FIXED CHARGE BI-CRITERION INDEFINITE QUADRATIC TRANSPORTATION PROBLEM WITH ENHANCED FLOW

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ABSTRACT
In the present paper a fixed charge bi-criterion quadratic transportation problem with enhanced flow is studied. An algorithm to find an efficient cost-time trade off pairs in a fixed charge bi-criterion quadratic transportation problem is presented. A related fixed charge bi-criterion quadratic transportation problem is formulated and the efficient cost-time trades off pairs to the given problem are shown to be derivable from this related problem. The algorithm is illustrated with the help of a numerical example.

Keywords: Transportation problem, cost-time trade off, bi-criterion quadratic transportation problem, restricted flow.
MSC: 90B06; 90C29

RESUMEN
En este trabajo un problema bi-cuadrático de transporte con cargas fijas y flujo mejorado es estudiado. Un algoritmo para hallar un acuerdo de costo–tiempo eficiente en el problema de transporte bi-cuadrático es presentado. Un problema de transporte bi-cuadrático de costos fijos relacional es formulado y se prueba que el acuerdo eficiente de pares costo-tiempo del problema dado es derivable del problema relacionado. El algoritmo es ilustrado con la ayuda de un ejemplo numérico.

1. INTRODUCTION

The fixed charge (for example, a setup cost) arises for a Transportation Problem in which each origin has a fixed-charge coefficient in addition to the usual cost coefficient. The fixed charge occurs only when the variable appears in the solution with a positive level. It is a nonlinear programming problem. It was originally formulated by G. B. Dantzig and W. Hirsch [10] in 1954. Later Ballinski [7] in 1961 studied fixed charge problems. Several procedures have been thereafter developed for solving fixed-charge transportation problems [1-3, 6-8, 13, 15-20]. The transportation problem where the total time is minimized have been studied many authors [11, 12]. Normally, bi-criteria problem arises when the user has to compromise between two criteria. Bhatia et. al. [9] in 1976 also calculated the time-cost trade off pairs in a classical transportation problem. Later Basu et al. [8] in 1994 developed an algorithm for the optimum time-cost trade off pair in a fixed charge bi-criterion transportation problem. Also, Thirwani et al. [20] in 1997 developed an algorithm for finding time-cost trade off pairs in a fixed charge bi-criterion transportation problem with restricted flow. The indefinite quadratic transportation problem was first studied by Arora and Khurana [4-5] in 2001. Later in 2006 Khurana et. al. [14] studied linear plus linear transportation problem for restricted and enhanced flow. Currently, there is lot of study going on fixed charge problems as well. Recently in 2010 [1], Adlakha et. al. gave a branching method for fixed charge transportation problem.

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In this paper, we find out various cost-time trade off pairs such that user can choose the pair which suits him the most. Sometimes, situations arise when because of extra demand in the market; the total flow needs to be enhanced, compelling some of the factories to increase their productions in order to be able to meet the extra demand. The total flow from the factories in the market is now increased by the amount of extra demand. This motivated us to develop an algorithm to find the cost-time trade off pairs for the case of enhanced flow for a quadratic transportation problem. In this paper we shall be discussing the case when the flow gets enhanced due to extra demand in the market for a fixed charge indefinite quadratic transportation problem and shall also find the cost-time trade off pairs for the problem.

2. PROBLEM FORMULATION AND THEORETICAL DEVELOPMENT

We know that linear functions are the most useful and widely used in modeling of mathematical optimization problems. Also quadratic functions and quadratic problems are the least difficult to handle out of all non-linear programming problems. A fair number of functional relationships occurring in the real world are truly quadratic. For example kinetic energy carried by a rocket or an atomic particle is proportional to the square of its velocity. In statistics, the variance of a given sample of observations is a quadratic function of the values that constitute the sample. So there are countless other non-linear relationships occurring in nature, capable of being approximated by quadratic functions.

Consider the Fixed Charge Bi-criterion Quadratic Transportation Problem given by

$$\text{Minimize} \quad \left\{ \left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} \right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij} \right) + \sum_{i\in I} F_i \,,\ \underset{\substack{i\in I\\ j\in J}}{\text{Max}} \left[t_{ij} \,\middle|\, x_{ij} > 0 \right] \right\}$$

$$\text{subject to} \quad \sum_{j\in J} x_{ij} = a_i \qquad i\in I$$

$$\sum_{i\in I} x_{ij} = b_j \qquad j\in J$$

$$x_{ij} \geq 0 \qquad i\in I,\ j\in J$$

where $I = \{1,2,\ldots\ldots\ldots..,m\}$ is the set of origins
$J = \{1,2,\ldots\ldots\ldots...,n\}$ is the set of destinations

x_{ij} = the quantity transported from the ith origin to the jth destination
c_{ij} = per unit cost in transporting goods from the ith origin to jth destination
d_{ij} = per unit depreciation cost in transporting goods from the ith origin to the jth destination
t_{ij} = the time of transporting the product from the ith origin to the jth destination which is independent of the amount of commodity transported , so long as $x_{ij} > 0$
F_i = the fixed cost associated with the ith origin which is independent of the amount of commodity transported, for $x_{ij} > 0$
= 0 for $x_{ij} = 0$
a_i = the amount available at the ith origin
b_j = the demand of the jth destination.
where x_{ij} are the variables and a_i, b_j, c_{ij} and F_i are the parameters in the above problem.
The total flow in the problem is $\sum_{i\in I} a_i = \sum_{j\in J} b_j$

In the above problem the cost of transporting one unit from *ith* origin to *jth* destination is $\sum_i\sum_j c_{ij}x_{ij}$, but while transporting goods from one origin to the other destination, some fraction of goods get damaged so the total cost of damaged goods is $\sum_i\sum_j d_{ij}x_{ij}$. Our aim is to minimize the two costs simultaneously; therefore we consider the product of two costs i.e., $\left(\sum_i \sum_j c_{ij} x_{ij}\right)$ $\left(\sum_i \sum_j d_{ij} x_{ij}\right)$. Also we need to minimize the fixed cost associated with ith origin and the time of transportation from *ith* origin to *jth* destination.

If in the above problem the total availability is not equal to the total demand, then some of the source and/or destination constraints are satisfied as inequalities. Sometimes because of extra demand in the market, the total flow from the factories in the market is increased. Let $P\ (> Max\ (\sum_{i\in I} a_i, \sum_{j\in J} b_j))$ be the enhanced flow. This flow constraint changes the structure of the transportation problem. The resulting fixed charge bi-criterion quadratic transportation problem with enhanced flow is

$$(P_1): \quad \text{Minimize} \left\{ \left(\sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij} x_{ij}\right) + \sum_{i\in I} F_i\ ,\ \underset{\substack{i\in I\\ j\in J}}{Max}\left[t_{ij} \middle|\ x_{ij} > 0\right] \right\}$$

subject to

$$\left.\begin{aligned}
&\sum_{j\in J} x_{ij} \ge a_i \quad , i \in I\\
&\sum_{i\in I} x_{ij} \ge b_j \quad , j \in J\\
&\sum_{i\in I}\sum_{j\in J} x_{ij} = P\ (> Max\ (\sum_{i\in I} a_i, \sum_{j\in J} b_j))\\
&x_{ij} \ge 0 \qquad , \ i \in I, j \in J\\
&\quad c_{ij}, d_{ij} \ge 0 \quad , \ i \in I, j \in J
\end{aligned}\right\} \qquad ....(1)$$

In order to solve the above problem (P_1) we separate it into two problems (P_1') and (P_1'') where

$$(P_1'): \quad \text{Minimize the cost function } \left\{ \left(\sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij} x_{ij}\right) + \sum_{i\in I} F_i \right\} \text{ subject to (1)}$$

$$(\mathbf{P_1''}): \quad \text{Minimize the time function } \left\{ Max\left[t_{ij} \middle|\ x_{ij} > 0\right]\right\} \text{ subject to (1)}$$

In the above problem, fixed charge is a periodic charge that does not vary with business volume as insurance or rent or mortgage payments etc. It includes overhead expenses such as rent, interest, and insurance which are related to the production capacity of a firm and not to its actual level of output. These charges may also include expenses fixed by agreements, such as pension fund contributions.

To formulate F_i *(i=1,2,.........,m)* we assume that F_i *(i=1,2,...........,m)* has *p* (*p* be any positive integer) number of steps so that

$$F_i = \sum_{r=1}^{p} \delta_{ir} F_{ir} \qquad i = 1,2,...........,m$$

$$= \delta_{i1}F_{i1} + \delta_{i2}F_{i2} + ... + \delta_{ip}F_{ip} \qquad , \ i = 1,2,...........,m$$

where $\delta_{i1} = 1 \qquad if \ \sum_{j=1}^{n} x_{ij} > K_{i1} \ , \quad i = 1,2,.........,m$

$\qquad = 0 \qquad$ otherwise ;

$$\delta_{i2} = 1 \quad if \sum_{j=1}^{n} x_{ij} > K_{i2} \quad , \quad i = 1,2,\dots\dots\dots,m$$

$$= 0 \quad \text{otherwise} \quad ;$$

and so on..

$$\delta_{ip} = 1 \quad if \sum_{j=1}^{n} x_{ij} > K_{ip} \quad , \quad i = 1,2,\dots\dots\dots,m$$

$$= 0 \quad \text{otherwise}$$

Here $0 = K_{i1} < K_{i2} < \dots\dots\dots\dots\dots\dots < K_{ip} < \max_i \{a_i\}$

Also $K_{i1}, K_{i2}, \dots\dots\dots\dots\dots, K_{ip}$ $(i = 1,2,\dots\dots.m)$ are constants and are chosen randomly as defined above. F_{ir} $(i = 1,2,\dots\dots,m;\ r = 1,2,\dots\dots,p)$ are fixed costs.

In order to deal with the flow constraints $\sum_{i\in I}\sum_{j\in J} x_{ij} = P$, a related fixed charge bi-criterion transportation problem is formulated by adding a fictitious factory with availability equal to $\left(P - \sum_{j\in J} b_j\right)$ and a fictitious destination with demand equal to $\left(P - \sum_{i\in I} a_i\right)$. Hence the related fixed charge bi- criterion quadratic transportation problem (P_2) associated with fixed charge bi- criterion quadratic transportation problem (P_1) is

$$(P_2): \quad \text{Minimize}\left\{\left(\sum_{i\in I'}\sum_{j\in J'} c'_{ij}x_{ij}\right)\left(\sum_{i\in I'}\sum_{j\in J'} d'_{ij}x_{ij}\right) + \sum_{i\in I'} F_i \,, \operatorname*{Max}_{\substack{i\in I' \\ j\in J'}} \left[t'_{ij} \,\middle|\, x_{ij} > 0 \right]\right\}$$

$$\left.\begin{aligned}
&\text{subject to} \sum_{j\in J'} x_{ij} = a'_i \quad , \quad i\in I' \\
&\sum_{i\in I'} x_{ij} = b'_j \quad , \quad j\in J' \\
&x_{ij} \geq 0 \quad , \quad i\in I' \,,\ j\in J' \\
&\text{where } I' = \{1,2,\dots\dots\dots,m+1\} = I \cup \{m+1\} \\
&J' = \{1,2\dots\dots\dots\dots,n+1\} = J \cup \{n+1\}
\end{aligned}\right\} \quad \dots\dots(2)$$

$$a'_i = a_i \quad i\in I \ , \quad a'_{m+1} = \left(P - \sum_{j\in J} b_j\right)$$

$$b'_j = b_j \quad j\in J \ , \quad b'_{n+1} = \left(P - \sum_{i\in I} a_i\right)$$

where $P\ (> Max\ (\sum_{i\in I} a_i , \sum_{j\in J} b_j\ ))$

$c'_{ij} = c_{ij}$, $(i, j)\in I\times J$; $d'_{ij} = d_{ij}$, $(i, j) \in I\times J$

$t'_{ij} = t_{ij}$, $(i, j)\in I\times J$

Let $c'_{im+1,j} = c_{lj}$ and $d'_{m+1,j} = d_{lj}$ such that $c_{lj}d_{lj} = \min_{i\in I}(c_{ij}d_{ij})$

$c'_{i,n+1} = c_{ik}$ and $d'_{i,n+1} = d_{ik}$ such that $c_{ik}d_{ik} = \min_{j\in J}(c_{ij}d_{ij})$

$t'_{i,n+1} = t'_{m+1,j} = 0$, $j\in J$; $t'_{m+1,n+1} > \operatorname*{Max}_{\substack{i\in I \\ j\in J}}\left[t_{ij} \mid x_{ij} > 0\right]$

$c'_{m+1,n+1} = M = d'_{m+1,n+1}$

$F_{m+1} = 0$ where M is a large positive number.

(P_2) is separated into two problems (P_2') and (P_2'') where

$(\mathbf{P_2'})$: Minimize $Z = \left\{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right) + \sum_{i\in I} F_i\right\}$

subject to (2)

$(\mathbf{P_2''})$: Minimize $T = \underset{\substack{i\in I\\ j\in J}}{\text{Max}}\left[t_{ij}\,\middle|\,x_{ij} > 0\right]$

subject to (2)

Consider the transportation problem where the objective function is the product of two linear functions.

(QTP): Minimize $Z = \left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i \qquad \forall\ i\in I$$

$$\sum_{i\in I} x_{ij} \geq b_j \qquad \forall\ j\in J$$

$$x_{ij} \geq 0 \qquad \forall\ i\in I,\ j\in J$$

Note: The objective function being the product of two affine functions is a quasiconcave function (for at least m or n > 1) will have its optimal solution at an extreme point.

Theorem 1: Let $X = \{x_{ij}\}$ be a basic feasible solution of (QTP) with basis matrix B. Then it will be an optimal basic feasible solution if

$$R_{ij} \geq 0 \qquad \forall\ \text{cells } (i,j) \notin B$$
$$= 0 \qquad \forall\ \text{cells } (i,j) \in B$$

where $R_{ij} = \theta_{ij}(z'_{ij} - d_{ij})(z_{ij} - c_{ij}) - Z_1(z'_{ij} - d_{ij}) - Z_2(z_{ij} - c_{ij})$

$$u_i + v_j = z_{ij} \qquad \forall\ \text{cells } (i,j) \notin B$$
$$u'_i + v'_j = z'_{ij} \qquad \forall\ \text{cells } (i,j) \notin B$$

Also

$$\left.\begin{aligned} u_i + v_j &= c_{ij} && \forall\ \text{cells } (i,j) \in B \\ u'_i + v'_j &= d_{ij} && \forall\ \text{cells } (i,j) \in B \end{aligned}\right\} \qquad \text{.....(3)}$$

Z_1 and Z_2 respectively equals the value of $\sum_i\sum_j c_{ij}x_{ij}$ and $\sum_i\sum_j d_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B and θ_{ij} is the level at which a non basic cell *(i, j)* enters the basis replacing some basic cell of B.

Note: u_i, v_j, u'_i, v'_j corresponds to the dual variables and are determined by using equations (3) and taking one of the u_i's or v_j's and u'_i's or v'_j's as zero.

Proof: Let Z^0 be the objective function value of the problem (QTP).
Let $Z^0 = Z_1Z_2$
Let $\hat{Z}$ be the value of the objective function at the current basic feasible $\hat{X} = \{x_{ij}\}$ corresponding to the basis B obtained on entering the cell *(i, j)* into the basis.

Then $\hat{Z} = \left[Z_1 + \theta_{ij}(c_{ij} - z_{ij})\right]\left[Z_2 + \theta_{ij}(d_{ij} - z'_{ij})\right]$

Now, $\widehat{Z} - Z^0 = Z_1 Z_2 + Z_1\theta_{ij}\left(d_{ij} - z'_{ij}\right) + Z_2\theta_{ij}\left(c_{ij} - z_{ij}\right) + \theta_{ij}{}^2\left(c_{ij} - z_{ij}\right)\left(d_{ij} - z'_{ij}\right) - Z_1 Z_2$

$$= Z_1\theta_{ij}\left(d_{ij} - z'_{ij}\right) + Z_2\theta_{ij}\left(c_{ij} - z_{ij}\right) + \theta_{ij}{}^2\left(c_{ij} - z_{ij}\right)\left(d_{ij} - z'_{ij}\right)$$

$$= \theta_{ij}\left[Z_1\left(d_{ij} - z'_{ij}\right) + Z_2\left(c_{ij} - z_{ij}\right) + \theta_{ij}\left(c_{ij} - z_{ij}\right)\left(d_{ij} - z'_{ij}\right)\right]$$

This basic feasible solution will give an improved value of Z if $\widehat{Z} < Z^0$, i.e., if
$\theta_{ij}\left[Z_1\left(d_{ij} - z'_{ij}\right) + Z_2\left(c_{ij} - z_{ij}\right) + \theta_{ij}\left(c_{ij} - z_{ij}\right)\left(d_{ij} - z'_{ij}\right)\right] < 0.$

Since $\theta_{ij} \geq 0$

$$\therefore \quad Z_1\left(d_{ij} - z'_{ij}\right) + Z_2\left(c_{ij} - z_{ij}\right) + \theta_{ij}\left(c_{ij} - z_{ij}\right)\left(d_{ij} - z'_{ij}\right) < 0 \qquad \ldots\ldots\ldots.(4)$$

Therefore one can move from one basic feasible solution to another basic feasible solution on entering the cell (i, j) into the basis for which condition (4) is satisfied.
It will be an optimal basic feasible solution if

$$R_{ij} = \theta_{ij}\left(z'_{ij} - d_{ij}\right)\left(z_{ij} - c_{ij}\right) - Z_1\left(z'_{ij} - d_{ij}\right) - Z_2\left(z_{ij} - c_{ij}\right) \geq 0$$

Also it can be easily seen that $R_{ij} = 0 \ \forall$ cells $(i,j) \in B$

3. DEFINITION

Corner Feasible Solution: When $y_{m+1,n+1}=0$ in the problem (P_2), then the basic feasible solution $\{y_{ij}\}$, $i \in I'$, $j \in J'$ to (P_2) is called a corner feasible solution (cfs) which implies that the bottom right corner cell of the transformed transportation matrix in the problem (P_2) must take the value zero.

Theorem2. Every corner feasible solution of (P_2) provides a basic feasible solution to (P_1) and conversely.

Proof: let $\{y_{ij}\}$, $i \in I'$, $j \in J'$ be a cfs to (P_2). Define $y_{ij} = x_{ij}$, $(i,j) \in I \times J$, $\{x_{ij}\}$ so defined can be established to be a basic feasible to (P_1).

Conversely, given $\{x_{ij}\}$ to be a basic feasible solution to (P_1) then
$\{y_{ij}\}$, $(i,j) \in I' \times J'$
where $I = \{1,2,\ldots\ldots,m+1\}$
$J' = \{1,2,\ldots\ldots,n+1\}$
defined by the transformation

$$y_{ij} = x_{ij}, \quad (i, j) \in I \times J$$

$$y_{i,n+1} = a_i - \sum_{j \in J} x_{ij}, \qquad i \in I$$

$$y_{m+1,j} = b_j - \sum_{i \in I} x_{ij}, \qquad j \in J$$

$$y_{m+1,n+1} = 0 \ \ (\text{as } c_{m+1,n+1} = M = d_{m+1,n+1})$$

can be shown to be a cfs to (P_2).
Remark1: A cfs of (P_2) is also a cfs of (P'_2).
Remark2: The value of the objective function of (P'_2) at a corner feasible solution is equal to the value of the objective function of (P'_1) at its corresponding basic feasible solution.
Remark 3: A non-corner feasible solution to (P'_2) cannot provide a feasible solution to (P'_1).

Remark 4: An optimal solution to (P_2') has to be a corner feasible solution.

Remark 5: Optimal corner feasible solution to (P_2') provides an optimal solution to (P_1').

4. ALGORITHM

Step 1: Given the fixed charge bi-criterion quadratic transportation problem. Separate it into two problems (P_1') and (P_1''). Let the flow be enhanced to P *(> Max (* $\sum_{i\in I} a_i, \sum_{j\in J} b_j$ *))*. Introduce an additional row with availability = P- $\sum_{j\in J} b_j$ and an additional column with demand = *P*- $\sum_{i\in I} a_i$.

Step 2: Form the problem (P_2'). Find its initial basic feasible solution *{* y'_{ij} *}*. Let *B* be its corresponding basis.

Step3: Calculate the fixed cost of the current basic feasible solution and denote it by F^1(current), where F^1(current) = $\sum_{i\in I} F_i$

Step4: Find $R_{ij}\ \forall (i,j) \notin B$

where $R_{ij}= \theta_{ij}(z'_{ij}-d_{ij})(z_{ij}-c_{ij})-Z_1(z'_{ij}-d_{ij})-Z_2(z_{ij}-c_{ij})$, and denote it by $(R_{ij})_1$

Step5: Find $A^1_{ij}=(R_{ij})_1\times(E_{ij})_1$

where A^1_{ij} is the change in the variable cost obtained on introducing a non basic cell *(i,j)* with value $(E_{ij})_1$ (for all $(i,j)\notin B$) into the basis.

Step6: Find F^1_{ij} (Difference) = Change in fixed cost = F^1_{ij}(NB) - F^1 (current) where F^1_{ij}(NB) is the total fixed cost obtained on introducing the variable x_{ij} with value $(E_{ij})_1$ (for all $(i,j)\notin B$) into the current basis to form a new basis.

Step7: Find $\Delta^1_{ij}=F^1_{ij}$(Difference) + $A^1_{ij}\ \ \forall\ (i,j)\notin B$

If all $\Delta^1_{ij}\geq 0$, then it is not possible to decrease the total cost i.e., (variable cost + fixed cost). Go to step 8 but if $\exists$ at least one $\Delta^1_{ij}<0$, find min$\{\Delta^1_{ij}/\Delta^1_{ij}<0,(i,j)\notin B\}$ $=\Delta_{pq}$ (say).Then the cell (p,q) enters the basis.

Step8: Let Z^1 be the optimal cost of (P_2') yielded by the basic feasible solution { y^1_{ij} }.

Step9: Find $T^1 = \underset{\substack{i\in I'\\ j\in J'}}{\text{Max}}\{t_{ij} \mid y^1_{ij}>0\}$ from the problem (P_2'')

Then the corresponding pair (Z^1, T^1) is the first cost- time trade off pair for the problem (P_2) and subsequently for the problem (P_1) .To find the next best cost-time trade off pair, go to step 10.

Step10: Define $c^1_{ij} = \begin{cases} M & \text{if } t_{ij}\geq T^1 \\ c_{ij} & \text{if } t_{ij}<T^1 \end{cases}$

where *M* is a sufficiently large positive number. Form the corresponding fixed charge quadratic transportation problem with variable cost c^1_{ij}. Repeat the above process till we get the problem to be infeasible.

The complete set of cost-time trade off pairs of (P_1) at the end of qth iteration are given by (Z^1,T^1) , (Z^2,T^2),................, (Z^q,T^q)
where $Z^1< Z^2 <\ldots\ldots\ldots<Z^q$ and $T^1> T^2 >\ldots\ldots\ldots>T^q$

Remark6: The pair (Z^l, T^q) with minimum cost and minimum time is the ideal pair, which cannot be achieved in practice except in some trivial case.

Convergence of the Algorithm: The algorithm will converge after a finite number of steps. In fact, the choice of c_{ij}^1 in step 10 will ensure an infeasible solution after a finite number of iterations.

Conclusion: In order to solve a bi-criterion quadratic fixed charge problem we separated the problem into two problems. One of them being an indefinite quadratic programming problem has its optimal solution extreme point. Also, using the concept of time, we calculated the various cost-time trade off pairs.

5. A NUMERICAL EXAMPLE

The Fixed Charge Bi-criterion Quadratic Transportation Problem with enhanced flow is:

$$\text{Minimize}\left\{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)+\sum_{i\in I}F_i\ ,\ \underset{\substack{i\in I\\ j\in J}}{\text{Max}}\left[t_{ij}\middle|\ x_{ij}>0\right]\right\}$$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i \ , \qquad i\in I$$

$$\sum_{i\in I} x_{ij} \geq b_j \ , \qquad j\in J$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P\ (>\max(\sum_{i\in I} a_i, \sum_{j\in J} b_j))$$

$$x_{ij} \geq 0 \ , \quad i\in I, j\in J$$

Consider the test problem below where Table-I gives the values of variable costs c_{ij}*'s* and d_{ij}*'s* *(i =1,2,3; j =1,2,3)* and Table-II gives the values of *time* t_{ij} *(i =1,2,3 ; j =1,2,3)*

Introducing a dummy source and a dummy destination in Table-I with
$c_{i(n+1)} = c_{iq}$ and $d_{i(n+1)} = d_{iq}$ such that

$$c_{iq}d_{iq} = \min_{j\in J} c_{ij}d_{ij} \quad \forall i\in I$$

$c_{(m+1)j} = c_{pj}$ and $d_{(m+1)j} = d_{pj}$ such that $c_{pj}d_{pj} = \min_{i\in I} c_{ij}d_{ij} \quad \forall j\in J$

$c_{44} = M = d_{44}$ where M is a large positive number

and $\quad b_4 = P - \sum_{i=1}^{3} a_i = 50 - 40 = 10$

$$a_4 = P - \sum_{j=1}^{3} b_j = 50 - 40 = 10$$

We form the corresponding problem (P_2'). Similarly, on introducing a dummy source and a dummy destination in Table-II with

$t_{i4} = \min_{j\in J} t_{ij}$, $\quad i = 1,2,3$; $t_{4j} = \min_{i\in I} t_{ij}$, $\quad j = 1,2,3$; $t_{44} > \underset{\substack{i\in I\\ j\in J}}{\text{Max}}\ t_{ij} = 16$

and taking $t_{44} = 18$ with $a_4 = 10$, $b_4 = 10$ we form the corresponding problem (P_2'').

				$a_i \downarrow$
$c_{11} \rightarrow$ / $d_{11} \rightarrow$	5 1	9 2	9 4	19
	4 3	6 7	2 4	10
	4` 2	1 9	2 5	11
$b_j \rightarrow$	15	10	15	

Table-I

$t_{11} \rightarrow$	8	14	2
	7	5	11
	12	9	16

Table-II

The fixed costs are $F_{11}=100$, $F_{12}=50$, $F_{13}=50$, $F_{21}=150$, $F_{22}=50$, $F_{23}=50$, $F_{31}=200$, $F_{32}=100$, $F_{33}=50$

The total cost, which is to be minimized, is given by $\left(\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij} \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}x_{ij}\right) + \sum_{i=1}^{3} F_i$

where $\quad F_i = \sum_{l=1}^{3} \delta_{il}F_{il}$ for $\quad i = 1,2,3$

where $\quad \delta_{i1} = 1 \quad$ if $\sum_{j=1}^{3} x_{ij} > 0$ for $i = 1,2,3$

$\quad = 0 \quad$ otherwise

$\quad \delta_{i2} = 1 \quad$ if $\sum_{j=1}^{3} x_{ij} > 7$ for $i = 1,2,3$

$\quad = 0 \quad$ otherwise

$$\delta_{i3} = 1 \qquad \text{if } \sum_{j=1}^{3} x_{ij} > 10 \text{ for } i = 1,2,3$$

$$= 0 \qquad \text{otherwise}$$

Let the flow be enhanced to P = 50

where $P = 50 > \max\left(\sum_{i=1}^{3} a_i = 40, \sum_{j=1}^{3} b_j = 40\right)$

Introducing a dummy source and a dummy destination in Table-I with

$c_{i(n+1)} = c_{iq}$ and $d_{i(n+1)} = d_{iq}$ such that $c_{iq}d_{iq} = \min_{j \in J} c_{ij}d_{ij} \quad \forall i \in I$

And $c_{(m+1)j} = c_{pj}$ and $d_{(m+1)j} = d_{pj}$ such that $c_{pj}d_{pj} = \min_{i \in I} c_{ij}d_{ij} \quad \forall j \in J$

$c_{44} = M = d_{44}$ where M is a large positive number

and $\qquad b_4 = P - \sum_{i=1}^{3} a_i = 50 - 40 = 10$

$a_4 = P - \sum_{j=1}^{3} b_j = 50 - 40 = 10$

We form the corresponding problem (P_2'). Similarly, on introducing a dummy source and a dummy destination in Table-II with

$t_{i4} = \min_{j \in J} t_{ij}$, $\quad i = 1,2,3$; $t_{4j} = \min_{i \in I} t_{ij}$, $\quad j = 1,2,3$; $t_{44} > \operatorname{Max}_{\substack{i \in I \\ j \in J}} t_{ij} = 16$

and taking $t_{44} = 18$ with $a_4 = 10$, $b_4 = 10$ we form the corresponding problem (P_2'').

Solving problem (P_2''). Its optimal solution is given in Table-III.

				F^1(current)↓
5 (9) 1	9 (10) 2	9 4	5 1	200
4 3	6 7	2 (10) 4	2 4	200
4 (1) 2	1 9	2 5	4 (10) 2	200
5 (5) 1	1 9	2 (5) 4	M M	0
				600

Table-III

Here $Z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} = 234$, $\quad Z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij} = 116$

For non- basic cells in Table-III, we compute the following

(i,j)	(1,3)	(1,4)	(2,1)	(2,2)	(2,4)	(3,2)	(3,3)	(4,2)
R_{ij}^{1}	812	0	594	747	309	550	116	430
E_{ij}^{1}	4	9	5	5	5	1	1	5
A_{ij}^{1}	4060	0	2970	3735	1545	550	116	2150
F_{ij}^{1} (NB)	550	650	600	550	550	600	600	600
F_{ij}^{1} (Diff.)	-50	50	0	-50	-50	0	0	0
Δ_{ij}^{1}	4010	50	2970	3685	1495	550	116	2150

Table-IV

Here $\Delta_{ij}^{1} \geq 0 \quad \forall\, (i,j) \notin B$

It is now not possible to decrease the total cost (variable cost + fixed cost),
minimum cost $Z^1 = (234 \times 116) + 600 = 27144 + 600 = 27744$ and corresponding time $= T^1 = 14$.

Hence the first cost time trade off pair is *(27744, 14)*.

$$\text{Define} \quad c_{ij}^{1} = \begin{cases} M & \text{if } t_{ij} \geq T^1 = 14 \\ c_{ij} & \text{if } t_{ij} < T^1 = 14 \end{cases}$$

On solving the problem, we get the optimal solution as below

				F^1(current)↓
5 (15) 1	M 2	9 4	5 (4) 1	200
4 3	6 7	2 (5) 4	2 (5) 4	150
4 2	1 (10) 9	M 5	4 (1) 2	300
5 1	1 9	2 (10) 4	M M	0
				650

Table-V

So the second cost-time trade off pairs is (29109, 11).

Again defining

$$c_{ij}^{2} = \begin{cases} M & \text{if } t_{ij} \geq T^1 = 11 \\ c_{ij} & \text{if } t_{ij} < T^1 = 11 \end{cases}$$

and on solving the problem, we get the third cost time trade off pair is (35705,9)

Again defining

$$c_{ij}^{3} = \begin{cases} M & \text{if } t_{ij} \geq T^{1} = 9 \\ c_{ij} & \text{if } t_{ij} < T^{1} = 9 \end{cases}$$

and on solving, the problem becomes infeasible.

Hence the cost-time trades off pairs are *(27744, 14), (29109, 11), (35705, 9)*

Remark7: We also solved various test problems for higher dimensions on GAMS (General Algebraic Modeling System) software on a P.C with Intel Pentium Processor 1.40GHz with 1.25GB RAM and it took less than 5 seconds to solve the problems. Also, because of choice of c_{ij}^{1} our algorithm would finally terminate in finite number of steps ending in infeasible solution. The numerical example taken here gives infeasible solution after third cost-time trade off pair.

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