i(\overline{b}^{n+1}) = \overline{b}^{n+1}$ and $k \neq i$ $acl_i(\overline{b}^{n+1}) = \overline{b}^{n+2}$), so by the finite character of acl_i we have $acl_i(\overline{a}) \subseteq \overline{a}$). Additionally, by a similar argument, $\overline{a}$ is a $L_1 \cup L_2$–structure, which we denote by N'. We may consider a sufficiently saturated model $\mathfrak{C} \models T_{fus}$, so there is a realization $N \models T_{fus}$ of the type $q_1 \cup q_2$. We use the fact that p_1 and p_2 are d_0–locally Schanuel for $\mathcal{K}_{fus}$ in this part, in order to guarantee that any realization of $p_1 \cup p_2$ satisfies the Schanuel condition. On the other hand, we have the subtuple $\overline{b}^n$ of N realizing the types $p_j^n(\overline{x}^n)$ $(j = 1, 2)$ then $\overline{b} := \overline{b}^0 \leq \overline{b}^n$ for every $n < \omega$. For $n = 0$, this is obvious. If we assume that for some $n < \omega$ we have $\overline{b} \leq \overline{b}^n$, then by construction $\overline{b}^{n+1} = acl_k(\overline{b}^n)$ for some

$k \in \{1,2\}$ and since $\overline{b}^{n+1} \setminus \overline{b}^n$ is j–independent over $\overline{b^n}$ $(j \in \{1,2\} \setminus \{k\})$ then $\overline{b}^n \leq acl_k(\overline{b}^n) = \overline{b}^{n+1}$ (proposition 4.38), and by the transitivity of $\leq$ we have $\overline{b} \leq \overline{b}^{n+1}$. As $N = \bigcup_{n<\omega} \overline{b}^n$ then by Fact 4.34 (3) we have $\overline{b} \leq N$.

□

Definition 5.3. Let T be a first order ω–stable theory, $M \models T$ and $A, B, C \subset |M|$ such that (without loose of generality) $C \subseteq A \cap B$. We say that A *does not fork* from B over C (which we denote by $A \mathop{\smile\!\!\!\!\!\!|}_C B$) if for every $\overline{a} \in A$ we have that $MR(\overline{a}/B) = MR(\overline{a}/C)$, where MR denotes the Morley Rank.

Fact 5.4. *Let T be a first order strongly minimal theory, M be a model of T, $B \subseteq |M|$ and $\overline{a} \in M$. Then $MR(\overline{a}/B) = d(\overline{a}/B)$, where d denotes the acl–dimension mapping.*

Reference. [27], theorem 6.2.19 □

Notation 5.5. $A \mathop{\smile\!\!\!\!\!\!|}_C^i B$ $(i \in \{1,2\})$ means that A does not fork from B over C, in the sense of the language L_i.

For the sake of completeness, we mention the following well known model–theoretic facts:

Fact 5.6. *If T is a strongly minimal theory, then T is ω–stable*

Fact 5.7. *If T is ω–stable, and $A, B \subset \mathfrak{C}$ (where $\mathfrak{C}$ is a monster model of T) then there exists $B_0 \subset_{finite} B$ such that A does not fork from B over B_0.*

Holland proved in [21] the following version of the *amalgamation property*:

Proposition 5.8 (Amalgams of Fusions)**.** *Let $M \leq N_i$ $(i = 1,2)$ be fusions. Then there are $N_i' \cong_M N_i$ and K fusions such that $N_i' \leq N_1' N_2' \leq K$.*

Proof. Let $M \leq N_i$ $(i = 1,2)$ be fusions. Without loose of generality, we may assume that $N_1 \cap N_2 = M$ and

$$N_1 \mathop{\smile\!\!\!\!\!\!|}_M^j N_2 \ (j = 1,2)\,. \qquad (*)$$

Consider an enumeration $\overline{m}$ of M and an enumeration $\overline{n}_i$ of $N_i \setminus M$ $(i \in \{1,2\})$. Consider $p_j := p_j(\overline{xy}_1\overline{y}_2)$ $(j = 1,2)$ a complete non–forking (over M) L_j–type extending $tp_j(\overline{mn}_1) \cup tp_j(\overline{mn}_2)$ (by non–forking extension property). Notice that p_j encodes the independence condition given in $(*)$. By Proposition 5.2 (as p_1 and p_2 are types inside a fusion), there exists a realization $\overline{b} = N'_1N'_2$ (isomorphic to N_1N_2 over M) of $p_1 \cup p_2$ and a fusion K such that $N'_1N'_2 \leq K$. On the other hand, taking $X \in [N'_1N'_2 \backslash N'_2]^{<\omega} = [N'_1 \backslash N'_2]^{<\omega}$ we have $X \cap N'_2 = \emptyset$.

Since $N'_1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{j}_{M} N'_2$, we have $d_i(X/N'_2) = d_i(X/M)$ (by definition 5.3 and fact 5.4).

So, by fact 4.31 we have that $d_0(X/M) = d_0(X/N'_2)$. As $M \leq N_1$, by corollary 4.36 we have $d_0(X/N'_2) = d_0(X/M) \geq 0$. By corollary 4.37 we have $N'_2 \leq N'_1N'_2$. By transitivity of $\leq$, we have $N'_2 \leq K$. We may show in a similar way that $N'_1 \leq K$.

□

Definition 5.9. Consider the following commutative diagram:

$$\begin{tikzcd}
M_0 \arrow[r, "f_0"] & N \\
M \arrow[u, "id"] \arrow[r, "id"'] & M_1 \arrow[u, "f_1"']
\end{tikzcd}$$

We say that the commutative diagram above is *smooth* if and only if we have that $f_0(M_0) \cap f_1(M_1) \leq f_i(M_i)$ for $i \in \{0, 1\}$.

In [34], using the techniques which Holland used in 5.8, we proved the following fact:

Fact 5.10. *Consider the following commutative diagram, where its base is smooth (see definition 5.9):*

$$\begin{tikzcd}
M_5 & & & \\
& M_4 \arrow[ul, "id"'] \arrow[rr, "id"'] & & M_6 \\
M_1 \arrow[uu, "id"] \arrow[rr, "f_{13}"] & & M_3 & \\
& M_0 \arrow[ul, "id"] \arrow[uu, "id"'] \arrow[rr, "id"'] & & M_2 \arrow[ul, "f_{23}"'] \arrow[uu, "id"']
\end{tikzcd}$$

where all the embeddings are inclusions, except f_{13} and f_{23}, the nodes correspond to fusions in disjoint languages $L_1 \cup L_2$ and additionally suppose that

$$M_j \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{i}_{M_1 \cap M_2} M_kM_l$$

($\{j, k, l\} = \{3, 5, 6\}$ where j, k and l are parwise disjoint and $i \in \{1, 2\}$). Then there exist a fusion M_7 and embeddings $f_{37} : M_3 \to M_7$, $f_{57} : M_5 \to M_7$ and $f_{67} : M_6 \to M_7$ such that the following diagram commutes:

$$\begin{tikzcd}
M_5 \arrow[rr, dotted, "f_{57}"] & & M_7 & \\
& M_4 \arrow[ul, "id"'] \arrow[rr, "id"'] & & M_6 \arrow[ul, dotted, "f_{67}"'] \\
M_1 \arrow[uu, "id"] \arrow[rr, "f_{13}"] & & M_3 \arrow[uu, dotted, "f_{37}"] & \\
& M_0 \arrow[ul, "id"] \arrow[uu, "id"'] \arrow[rr, "id"'] & & M_2 \arrow[ul, "f_{23}"'] \arrow[uu, "id"']
\end{tikzcd}$$

In this way, assuming that every square is smooth, using ideas of [12], Villaveces and the author proved in [34] that the class of Hrushovski fusions over disjoint and countable languages are $\aleph_0$–tame.

We focus on the technique used in the proof of the tameness of Hrushovski fusions, because it does not depend of this particular class. Further works in this way should take us to prove the tameness of general Hrushovski constructions which satisfy at least 3–amalgamation property of smooth fusions (5.9).

6 Some open problems

Most of studies in Hrushovski constructions had just included the theory of the generic model. However, Villaveces and the author studied the class of all Hrushovski fusions (over disjoint languages) as an AEC.

Zilber studied in [42] the class of covers of the multiplicative group of a field of characteristic 0 and studied in [39] the class of fields with a pseudo–exponentiation. Actually, these classes are quasi–minimal excellent. Zilber also proved in [41] that any quasi–minimal excellent class is categorical in every uncountable cardinality. Quasi–minimal excellence of the class of covers strongly depends of algebraic arguments (the key result in that setting is the *Thumbtack lemma*, see [42]). Since quasi–minimal excellence is a specific example of excellence, the class of covers is tame (by [12]). We conjecture that we can use the techniques used in [34] for proving the tameness of more general Hrushovski constructions which include the Zilber's covers class, avoiding the algebraic arguments.

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References

[1] J. Baldwin, *Rank and homogeneous structures*, preprint, Department of Mathematics Statsitics and Computer Science, University of Illinois at Chicago 2001).

`http://www.math.uic.edu/~jbaldwin/pub/wurzart2.pdf`

[2] J. Baldwin, *Abstract elementary classes: some answers, more questions*, preprint, Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago (2006).

`http://www.math.uic.edu/~jbaldwin/pub/turion2.pdf`

[3] J. Baldwin, *Categoricity*, University Lecture Notes **51** (American Mathematical Society, 2009).

`http://www.math.uic.edu/~jbaldwin/pub/AEClec.pdf`

[4] J. Baldwin and K. Holland, *Constructing ω–stable structures: Rank 2 fields*, J. Symb. Logic **65**, 371–391 (2000).

[5] J. Baldwin and K. Holland, *Constructing ω–stable structures: Rank k fields*, NotreDame J. Formal Logic **44**, 139–147 (2003).

[6] J. Baldwin, D. Kueker and M. VanDieren, *Upward stability transfer for tame abstract elementary classes*, Notre dame J. Form. Logic **47**, 291–298 (2006).

[7] A. Baudisch, A. Martén–Pizarro and M. Ziegler, *On fields and colors*, preprint, Mathematisches Institut, Albert–Ludwigs–Universität Freiburg (2005).

`http://home.mathematik.uni-freiburg.de/ziegler`
`/Preprints.html`

[8] A. Baudisch, A. Martén–Pizarro and M. Ziegler, *Fusion over a vector space*, preprint, Mathematisches Institut, Albert–Ludwigs–Universität Freiburg (2007).

`http://home.mathematik.uni-freiburg.de/ziegler`
`/Preprints.html`

[9] M. Bays, J. Kirby and A. J. Wilkie, *A Schanuel property for exponentially transcendental powers*, preprint, Mathematical Institute, University of Oxford (2008).

`http://arxiv.org/0810.4457`

[10] J. D. Caycedo and B. Zilber, *Green points in the complex numbers revisited*, preprint, Mathematical Institute, University of Oxford (20087).

`http://people.maths.ox.ac.uk/caycedo/docs/green.pdf`

[11] R. Grossberg, *Classification theory for abstract elementary classes*, Logic and Algebra, ed. Yi Zhang, Contemporary Mathematics **302**, AMS, (2002), 165 – 204.

[12] R. Grossberg and A. Kolesnikov, *Excellent abstract elementary classes are tame*, preprint, Department of Mathematical Sciences, Carnegie Mellon University (2005.

`http://www.math.cmu.edu/~rami/AtameP.pdf`

[13] R. Grossberg and M. VanDieren, *Galois–stability for tame abstract elementary classes*, J. Math. Logic **6**, 25–49 (2006).

[14] R. Grossberg and M. VanDieren, *Categoricity from one succesor cardinal in tame abstract elementary classes*, J. Math. Logic **6**, 181–201 (2006).

[15] R. Grossberg and M. VanDieren, *Shelah's categoricity conjecture from a successor for tame abstract elementary classes*, J. Symb. Logic **71**, 553–568 (2006).

[16] R. Grossberg, M. VanDieren and A. Villaveces, *Uniqueness of limit models in classes with amalgamation*, preprint, Department of Mathematical Sciences, Carnegie Mellon University (2008).

`http://www.math.cmu.edu/~rami/gvv_11_5.08.pdf`

[17] A. Hasson, *Some questions concerning Hrushovski's amalgamation constructions*, J. Inst. Math. Jussieu **7**, 793–823 (October, 2008).

[18] A. Hasson and M. Hils, *Fusions over sublanguages*, J. Symbolic Logic **71**, 361–398 (2006).

[19] M. Hils, *La fusion libre: le cas simple*, J. Inst. Math. Jussieu **7**, 825–868 (2008.

[20] Å. Hirvonen, *Categoricity in homogeneous complete metric spaces*, Ph. L. thesis, Department of Mathematics and Statistics, University of Hlesinki (2006).

`http://www.helsinki.fi/~asaekman/paperit/categoricityv2.pdf`

[21] K. Holland, *Introduction to fusion of strongly minimal sets: the geometry of fusions*, Arch. Math. Logic **34**, 395–413 (1995).

[22] K. Holland, *Model–completeness of the new strongly minimal sets*, J. Symb. Logic **64**, 946–962 (1999).

[23] E. Hrushovski, *Strongly minimal expansions of algebraically closed fields*, Israel J. Math. **79**, 129–151 (1992).

[24] E. Hrushovski, *A new strongly minimal set*, Ann. Pure Appl. Logic **62**, 147–166 (1993).

[25] J. Kirby, *A Schanuel condition for Weierstrass equations*, J. Symb. Logic **70**, 631–638 (2005); *corrigendum*, *ibid*, 1023.

[26] J. Kirby, *Exponential and Weierstrass equations*, preprint, Mathematical Institute, University of Oxford (2006).

`http://people.maths.ox.ac.uk/~kirby`

[27] D. Marker, *An introduction to Model Theory* (Springer, 2002).

[28] B. Poizat, *Le carré l'egalité*, J. Symb. Logic **64**, 1339–1356 (1999).

[29] B. Poizat, *L'egalité au cube*, J. Symb. Logic **66**, 1647–1676 (2001).

[30] S. Shelah, *Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel J. Math. **46**, 212–240 (1983).

[31] S. Shelah, *Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel J. Math. **46**, 241–273 (1983).

[32] S. Shelah, *Classification of nonelementary classes. II. Abstract elementary classes*, in Classification Theory: 1985, Lect. Notes Math. **1292**, 419–497 (1987).

[33] S. Shelah, *Categoricity of abstract class with amalgamation*, Ann. Pure Appl. Logic **98**, 261–294 1999.

[34] A. Villaveces and P. Zambrano, *Hrushovski constructions and tame abstract elementary classes*, preprint, Departamento de Matemáticas, Universidad Nacional de Colombia (2009).

[35] A. Villaveces and P. Zambrano, *Around stability in metric abstract elementary classes*, preprint, Departamento de Matemáticas, Universidad Nacional de Colombia (2009).

[36] B. Zilber, *Raising to powers in algebraically closed fields*, J. Math. Logic **3**, 217–238 2003.

[37] B. Zilber, *Bi–coloured fields on the complex numbers*, J. Symb. Logic **69**, 1171–1186 (2004).

[38] B. Zilber, *Complex and pseudo–analytic structures*, Oberwolfach tutorial slides, Mathematical Institute, University of Oxford (2004).

`http://people.math.sox.ac.uk/zilber.oberwolfach.ps`

[39] B. Zilber, *Pseudo–exponentiation on algebraically closed fields of characteristic zero*, Ann. Pure Appl. Logic **132**, 67–95 (2004).

[40] B. Zilber, *Analytic and pseudo–analytic structures*, in Logic Colloquium 2000, Paris, Lect. Notes Logic **19**, 392–408 (2005).

[41] B. Zilber, *A categoricity theorem for quasi–minimal excellent classes*, J. London Math. Soc. **74**, 41–58 (2006).

[42] B. Zilber, *Covers of the multiplicative group of an algebraically closed field of characteristic zero*, J. London Math. Soc. **74**, 41–58 (2006).