Boundedness in third order nonlinear differential equations with bounded delay

Cemil Tunç[1]

Faculty of Arts and Sciences, Department of Mathematics
Yüzüncü Yıl University, 65080, Van, Turkey

Se presenta un teorema que incluye condiciones suficientes para el acotamiento de soluciones de una cierta clase de ecuaciones diferenciales no lineales de tercer orden con un retardo constante. Nuestros resultados extienden un resultado reciente de Omeike (2009).

Palabras Claves: Acotamiento, funcional de Lyapunov ecuaciones diferenciales no lineales, tercer orden, retardo.

A theorem is presented which includes sufficient conditions for the boundedness of solutions of a certain class of third order nonlinear differential equations with a constant delay. Our result extends a recent result by Omeike (2009).

Keywords: Boundedness, Lyapunov functional, nonlinear differential equations, third order, delay.

MSC: 34K20.

1 Introduction

A problem of considerable interest in qualitative theory of ordinary differential equations of higher order, with or without delay, is the determination of the boundedness of solutions. In the last three decades, specially in recent years, some authors dealt with the problem for various nonlinear delay differential equations of third order. In particular, we refer readers to the papers of Afuwape and Omeike [2], Sinha [4], Tejumola and Tchegnani [5], Tunç [6, 7, 8, 9, 10, 11, 12, 13, 14], Zhu [15], and the references therein for some results achieved on the boundedness of solutions of certain nonlinear delay differential equations of third order.

[1] cemtunc@yahoo.com

In the meantime Omeike, in 2009, [3] considered the following nonlinear differential equation of third order, with a constant deviating argument r,

$$\frac{d^3x}{dt^3}(t) + a(t)\,\frac{d^2x}{dt^2}(t) + b(t)\,g\left(\frac{dx}{dt}(t)\right) + c(t)\,h(x(t-r)) = p(t)\,.$$

Omeike discussed the stability and boundedness of solutions of this equation when $p(t) = 0$ and $p(t) \neq 0$.

In this paper, instead of the above equation, we consider the following non–autonomous differential equation of third order with a constant deviating argument, r, given by:

$$\begin{aligned} &\frac{d^3x}{dt^3}(t) + a(t)\,\psi\left(\frac{dx}{dt}(t)\right)\frac{d^2x}{dt^2}(t) + b(t)\,g\left(\frac{dx}{dt}(t)\right) \\ &+c(t)\,h(x(t-r) \\ = \;& p\left(t,\, x(t),\, x(t-r),\, \frac{dx}{dt}(t),\, \frac{dx}{dt}(t-r),\, \frac{d^2x}{dt^2}(t)\right), \end{aligned} \tag{1}$$

whose associated system is

$$\begin{aligned} \frac{dx}{dt}(t) &= y(t)\,, \\ \frac{dy}{dt}(t) &= z(t)\,, \\ \frac{dz}{dt}(t) &= -a(t)\,\psi(y(t))\,z(t) - b(t)\,g(y(t)) - c(t)\,h(x(t)) \\ &\quad +c(t)\int_{t-r}^{t}\frac{dh}{dt}(x(s))\,y(s)\,ds \\ &\quad +p(t,\, x(t),\, x(t-r),\, y(t),\, y(t-r),\, z(t))\,, \end{aligned} \tag{2}$$

where r is a positive constant; the functions $a(t)$, $b(t)$, $c(t)$, $\psi(y)$, $g(y)$, $h(x)$ and p depend only on the arguments displayed explicitly; $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$. The functions $a(t)$, $b(t)$, $c(t)$, $\psi(y)$, $g(y)$, $h(x)$ and p are assumed to be continuous for their respective arguments on $\mathbb{R}^+$, $\mathbb{R}^+$, $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^5$, respectively; $g(0) = 0$, $h(0) = 0$; the derivatives $\frac{da}{dt}(t)$, $\frac{db}{dt}(t)$, $\frac{dc}{dt}(t)$ and $\frac{dh}{dx}(x)$ exist and are continuous. It should be noted that the continuity of the functions $a(t)$, $b(t)$, $c(t)$, $\psi(y)$, $g(y)$, $h(x)$ and p guarantees the existence of the solution of equation (1). Besides, it is assumed that the functions $\psi(y)$, $g(y)$, $h(x)$ and $p(t, x, x(t-r), y, y(t-$

$r), z)$ satisfy a Lipschitz condition in x, y, z, $x(t-r)$ and $y(t-r)$. Hence the solution is unique. Finally, all solutions considered are supposed to be real valued and throughout the paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively.

The motivation of this paper has come from the result by Omeike [3, Theorem 2] and the papers mentioned above. Our purpose here is to extend the result established by Omeike [3, Theorem 2] to the preceding non–autonomous differential equation with the deviating argument r for the boundedness of all solutions. Clearly, the equation discussed in Omeike [3, Theorem 2] is a special case of our equation, equation (1).

2 Main result

We prove the following theorem.

Theorem. *In addition to the basic assumptions imposed on the functions $a(t)$, $b(t)$, $c(t)$, $\psi(y)$, $g(y)$, $h(x)$ and p, let us assume that there exist positive constants $\delta(0)$, δ_1, δ_2, a, b, c, α and L such that the following conditions hold:*

i.

$$\begin{aligned}
&\frac{h(x)}{x} \geq \delta(0)\,, \quad x \neq 0\,,\\
&\frac{dh}{dx}(x) \leq c\,,\\
&\frac{g(y)}{y} \geq b\,, \quad y \neq 0\,,\\
&\psi(y) \geq 1\,,
\end{aligned}$$

ii.

$$\begin{aligned}
&0 < a \leq a(t) \leq L\,,\\
&0 < \delta_1 \leq c(t) \leq b(t) \leq L\,,\\
&-L \leq \frac{dc}{dt}(t) \leq \frac{db}{dt}(t) \leq 0\,,\\
&\frac{1}{2}\,\frac{da}{dt}(t) \leq \delta_2 < \delta_1(b - \alpha c)\,,\\
&\frac{b}{c} > \alpha > \frac{1}{a}\,,
\end{aligned}$$

iii.

$$|p(t,\, x,\, x(t-r),\, y,\, y(t-r),\, z)| \leq q(t)\,,$$

where $q \in L^1(0,\infty)$*,* $L^1(0,\infty)$ *is the space of Lebesgue integrable functions. Then, there exists a finite positive constant* K *such that the solution* $x(t)$ *of equation (1) defined by the initial function*

$$\begin{aligned} x(t) &= \phi(t)\,, \\ \frac{dx}{dt}(t) &= \frac{d\phi}{dt}(t)\,, \\ \frac{d^2x}{dt^2}(t) &= \frac{d^2\phi}{dt^2}(t)\,, \end{aligned}$$

satisfies

$$\begin{aligned} |x(t)| &\leq \sqrt{K}\,, \\ \left|\frac{dx}{dt}(t)\right| &\leq \sqrt{K}\,, \\ \left|\frac{d^2x}{dt^2}(t)\right| &\leq \sqrt{K}\,, \end{aligned}$$

for all $t \geq t(0)$*, where* $\phi \in C^2([t(0)-r, t(0)], \mathbb{R})$*, provided that*

$$r < \min\left\{\frac{2\delta_5}{(2+\alpha)\,L\,c}\,,\, \frac{2(\alpha a - 1)}{L\,\alpha\,c}\right\}\,.$$

Proof. We define a Lyapunov functional $V = V(t, x(t), y(t), z(t))$ by

$$\begin{aligned} 2\,V(t,\, x(t),\, y(t),\, z(t)) &= \alpha\, z^2 + 2\,y\,z + 2\,\alpha\, b(t) \int_0^y g(\eta)\, d\eta \\ &\quad +2\,a(t) \int_0^y \psi(\eta)\,\eta\, d\eta + 2\,\alpha\, c(t)\, h(x) \\ &\quad +2\,c(t) \int_0^x h(s)\, ds \\ &\quad +\lambda \int_{-r}^0 \int_{t+s}^t y^2(\theta)\, d\theta\, ds\,, \end{aligned} \qquad (3)$$

where λ is a positive constant which will be determined later in the proof.

By the assumptions $a(t) > 0$ and $\psi(y) \geq 1$, from (3) we have

$$
\begin{aligned}
2\,V(t,\,x(t),\,y(t),\,z(t)) \;\geq\;& \alpha\,z^2 + 2\,y\,z \\
&+2\,\alpha\,b(t)\int_0^y g(\eta)\,d\eta + a(t)\,y^2 \\
&+2\,\alpha\,c(t)\,h(x)\,y + 2\,c(t)\int_0^x h(s)\,ds \\
&+\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)\,d\theta\,ds\,. \qquad (4)
\end{aligned}
$$

Taking into account the assumptions of the theorem and the discussion in Omeike [3, Theorem 2], from (4) one can conclude that

$$
\begin{aligned}
V(t,\,x(t),\,y(t),\,z(t)) \;\geq\;& \frac{\delta_0\delta_1\delta_4}{2}\,x^2 + \frac{\delta_3}{4}\,y^2 + \frac{\delta_3}{4}\,z^2 \\
&+\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)\,d\theta\,ds \\
\geq\;& D_1\left(x^2+y^2+z^2\right)\,, \qquad (5)
\end{aligned}
$$

for some positive constants δ_0, δ_1, δ_3 and δ_4, where

$$
D_1 = \min\left\{\frac{1}{2}\,\delta_0\,\delta_1\,\delta_4,\;\frac{1}{4}\,\delta_3\right\}\,.
$$

For the time derivative of the Lyapunov functional $V(t, x(t), y(t), z(t))$, along a trajectory of the system (2), we have

$$
\begin{aligned}
\frac{dV}{dt} \;=\;& \frac{dc}{dt}(t)\int_0^x h(s)\,ds + \alpha\,\frac{db}{dt}(t)\int_0^y g(\eta)\,d\eta \\
&+\alpha\,\frac{dc}{dt}(t)\,h(x)\,y + \frac{da}{dt}(t)\int_0^y \psi(\eta)\,\eta\,d\eta \\
&-\left[b(t)\,\frac{g(y)}{y} - \alpha\,c(t)\,\frac{dh}{dt}(x) - \lambda\,r\right]\,y^2 \\
&-\left[\alpha\,a(t)\,\psi(y) - 1\right]\,z^2 + c(t)\,y\int_{t-r}^t \frac{dh}{dx}(x(s))\,y(s)\,ds \\
&+\alpha\,c(t)\,z\int_{t-r}^t \frac{dh}{dx}(x(s))\,y(s)\,ds - \lambda\int_{t-r}^t y^2(s)\,ds \\
&+(y+\alpha\,z)\,p(t,\,x,\,x(t-r),\,y,\,y(t-r),\,z)\,. \qquad (6)
\end{aligned}
$$

Hence, subject to the assumptions of the theorem, from Omeike [3, Theorem 1], it follows that

$$\frac{dc}{dt}(t)\int_0^x h(s)\,ds+\alpha\,\frac{db}{dt}(t)\int_0^y g(\eta)\,d\eta+\alpha\,\frac{dc}{dt}(t)\,h(x)\,y\le 0\,.$$

Applying the assumptions of the theorem and the inequality $2\,|mn|\le m^2+n^2$, we obtain the following inequalities:

$$\begin{aligned}\left[b(t)\,\frac{g(y)}{y}-\alpha\,c(t)\,\frac{dh}{dx}(x)-\lambda\,r\right]y^2 &\ge \left(b^2(t)-\alpha\,c^2(t)-\lambda\,r\right)\,y^2\\ &= c(t)\,\left(\frac{b^2(t)}{c(t)}-\alpha\,c\right)\,y^2\\ &\quad -\lambda\,r\,y^2\\ &\ge \delta_1\,(b-\alpha\,c)\,y^2-\lambda\,r\,y^2\\ &= \delta_1\,\left(b-\alpha\,c-\frac{1}{\delta_1}\,\lambda\,r\right)\,y^2\,,\end{aligned}$$

$$\begin{aligned}&\frac{da}{dt}(t)\int_0^y\psi(\eta)\,\eta\,d\eta-\left(b(t)\,\frac{g(y)}{y}-\alpha\,c(t)\,\frac{dh}{dx}(x)-\lambda\,r\right)\,y^2\\ \le\;& \frac{1}{2}\,\frac{da}{dt}(t)\,y^2-\left(b(t)\,\frac{g(y)}{y}-\alpha\,c(t)\,\frac{dh}{dx}(x)-\lambda\,r\right)\,y^2\\ \le\;& \delta_2\,y^2-\delta_1\,\left(b-\alpha\,c-\frac{1}{\delta_1}\,\lambda\,r\right)\,y^2\\ =\;& -\left[-\delta_2+\delta_1\,(b-\alpha\,c)-\lambda\,r\right]\,y^2\\ =\;& -\left(\delta_5-\lambda\,r\right)\,y^2\,,\end{aligned}$$

$$\delta_5=-\delta_2+\delta_1\,(b-\alpha\,c)>0\,,$$

$$(\alpha\,a(t)\,\psi(y)-1)\,z^2\ge(\alpha\,a-1)\,z^2\,,$$

$$c(t)\,y\int_{t-r}^t\frac{dh}{dx}(x(s))\,y(s)\,ds\le\frac{1}{2}\,L\,c\,r\,y^2+\frac{1}{2}\,L\,c\int_{t-r}^t y^2(s)\,ds\,,$$

$$\alpha\,c(t)\,z\int_{t-r}^t\frac{dh}{dx}(x(s))\,y(s)\,ds\le\frac{1}{2}L\,\alpha\,c\,r\,z^2+\frac{1}{2}\,L\,\alpha\,c\int_{t-r}^t y^2(s)\,ds\,,$$

$$
\begin{aligned}
& (y+\alpha\, z)\, p(t,\, x,\, x(t-r),\, y,\, y(t-r),\, z) \\
\le\; & |y+\alpha\, z|\; |p(t,\, x,\, x(t-r),\, y,\, y(t-r),\, z)| \\
\le\; & (|y|+\alpha\; |z|)\, q(t) \\
\le\; & \left(1+\alpha+y^2+\alpha\, z^2\right)\; q(t) \\
\le\; & (1+\alpha)\, q(t) + D_2\; \left(y^2+z^2\right)\; q(t) \\
\le\; & (1+\alpha)\, q(t) + \frac{D_2}{D_1}\, V(t,\, x(t),\, y(t),\, z(t))\, q(t)\,,
\end{aligned}
$$

where $D_2 = \max\{1, \alpha\}$.

The replacement of the preceding inequalities in (6) gives

$$
\begin{aligned}
\frac{dV}{dt}(t,\, x(t),\, y(t),\, z(t)) \;\le\; & -\left[\delta_5 - (\lambda + \frac{1}{2}\, L\, c)\, r\right]\; y^2 \\
& -\left[(\alpha a - 1) - \frac{1}{2}\, L\, \alpha\, c\, r\right]\; z^2 \\
& +(1+\alpha)\, q(t) \\
& +\frac{D_2}{D_1}\, V(t,\, x(t),\, y(t),\, z(t))\, q(t) \\
& +\left[\frac{1}{2}\,(1+\alpha)\, L\, c - \lambda\right]\; \int_{t-r}^{t} y^2(s)\, ds\,.
\end{aligned}
$$

Choose $\lambda = \frac{1}{2}(1+\alpha)L\, c$. Hence,

$$
\begin{aligned}
\frac{dV}{dt}(t,\, x(t),\, y(t),\, z(t)) \;\le\; & -\left[\delta_5 - \frac{1}{2}\,(2+\alpha)\, L\, c\, r\right]\; y^2 \\
& -\left[(\alpha\, a - 1) - \frac{1}{2}\, L\, \alpha\, c\, r\right]\; z^2 \\
& +(1+\alpha)\, q(t) \\
& +\frac{D_2}{D_1}\, V(t,\, x(t),\, y(t),\, z(t))\, q(t)\,.
\end{aligned}
$$

The last inequality implies

$$
\begin{aligned}
\frac{dV}{dt}(t,\, x(t),\, y(t),\, z(t)) \;\le\; & -D_3\, y^2 - D_4\, z^2 + (1+\alpha)\, q(t) \\
& +\frac{D_2}{D_1}\, V(t,\, x(t),\, y(t),\, z(t))\, q(t)\,,
\end{aligned}
$$

for some positive constants D_3 and D_4, provided that

$$r < \min\left\{\frac{2\delta_5}{(2+\alpha)L\,c},\ \frac{2(\alpha\,a-1)}{L\alpha\,c}\right\}.$$

Thus, we have

$$\begin{aligned}\frac{dV}{dt}(t,\,x(t),\,y(t),\,z(t)) &\leq (1+\alpha)\,q(t)\\ &\quad+\frac{D_2}{D_1}\,V(t,\,x(t),\,y(t),\,z(t))\,q(t)\,.\end{aligned}$$

Integrating the last inequality from 0 to t, using the assumption $q \in L^1(0,\infty)$ and the Gronwall–Reid–Bellman inequality (see Ahmad and Rama Mohana Rao [1]), we obtain

$$\begin{aligned}V(t,\,x(t),\,y(t),\,z(t)) &\leq V(0,\,x(0),\,y(0),\,z(0))\\ &\quad+(1+\alpha)\int_0^t q(s)\,ds\\ &\quad+\frac{D_2}{D_1}\int_0^t V(s,\,x_s,\,y_s,\,z_s)\,q(s)\,ds\\ &\leq \{V(0,\,x(0),\,y(0),\,z(0))+(1+\alpha)\,A\}\\ &\quad\times\exp\left(\frac{D_2}{D_1}\int_0^t q(s)\,ds\right)\\ &= K_1<\infty\,, \end{aligned}\tag{7}$$

where

$$K_1=\{V(0,\,x(0),\,y(0),\,z(0))+(1+\alpha)\,A\}\,\exp\left(\frac{D_2}{D_1}\,A\right),$$

is a constant and $A=\int_0^\infty q(s)ds$.

Hence, the inequalities (5) and (7) together imply that

$$x^2(t)+y^2(t)+z^2(t)\leq\frac{1}{D_1}\,V(t,\,x(t),\,y(t),\,z(t))\leq K\,,$$

where $K=\frac{1}{D_1}K_1$. As a result, for solutions of the system (2), we can conclude that

$$\begin{aligned} |x(t)| &\leq \sqrt{K}\,, \\ |y(t)| &\leq \sqrt{K}\,, \\ |z(t)| &\leq \sqrt{K}\,, \end{aligned}$$

for all $t \geq 0$. That is,

$$\begin{aligned} |x(t)| &\leq \sqrt{K}\,, \\ \left|\frac{dx}{dt}(t)\right| &\leq \sqrt{K}\,, \\ \left|\frac{d^2x}{dt^2}(t)\right| &\leq \sqrt{K}\,, \end{aligned}$$

for all $t \geq 0$. This shows boundedness of all solutions of equation (1).

The proof of the theorem is now complete.

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