算數書

Suan Shu Shu

(A Book on Numbers and Computations)

TWO PROBLEMS IN COLLATING, INTERPRETING AND TRANSLATING THE *SUAN SHU SHU*

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In honor of Mariano Hormigón, to whose memory and with thanks for his many contributions to the international appreciation of the history of mathematics, across all cultures and times, this study is respectfully dedicated.

ABSTRACT

In December and January of 1983-1984, a team of archaeologist excavating an early Han dynast tomb at Zhangjiashan in Hubei Province, China, discovered the earliest yet-known mathematical work from ancient China, the Suan Shu Shu (on Numbers and Computations). This work, on nearly 200 bamboo strips, reflects largely the state of mathematics in pre-Qin China, and includes more than 60 problems dealing with various matters of arithmetic and geometry. Two of these have been open to diverse and especially divergent interpretations, namely a. pair of seemingly related problems, 以· 材方 Yi Yuan Cai Fang and 以方材· Yi Fang Cai Yuan, which have generated considerable disagreement about whether they are inverse or quite different problems. Virtually everyone who has approached these two problems has understood them differently in trying to account for the statements, answers, and methods as given in the Suan Shu Shu. This paper, which I am pleased to dedicate to the memory of Mariano Hormigón, would have pleased him, I would like to think, because it shows that there may not always be a single, mathematically consistent interpretation of a given historical document. What follows is devoted to discussion of the various collations and explanations offered for these especially challenging parts of the Suan Shu Shu, and what they may tell us about early Chinese mathematics in general.

Squaring Circles: Two Problems, Inverses or Not?

Among the most challenging problems given in the *Suan Shu Shu* for collation, interpretation, and translation are two that come very near the end of the presently accepted editions, one devoted (apparently) to the dimensions of a square either cut from or inscribed in a circle, the other devoted (apparently) to the dimensions of a circle either cut from or inscribed in (or possibly circumscribed around) a square.[1] This is how the two problems were first published in the journa of Chinese culture, *Wenwu*, in 2000, hereafter referred to as [WW 2000, p. 83]:

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1 The first publication of the complete text of the *Suan Shu Shu* in the journal *Wenwu* in 2000 did not number the problems, but in its collation of that same year, the Tongxun group did, and here the numbering of the problems to be found in [Tongxun 2000] is followed. None of the later collations has numbered all of the problems, so the Tongxun numbering is still the most convenient for easy reference.

[61] **以· 材方** 以· 材為方材, 曰大四韋二寸廿五分寸十四, 為方材幾何？曰: 方七寸五分寸三。術曰: 因而五之為實, 令七而一四…

(For now, what follows is as literal a translation of the above as possible):

[61] From a circle cut a square: in order to turn a circular (piece of) wood into a square (piece of) wood, say its size is 4 *wei* 2 14/25 *cun*, how large is the square (piece of) wood? (The answer) says: the square is 7 3/5 *cun*. The method says: (obtain) the result by multiplying by 5 as the dividend, and divide by 7, 4 (note the 4 at the end of this line is not followed with any final punctuation, nor are there any instructions as to what its role in the problem should be).

[62] **以方材·** 以方為· 曰材, 方七寸五分寸三為· 材幾何？ 曰: 四韋二寸廿五分十四。·術曰: 方材之一面即· 材之徑也, 因而四之以為實, 令五而成一。

[62] From a square cut a circle: in order to turn a square into a circle, say the wooden square is 7 3/5 *cun*, how large is the circular (piece of) wood? (The answer) says: 4 *wei* 2 14/25 *cun*. The method says: one side of the square (piece of) wood is the diameter of the circular (piece of) wood; (obtain) the result by multiplying by 4 as the dividend, and divide by 5.

Based upon the [WW 2000] version of the text, a team of scholars at National Taiwan Normal University issued one of the first interpretations of the *Suan Shu Shu* in 2000, as did Guo Shirong of Inner Mongolia Normal University and Guo Shuchun of the Institute for History of Natural Science in Beijing, both of the latter in 2001. Their interpretations of these two passages now follow, after which we consider their various points of view and the subsequent versions of the text issued by Peng Hao, along with his interpretation of these passages and the later, definitive edition of the text as published by the Zhangjiashan editorial group, also in 2001. These are followed by a synoptic overview of all these interpretations of the two problems, along with yet another interpretation of their meaning offered by Duan Yaoyong and Zou Dahai in 2003, after which are offered yet another two possible interpretations of the texts in question.

HPM 通訊 *HPM Tongxun* (2000)

In November of 2000 the first detailed study of the text of the *Suan Shu Shu* was presented in *HPM Tongxun* (HPM Newsletter of Taiwan Normal University) by a team of seven scholars: Su Yiwen, Su Junhong, Su Huiyu, Chen Fengzhu, Lin Cangyi, Huang Qingyang, and Ye Jihai (hereafter referred to as the Tongxun group, or [Tongxun 2000]). Numbering these problems as [61] and [62] following the ordering of the problems as given in the original *Wenwu* 2000 publication of the *Suan Shu Shu*, the group only makes one emendation of the text as published in *Wenwu*, namely at the end of Problem 61 where they say the abandoned "4" should mean "divide by 4" ("in order to conform with the reasoning of the original problem," [Tongxun 2000, p. 17, note 175]), and thus they revise the original text to read as follows:

[Tongxun 61] From a circle cut a square: in order to turn a circular (piece of) wood into a square (piece of) wood, say its size is 4 *wei* 2 14/25 *cun*, how large is the square (piece of) wood? (The answer) says: the square is 7 3/5 *cun*. The method says: (obtain) the result by multiplying by 5 as the dividend, and divide by 7, [then] **divide by** 4.

In note 174, explaining the meaning of Problem [61] and its solution procedure, the Tongxun group interprets the situation as follows: "This problem discusses the relation between the lengths of the circumference of a circle and the perimeter of its inscribed square. Its method of calculation is: 42 14/25 × 5 × 1/7 × 1/4 = 7 3/5; probably this is a mistake with respect to the relation between the side and diagonal of the square in the ratio 5:7 (side 5 diagonal 7), and a miscalculation of the ratio of the perimeter of the inscribed square and the circumference of the circle, in order to reach the length of the side of the (inscribed) square which is sought (and calculated in the answer to be) 7 3/5. What is important to pay attention to here is that the correct calculation should proceed from a ratio of the (circumference of the) circle to the perimeter of its inscribed square of $7\pi:20$ (at that time π was approximated as 3)" [Tongxun 2000, p. 17, note 174].

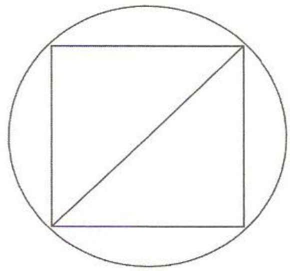

Figure 61a

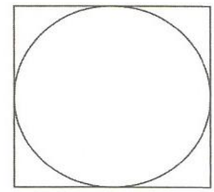

Figure 62a

With respect to Figure 61a, the Tongxun group's interpretation of this problem seems to be as follows, based upon everything said above: given a circle of circumference C= 42 14/25 *cun*, what is the length of the edge of the inscribed square and correspondingly its perimeter? The group does not directly address the significance of the final "4" or the instruction as they revise it, "divide by 4," but their commentary suggests that the original author or copyist of Problem 61 mistook the ratio of the circumference of the circle and the perimeter of its inscribed square to be 7:5, rather than the correct ratio 21:20 (assuming π=3). Had this ratio been used, then the calculation, (42 14/25)(20/21) would have given the perimeter of the inscribed square, and 1/4 of that would have given the correct value of the length of the edge of the inscribed square.

To put this another way, proceeding from their suggestion that this problem is about a comparison of the circumference of the circle and the perimeter of its inscribed square, and assuming that $\pi = 3$, the diameter d of the circle would then be (1/3)(42 14/25). Since the ratio of the diagonal d to the side s of the square is taken as approximately d/s = 7:5 (a fair approximation to $\sqrt{2}$ if $\pi = 3$), then the side of the inscribed square would be (5/7)(1/3)(42 14/25), and the perimeter of the inscribed square would be four times this, or (4)(5/7)(1/3)(42 14/25), and this of course gives the ratio the Tongxun group notes is correct for P:C = 20/21, taking $\pi = 3$. If all of these assumptions are accepted, then the correct solution to the problem should have been an inscribed square with a perimeter of 40 8/15 *cun*, or an edge of 10 2/15 *cun*. Rather than carry out this calculation based upon its understanding of the problem, the Tongxun group instead gives what it takes to be the incorrect calculation as only partially described in the method of the problem (as amended above), with a note to the effect that had the correct ratio been applied for computing the perimeter of the inscribed square from the given circumference of the circle, this would have led to the correct answer, which the group also does not specify

However, the major difficulty with the Tongxun group's interpretation of Problem 61 is that nowhere does the text refer explicitly to either the circumference of a circle (· 的周· *yuan de zhou chang*) or to the perimeter of an inscribed square (方的周· *fang de zhou chang*). This is problematic given the immediately following Problem 62, which refers specifically both to the diameter of a circle and to the side of a square, but not to circumferences or perimeters. At this point, there are too many open questions about exactly what dimensions of the square and circle are meant to be understood in Problem 61 to say more about it here; but this is a matter to which we shall shortly provide a very different interpretation.

As for Problem 62, the Tongxun group offers the following explanation of the problem and its method of solution: "The original problem discusses the relation of a square figure and the circumference of its inscribed circle. Its method of calculation is the inverse of the calculation of Problem 61: 7 3/5 × 4 × 1/5 × 7 = 42 14/25; nevertheless, because the original problem discusses the relation of a square figure and its inscribed circle, and Problem 61 discusses the connection between a circle and its inscribed square figure, these are definitely not similar or inverse (problems); it may be because the author or whomever copied the book did not understand the meaning of this problem that they could give this kind of a calculation procedure" [Tongxun 2000, p. 17, note 176].

This, however, does not really offer an explicit interpretation of Problem 62. Although the Tongxun group says that it does not regard the two Problems 61 and 62 as inverse problems, it does seem to regard them as similar in relating perimeters, i.e. Problem 62 is based on the ratio of the lengths of the perimeter of the square to the circumference of its inscribed circle, even if the ratio used in the *Suan Shu Shu* is incorrect (mistaken this time as 7/5). The Tongxun group does explicitly give the instructions for solving the problem given in Problem 62 as they reconstruct it, which is exactly the inverse of the method given for Problem 61: 7 3/5 × 4 × 1/5 × 7 = 42 14/25. Judging from this and the Tongxun group's interpretation of Problem 61, the problem here seems to be understood as: given the edge of the square, find the circumference of its inscribed circle. Thus if 7 3/5 is the edge of the square, 7 3/5 × 4 gives the length of its perimeter; applying the correct ratio for the perimeter of the square to the circumference of its inscribed circle, 4:3, requires multiplication by 3/4 to provide a correct solution to this problem, i.e. 7 3/5 × 4 × 3/4 = 22 4/5. However, since the method of Problem 62 says explicitly that "one side of the square (piece of) wood is the diameter of the circular (piece of) wood," and since throughout the *Suan Shu Shu* the ratio of the diameter to the circumference of the circle is taken as 3, the solution to Problem 62, if the 7 3/5 is the edge of the square, means that to find the circumference of the inscribed circle should only require multiplication of the length of the edge by 3. So there are a number of problems here with respect to Problem 62 that the Tongxun group does not address.

This is complicated by the fact that the Tongxun group does not provide its version of either what it takes to be the correct procedure or answer for Problem 62. (Figure 62a captures the essence of the Tongxun group's interpretation of the problem, although neither Figure 61a nor 62a is given with the Tongxun discussion of either problem.) Furthermore, concerning Problem 62, the Tongxun group warns that at the end of the problem "the original text is definitely not complete, perhaps there is a missing part" [Tongxun 2000, p. 17, note 177]. In the absence of any indication of exactly what the Tongxun group believes to be missing, it is perhaps best not to guess further how the Tongxun group might render a corrected collation for Problem 62. In its edition of the text, despite its interpretation of the problem as reflected in the above discussion, the Tongxun collation offers no changes from the original text for Problem 62 from that given in the *Wenwu* 2000 publication; the following is Problem 62 as it appears in [Tongxun 2000, p. 17].

One very direct if not very interesting interpretation of Problem 62 is that it is really no more complicated than its statement would suggest, although the author or copyist certainly got the answer and procedure entirely wrong. But if the problem should proceed directly as stated, from the length of the edge of a square to find the diameter of the inscribed circle, then as the problem says, since the edge and diameter are the same, the answer follows directly, and the diameter of the circle is 7 3/5 *cun*. Making the problem only slightly more interesting would be to ask for the circumference C of the inscribed circle, which would seem to be the solution for which the Tongxun group opts. But we already know from the above that based on the assumption that $\pi = 3$, the circumference of the inscribed circle is simply equal to 3 times the edge of the square, or 3(7 3/5) = 22 4/5 *cun*. This does not seem to amount to much of a problem mathematically. In any case, none of these alternatives manages to account for the computation as prescribed in the method of solution for Problem 62 as it appears on the surviving bamboo slips. There are, however, more interpretative possibilities to consider.

Guo Shirong (2001)

Writing in the third number of the *Journal of Inner Mongolia Normal University* for 2001, Guo Shirong offered "A Collation to the *Suanshushu* (A Book of Arithmetic)." As noted in his English abstract, "Based on analyses in mathematical principles of the text, the author collates the text and corrects all kinds of mistakes in the original bamboo text, such as mistakes of formulas, mistakes in calculation, swapping combinations of bamboo slips, missing and redundant words, transposition of words, errors in copy, and so on. Meanwhile, 53 Chinese characters in some fragments of the text that could not be identified before have been restored" [Guo Shirong 2001, 285]. Guo Shirong does not consider every problem or section of the *Suan Shu Shu*, and he numbers consecutively only those he does treat, following the order of the original text as published in [Wenwu 2000]. Consequently the problems dealing with the inscribed squares and circles occur as Problems 43 and 44 in his collation.

Guo Shirong begins his analysis of Problem 61/43 with the following comment:

> In this question the problem and data in the answer contain copyist errors, and the method is also problematic. The meaning of the original problem is: wishing to turn a

column-shaped circular (piece of) wood into a square-shaped (piece of) wood, and knowing 'the *da si wei* is 2 14/25 *cun*,' the question is how much is the side of the square (piece of) wood? The problem's assumed condition is that the amount of the difference between the diameter of the circular (piece of) wood and the side of the square (piece of) wood should be understood to be 2 14/25 *cun*. According to the text of the method, it is known that (the square) inscribed in the circle is determined in accordance with the approximate computational formula 'side 5 diagonal 7,' but the difference between the computational result and the answer is too large.[2] This author thinks that '2 14/25 *cun*' is wrong (and should be) '2 24/25 *cun*,' and that the answer 'the square is 7 3/5 *cun*' is wrong (and should be) 'the square is 7 2/5 *cun*,' both of which are copyist errors. In the method, "divide by 7, 4" should be "multiply by 7, and divide by 14.' If x represents the edge of the square, then $(2\ 24/25 + x) \times 5/7 = x$, namely $x = (2\ 24/25) \times 5/7 \times 7/2 = (2\ 24/25) \times (5 \times 7)/(7 \times 2) = 7\ 2/5$ *cun*. In this way, the question, answer, and method, all three square with one another" [Guo Shirong 2001, p. 283, note to Problem 43].

In light of the above, Guo Shirong revises the original text as follows:

[Guo Shirong 61/43] From a circle cut a square: in order to turn a circular (piece of) wood into a square (piece of) wood, say its *da si wei* is 2 **24**/25 *cun*, how large is the square (piece of) wood? (The answer) says: the square is 7 2/5 *cun*. The method says: (obtain) the result by multiplying by 5 as the dividend, [then] **multiply by 7, and divide by 14**.

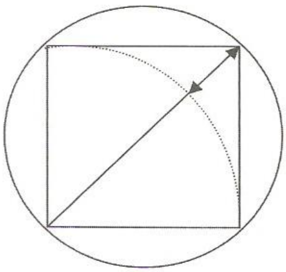

Figure 61b

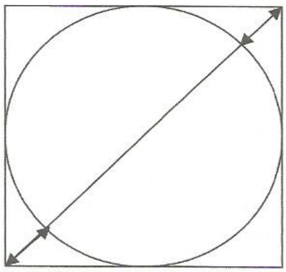

Figure 62b

Now this is a very ingenious interpretation of Problem 61, but it is also makes a number of assumptions, the most unconventional being the interpretation of "*da si wei* is 2 [24]/25 *cun*." Neither the Tongxun group nor Guo Shirong comments on exactly how 大四韋 *da si wei* should be understood here, so we shall leave this aside for now, but note that on Guo Shirong's interpretation of the rest of this problem, what should be at issue here is the difference between the length of the diameter of the circle and the side of the inscribed square, which Guo Shirong takes to be 2 [24]/25 *cun* in his revised reading of the problem [Figure 61b]. But what is there in the logic of the problem and its solution that would explain the reading of the method: "divide by 7, 4," as "multiply by 7, and divide by 14"? There is nothing in the general character of this problem to compel a division by 14 (and for that matter, if instructed to multiply by 7 and divide by 14, why not just say "divide by 2"?) This of course is what Guo Shirong wants to get at, namely the numbers given in the method for solution of the problem are 5, 7 and 4. On his reading of the problem, the method should boil down to multiplying the given "difference" between the diagonal and edge of the square by 5/2 to give the length of the edge of the square. But the method as given in the bamboo text says to multiply by 5, divide by 7, and then there is the lonesome "4." But if we take care of the 7 and 4 by saying it is all a

2 Approaching the problem as Guo Shirong does, the length of the side of the square should be 5/2 the difference of the diagonal and side of the square, or (5/2)(2 14/25) = 6 2/5 *cun*. Regarding this as too large a difference from the given answer, 7 3/5 *cun*, it seems that Guo Shirong then tried 2 24/25 as the difference and obtained a result nearer to that given in the text, namely (5/2)(2 24/25) = 7 2/5 *cun*, and so he emended the text accordingly.

mistake for "multiply by 7 and divide by 14," then what we have done in effect is to multiply the "difference" of 2 [24]/25 by 5/2, which gives the needed result.

This, however, is completely arbitrary, and there is nothing that would explain why in the general statement of the method for working out such problems a general method should say "multiply by 7 and divide by 14." Given this apparent level of arbitrariness and the fact that that there is nothing in the language of the text *per se* to indicate that this problem concerns the difference between the lengths of the diameter/diagonal and side in question, we are left with a number of open questions concerning this reading of Problem 61. Moreover, the number of changes subsequently required in the data given in this problem in order to reach the author's conclusion also raises questions about whether this interpretation is really as faithful to the original meaning of the problem as might be wished.

With these questions in mind, let us now turn to the companion Problem 62/44. Guo Shirong interprets this as follows:

> This problem and the "Yi Yuan Cai Fang" problem are mutually converse. Here "si wei" refers to the part in excess of the difference between the diagonal[3] of the square and the diameter of the circle. And as in the "Yi Yuan Cai Fang" problem, "3/5 *cun*" is wrong and should be "2/5 *cun*"; in "2 14/25 [*cun*] " the "14" is wrong and should be "24, " not to mention that there is a character missing for "*cun*." Moreover, in the method "multiplying by 4" is wrong for "multiplying by 2." Also, in the phrase "yi fang wei yuan yue cai" the two characters "yue cai' are reversed, and should be "cai yue." The punctuation should be: "yi fang [cai] wei yuan cai, yue fang qi cun wu fen cun san." If y represents the "si *wei*," the equation for this problem is $y = (7\ 2/5) \times 5/7 - 7\ 2/5 = (7\ 2/5) \times (2/5) = 2\ 24/25$ (*cun*) [Guo Shirong 2001, p. 284].[4]

In light of the above, Guo Shirong has made the following changes in the original text, to conform primarily to his understanding of the meaning of the problem:

> [Guo Shirong 44/62] From a square cut a circle: in order to turn a square [**piece of wood**] into a circular [**piece of wood**], say the square is 7 **2**/5 *cun*, how large is the circular (piece of) wood? (The answer) says: 4 *wei* 2 **24**/25 *cun*. The method says: one side of the square (piece of) wood is the diameter of the circular (piece of) wood; (obtain) the result by multiplying by **2** as the dividend, and divide by 5.

This interpretation for Problem 62 is just as ingenious as that given for the previous problem, and here Guo Shirong makes explicit that he takes "si wei" to refer somehow to the difference between the diagonal of the square and the shorter length of the diameter of the inscribed circle, those portions of [Figure 62b] indicated by the bold double-ended arrows. But no rationale is offered for why "si wei" should be understood here as a difference, let alone the difference between the diagonal and edge of the square in this problem, whereas "da si wei" was taken to represent the difference between the diagonal and edge of the square in the previous problem. In fact, given the identity of the edge of the square and the diameter of the circumference of the inscribed circle, both Problems 61 and 62 are really the same problem, namely one dealing with the inverse relations between the side and diagonal of the square.

This time, given the (revised) length of the side of the square as 7 2/5 *cun*, we are to find the difference between this and the diagonal of the square, which is simply 2/5 that of the edge. Given that the original method for this problem is explicit in directing that the edge be multiplied by 4, then divided by 5, Guo Shirong has no choice here but to say that the "4" is wrong and should be a "2," which immediately gives the necessary ratio, 2/5, needed to determine the difference between the length of the diagonal and side of the square. Nevertheless, this is just as arbitrary a change as before in order to make the method suit the interpretation of the problem given here.

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[3] Note that here there is a misprint in the text; instead of *qian* (money), the text clearly intends *xian* (diagonal) [Guo Shirong 2001, p. 284, comments on Problem 44 [Problem 62].

[4] At this point, Guo Shirong mistakenly reverses the ratio that is needed between the side/diagonal of the square, which here should be 7/5, not 5/7.

Again, considering the extent to which Guo Shirong's version of Problem 62/44 has to tamper with the numbers, we are left with the same questions as pertain to his interpretation of the preceding problem. On these matters Guo Shuchun's approach to both Problems 61 and 62 offers further help in understanding what may be involved.

Guo Shuchun (2001)

Basically, Guo Shuchun agrees with Guo Shirong's understanding of Problems 61 and 62 of the *Suan Shu Shu*, but he interpolates four new characters into the text and makes some slightly different changes to the given data. Where the original statement of Problem 61 gives the answer for the length of the side of the inscribed square as 7 3/5 *cun*, and Guo Shirong emends this to 7 2/5 *cun*, Guo Shuchun corrects the text to 6 2/5 *cun*.[5] Guo Shuchun also maintains that there are four missing characters following announcement of the method, "shu yue: zhi da si wei," a change which serves to bring the method into a form consistent with the original statement of the problem. Guo Shuchun then offers the following explanation for his understanding of the problem in general:

> Both Guo Shirong and I think this problem is, knowing that the diameter of the circle is greater than the length of the side of its inscribed square by 2 14/25 (or 2 24/25), and using/depending on (the ratio) square 5 diagonal 7 (approximating $\sqrt{2}$ or the side/diagonal ratio), find the length of the edge of the square. Su Yiwen *et al.* [Tongxun 2000] after the "4" add two characters, 而一 *er yi* (divide by)" [Guo Shuchun 2001, p. 215, note 3 to "Yi Yuan Cai Fang"].

Guo Shuchun ends his comment by quoting the interpretation of the problem given by Su Yiwen *et al.* above, noting that in [Tongxun 2000] the problem is taken to involve the relation of a circle to the perimeter of its inscribed square. He also gives the method of solution along with the Tongxun group's explanation of where the text seems mistaken, but with no further comment by Guo Shuchun as to why he disagrees with this particular approach to the problem. Taking all of Guo Shuchun's comments on Problem 61 into account, his version reads as follows:

> [Guo Shuchun 61] From a circle cut a square: in order to turn a circular (piece of) wood into a square (piece of) wood, say its *da si wei* is 2 14/25 *cun*, how large is the square (piece of) wood? (The answer) says: the square is **6 2**/5 *cun*. The method says: **given the *da si wei***, (obtain) the result by multiplying by 5 as the dividend, [then] **multiply by 7 divided by 14**.
>
> While Guo Shuchun's reading of this problem in a sense improves upon Guo Shirong's interpretation—it requires only a change in the data concerning the length of the edge of the inscribed square as given in the answer, leaving the other numbers as in the original text—there is still no rationale offered for why "da si wei" should be interpreted as the difference between the diagonal of the square and its edge. And while he resists the need to emend the end of the text, the "hanging" 4 is still a problem, since by ending the phrase "ling qi er yi si" with a period, the method concludes with the instruction "divide 7 by 14." Thus Guo Shuchun's version of Problem 61 works out as: (2 14/25)(5)(7/14) = 6 2/5 *cun*. While this indeed gives an answer consonant with Guo Shirong's reading of the text, and gives the answer Guo Shuchun has anticipated, it runs into the same sort of difficulties as noted above with the statement of the method, for there is no reason why it should end with an instruction to "divide 7 by 14" as a natural result of the relation between circles and squares (or the diagonal and side of a square). If Guo Shirong and Guo Shuchun are correct in their approach to this problem, the method should simply direct immediately, "multiply by 5, divide by 2." Working out the problem as Guo Shuchun understands it shows where all this comes from, since on his reading of the problem it requires multiplication of the given edge of the square by a factor of 5/2, i.e.: (5)(7)(1/14) = 5/2, but this seems to be working backwards from the numbers, rather than

[5] Whereas Guo Shirong, calculating from the given "difference" of 2 14/25 *cun* that the result 6 2/5 *cun* was too far from the given answer of 7 3/5 *cun* to be correct, therefore changed the given "difference" to 2 24/25 *cun*, Guo Shuchun seems to have preferred accepting the original data as given, and then changing instead the answer from 7 3/5 to 6 2/5 *cun*.

giving an explanation that follows from some general method that could be used to solve all such problems. Nevertheless, if the numbers Guo Shuchun introduces are computed as indicated, the result is indeed (5/2)(64/25) = 32/5 = 6 2/5 *cun*.

Although there is nothing inherent in this problem to explain why the method should call for multiplication by 7 and division by 14, Guo Shuchun's reading does leave the end of the problem intact as stated, albeit the reading of *yi si* as 1-4, which should properly be *shi si* 14, must be explained away either as a copyist's error, or perhaps as the copyist's shorthand for what one would have read off from the counting board if this problem were actually being calculated, i.e. "one in the ten's place, and four," hence "one-four," i.e. 14, rather than "ten-four," i.e. 10+4, or 14. This still leaves unexplained why the method should call for 7/14, and why the problem should concern the *difference* between the diagonal and edge of the square, and not the diameter of the circle and the edge of its inscribed square, a possibility we shall consider shortly.

With Guo Shuchun's interpretation of Problem 61 in mind, consider now his approach to Problem 62. As expected, he reads the given length of the square as 6 2/5 *cun*, although he notes that Guo Shirong has changed the text to read 7 2/5 *cun*, and also changes the following part of the text to read "2 24/25 *cun*" rather than leaving it as "2 14/25 *cun*," a reading that Guo Shuchun retains. He does follow Guo Shirong's emendation of the method to "multiplying by 2" rather than by "4," and then offers the following comment: "This problem is, knowing the length of the edge of a square, to find the part in excess between its diagonal and the diameter of its inscribed circle" [Guo Shuchun 2001, p. 215]. He then goes on to quote from the Tongxun solution to this problem, but again with no indication as to why he disagrees with the group's interpretation. Given the above, Guo Shuchun's reading of Problem 62 may be rendered as follows:

> [Guo Shuchun 62] From a square cut a circle: in order to turn a square into a circle, say the wooden square is **6 2**/5 *cun*, how large is the circular piece of wood? (The answer) says: 4 *wei* 2 14/25 *cun*. The method says: one side of the square (piece of) wood is the diameter of the circular (piece of) wood; (obtain) the result by multiplying by **2** as the dividend, and divide by 5.

Again, Guo Shuchun's interpretation of Problem 62 has the advantage that it does not involve a change in the answer given in the text, although it does require a change in the given length of the edge of the square, and adopts the same change in the method that Guo Shirong recommends. (Even so, the other changes Guo Shirong offers are purely cosmetic and do not really affect the substance of the problem or its solution.) But in a sense there is only a slight trade-off between the solutions offered by Guo Shirong and Guo Shuchun. Assuming with Guo Shuchun that the edge of the square is 6 2/5 *cun*, then to find the difference between the diagonal which is 7/5 of the edge and the edge itself, is simply a matter of computing 2/5 of the edge, i.e. (2/5)(6 2/5) = (64/25) = 2 14/25 *cun*. Again, where the original method given clearly instructs "multiply by 4, divide by 5," all that need be done is to change the "4" to a "2" and we have the necessary reading to yield a correct solution. As before, however, this all has a feeling of artificiality about it, clever though this solution may be. But there are too many changes in the data of the original text, even in Guo Shuchun's slightly different reading of the numbers, to be convincing that the problem really is about differences between diagonals and edges, and not more essentially about circles and squares as the titles of both Problems 61 and 62 seem to suggest. But before considering this further, Peng Hao offers yet another interpretation of these two problems.

Peng Hao (2001)

In the same year that Guo Shirong and Guo Shuchun presented their collations with commentaries of the *Suan Shu Shu*, Peng Hao, a member of the team responsible for the original study, arrangement and first publication of the bamboo text in 2000, published his own collation with commentary. Some texts were also revised, considerably more punctuation was added, and in Peng Hao's collation there are three color plates of 9 of the bamboo strips, with black-and-white reproductions of 63 more, but not including the strips 153, 154 and 155 on which Problems 61 and 62 are written.

Peng Hao's reading of the two problems does not follow the approach taken by either Guo Shirong or Guo Shuchun, and resembles more the analysis offered by the Tongxun group, although with some major differences. Peng Hao begins with an explanation of his reading of the first problem as follows: "The original problem is a calculation involving a circular (piece of) wood transformed into a square (piece of) wood. This could be, already knowing the length of the circumference of the circle, to find the length of the side of the square inscribed in the circle" [Peng Hao 2001, p. 111, note 1 to "Yi Yuan Cai Fang"]. To help visualize what this problem involves, Peng Hao offers a diagram no different from that given in [Figure 61a].

As for the answer, Peng Hao notes that it is mistaken, and says that it should be "10 14/105 *cun*," a number we have not as yet seen in connection with this problem [Peng Hao 2001, p. 111, note 2 to "Yi Yuan Cai Fang"]. He then goes on to elaborate several additional changes he has to make in the original text:

> At the end of the sentence, "wei fa" is missing. The "4" at the end of the sentence should be made into a "3." Because the circumference of the circle is already known, to find the diameter should follow using the ratio for the circumference of the circle, at that time approximated by taking the value of 3. The "5" and "7" in the method are the relation[6] for the side and hypotenuse of an (isosceles) right triangle. Both *Zhang Qiujian's Mathematical Manual* and *Master Sun's Mathematical Manual* include "knowing the diagonal, to find the side, multiply by 5 and divide by 7,' the same as in this problem (in the *Suan Shu Shu*).[7] Nevertheless in comparison with the *gou-gu* theorem, this is (only) a rough approximation.[8] But as a result of this problem (in the *Suan Shu Shu*), it permits us to understand how the *Suan Shu Shu* relates to the level of knowledge of the *gou-gu* relation. On the basis of changes at the end of the (part of the) text on method, the calculation proceeds as follows, following the earlier example (where) 1 *wei* is 1 *chi*, the length of the circumference of the circular (piece of) wood is 42 14/25 *cun*. (The calculation): (42 14/25) × 1/3 × 5/7 = 10 14/105 (*cun*) [Peng Hao 2001, p. 111, note 3 to "Yi Yuan Cai Fang"].

Thus, with all of the above in mind, the original Problem 61, as Peng Hao understands it, should read as follows:

> [Peng Hao 61] From a circle cut a square: in order to turn a circular (piece of) wood into a square (piece of) wood, say its *da si wei* is 2 14/25 *cun*, how large is the square (piece of) wood? (The answer) says: the square is **<u>10 14/105</u>** *cun*. The method says: (obtain) the result by multiplying by 5 as the dividend, [then] divide by 7, (with) <u>3</u> as the divisor.

Something Peng Hao does not discuss here, but notes earlier in his edition of the *Suan Shu Shu*, is how the character *wei* should be understood—as a technical term for a measure of the circle in *chi*, thus 4 *wei* translates to 4 *chi* or 40 *cun*, so we now know that what we have here is a measure for the

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6 Here Peng Hao uses the word *bi* (compare/relate/differ), as opposed to the word we might expect to find here, *lü* (ratio), which would express the mathematical relation between the side and diagonal of the square, or as Peng Hao prefers to describe it, the side and hypotenuse of an isosceles right triangle.

7 Both the *Zhang Qiujian Suan Jing* and the *Sun Zi Suan Jing* were later counted among the "ten classics" of ancient Chinese mathematics. The former is usually dated to the 5th century CE, the latter to the 3rd-4th century CE. Why Peng Hao should refer here to right triangles in relation to the *Suan Shu Shu*, however, is problematical; nowhere in this text are there any examples of problems dealing with right triangles, and this seems to have been an innovation that was first given any comprehensive treatment in Chapter 9 of the *Nine Chapters*, the chapter devoted to problems whose solutions depend upon properties of right triangles and the *gou-gu* ("Pythagorean") relation of the sides and hypotenuse of such triangles [Qian 1963, vol. 1, pp. 241-258]. Right triangles also appear in the astronomical treatise, *Zhou Bi Suan Jing* [Qian 1963, vol. 1, pp. 11-90]. For studies in Chinese, see Guo 1990 and 2001. For translations of the *Nine Chapters*, see Berezkina 1957, Chemla 2004, Shen 1999, and Vogel 1968; for the *Zhou Bi*, see Cullen 1996. For studies of both, see Martzloff 1987, 1997; and Li and Du 1987.

8 It was indeed clear to Liu Hui—fully cognizant in 263 CE, when he completed his commentary on the *Nine Chapters*, of how mathematically unacceptable "3" was as a value for π or the ratio of the circumference to the diameter of the circle—that 7/5 was only a very "rough" approximation for $\sqrt{2}$. Given his thorough understanding of the properties of right triangles, Liu Hui's commentary on the relation between squares and circles makes clear he understood the ratio 5:7 for the side:diagonal of the square would have been too small, and that the resulting ratio of the areas of the square and that of its hypotenuse should be 1:2 or 25:50, not 25:49 as would be the case assuming the 5:7 as the ratio between the side and diagonal of the square. But again, given the fact that there is absolutely no evidence of any appreciation of the mathematical properties of right triangles in the *Suan Shu Shu*, this discussion seems anachronistic and does not really help to understand the *Suan Shu Shu* on it own terms.

circle not of 2 14/25 *cun* but of 42 14/25 *cun*.[9] And if Peng Hao is correct in understanding the problem to involve determining the edge of the square inscribed in a circle of circumference 42 14/25 *cun*, then his solution of the problem and conclusion that it yields a square of side 10 14/105 *cun* follow accordingly. It should be noted, however, that this is not the most simplified solution; that would have been given as 10 2/15 *cun*, a more appealing answer. But again, the fact that neither Problem 61 nor 62 mentions circumferences of circles explicitly is troublesome. Moreover, if the problem were to begin with the circumference of the circle in question, shouldn't it have been immediately obvious to the author or copyist of the book that the first thing to be determined would be the diameter of the circle, which would require division by 3 at the outset. As Peng Hao reconstructs the problem, the method begins with the circumference of the circle, multiplies by 5 and divides by 7 (to determine the length of the edge from the diameter of the circle) and only then divides by 3 (to recover the diameter from the given length of the circumference). Why go from the circumference of the circle to only the edge of the inscribed square? The problem would seem much more satisfying if it were about the computation of the *perimeter* of the inscribed square given the circumference of a circle. But then we are back with the inconsistency of the numbers given in the method of solution for this problem. Nevertheless, Peng Hao's solution is as yet the most straightforward, even if it requires significant changes in the data.

What about his solution for the difficulties concerning Problem 62? He begins by explaining the problem as "knowing the length of the edge of a square, find the area of its inscribed circle" [Peng Hao 2001, p. 112, note 1 to "Yi Fang Cai Yuan"]. Unlike the Tongxun group, which interprets this problem as one of finding the circumference of the inscribed circle, Peng Hao opts for the area of the circle. Now, based on the calculation procedure itself, he makes the following changes in data: the "2 *cun*" should be "3 *cun*," the "14" should be "8," and the character *cun* is missing after the *fen*. The answer should be 4 *wei* 3 8/25 *cun* [Peng Hao 2001, page 112, note 2 to "Yi Fang Cai Yuan"]. Peng Hao offers the following diagram to accompany his explanation [Peng Hao 2001, p. 112, note 3 to "Yi Fang Cai Yuan"]:

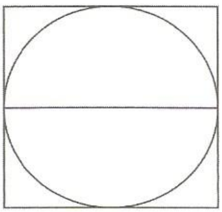

Figure 62c

Next Peng Hao relies upon the *Jiu Zhang Suan Shu* (The Nine Chapters) for its formula for the area of a circle: "multiply the diameter by itself, multiply by 3, divide by 4." Peng Hao suggests that with this in mind, in the *Suan Shu Shu* bamboo text, there are four characters missing: "multiply the diameter by itself." The bamboo text is also wrong in instructing to "multiply by 4," which should be "multiply by 3" according to Peng Hao [2001, page 112, note 4 to "Yi Fang Cai Yuan"].

[9] The term *wei* occurs only in one other problem in the *Suan Shu Shu*, Problem 36, *Qu Xi Cheng* (Norms for Getting Hemp), in which a "3 *wei*" bundle of hemp is described. Peng Hao there relies upon a commentary by the Jin Dynasty scholar, Li Yi, on the ancient classic text, the *Zhuang Zi*, to argue that *wei* should be taken to mean "diameter in *chi*," that is, "the diameter in *chi* is equivalent to *wei*"). This is thoroughly in keeping with the statement and data given in the rest of the problem, which makes it clear that as a technical term, *wei* is related to the diameters in question, not the circumferences. For details, see [Peng Hao 2001, p. 82, note 2]. We leave aside for now the fact that Peng Hao here conflates *wei* with the circumference of the circle in his interpretation of Problem 61. It should be noted, however, that in his consideration of Problem 36, Guo Shuchun explains that *wei* is a unit of measure for the circumference of tree trunks, with a reference to the appearance of the character in the *Zihui bu* (*wei bu*), but without any indication of a technical or mathematical meaning for the term *wei*. See [Guo Shuchun 2001, p. 211, note 2 for "Qu Xi Cheng"]. Guo Shirong also takes *wei* to be a measure of length in relation to the circle in his reading of Problem 36, specifically 1 *wei* = 10 *cun*, i.e. 3 *wei* = 30 *cun*, but again in terms of diameters rather than circumferences; there is no mention in his discussion that *wei* should be understood as the difference between diagonals/diameters and the edge of a square [see Guo Shirong 2001, p. 284, note to (his numbering) Problem 25: *Qu Xi Cheng* (Norms for Getting Hemp)]. But in discussing Problem 36, Guo Shirong makes no mention of any further distinctions either between *si wei* or *da si wei*.

Peng Hao's final comment and summary of the problem is as follows:

> According to the logic of the calculation, the "5" is wrong for "4. " Throughout the *Suan Shu Shu* the ratio for the circumference of the circle (π) is always 3, and from this it can be deduced that the ratio of the areas of the square to its inscribed circle is 4:3. In this work and in other writings we have not seen anything else. Thus we conclude that the copyist of the bamboo text has made some errors. At the end of the text *wei fa* (as divisor) is missing. Thus the sentence should be: "multiply the diameter by itself, multiply the result by 3 and use it as the dividend, let 4 be the divisor and then divide. " The text for the method of this problem is, from the ratio of the areas of a square and its inscribed circle, find the area of the circle [Peng Hao 2001, page 112, note 5 to "Yi Fang Cai Yuan"].

Peng Hao then refers to [Figure 62c] to explain how the ratio of the perimeter P of the square to the circumference C of the circle follows as P:C=4:3. From the data stated in Problem 62, that the given length of the side of the square is 7 3/5 *cun*, Peng Hao calculates that the area of the inscribed circle must then be $(3/4)(38/5)^2 = 4332/1000 = 43\ 8/25$ *cun*^{2}. He further notes that given the terminology of Problem 36 and the use of *wei* there, this answer could also be given as 4 *wei* 3 8/25 *cun*^{2} (note a minor error in the printed text, where in writing out the fraction in Chinese, it mistakenly writes this as 4/25 instead of 8/25 [Peng Hao 2001, p. 113, note 5 to "Yi Yuan Cai Fang"]). He also adds that since the problem uses a value of 3 for π, the calculation gives only an approximately correct value.

Peng Hao concludes his commentary on Problem 62 with the following general observation: "From this problem it is possible to see that at that time (when the *Suan Shu Shu* was written), there was already a preliminary understanding of the relation between the square and its inscribed circle, and was a good basis from which to advance to a more precise determination of the ratio of the (diameter) and circumference of the circle" [Peng Hao 2001, p. 113, note 5 to "Yi Yuan Cai Fang"].

With all of the above in mind, we can now rewrite Peng Hao's interpretation of Problem 62 as follows:

> [Peng Hao 62] From a square cut a circle: in order to turn a square into a circle, say: the woodwn square is 7 3/5 *cun*, how large is the circular (piece of) wood? (The answer) says: 4 *wei* **3 8**/25 *cun*. The method says: one side of the square (piece of) wood is the diameter of the circular (piece of) wood, **multiply the diameter by itself**, (obtain) the result by multiplying by **3** as the dividend, and divide by **4 as divisor**.

Thus far, Peng Hao is the only author to suggest that Problem 62 is concerned with a square and the area of its inscribed circle. That this is not unreasonable is suggested by the fact that from the given edge of the square in Problem 62, 7 3/5 *cun*, we know the answer as given in the problem is rendered in terms of a fraction in 25$^{\text{th}}$s, and this could certainly arise naturally from squaring the length of the edge of the square (which as the problem notes is also the diameter of the inscribed circle), to give the area of the circle. Among the rules given in the *Nine Chapters* for determining the area of a circle is to square the diameter, and then take 3/4 of the result, and indeed, $(3/4)(7\ 3/5)^2 = (3/4)(38/5)^2 = 5332/100 = 43\ 8/25$ *cun*^{2}. But the 25 of the denominator in the answer could as easily have come from applying the ratio of the length of the diagonal to the side of the square as 7:5, which might also be interpreted as yielding the diameter of a circle, but in this case it would have to be the circle *circumscribed* around the square in question. But we shall consider this possibility shortly. For now, given nothing in the language of Problem 62 to suggest that it is concerned with the area of the inscribed circle, and considering the number of changes in the given data that Peng Hao must make in order for his interpretation to provide consistent results between his interpretation of the problem and the working out of its solution, we go on to consider yet another interpretation of this twin set of problems, one that follows almost exactly the analysis just provided by Peng Hao.

Zhangjiashan 2001 (Study Group for the Bamboo Text of the *Suan Shu Shu*)

It was also in 2001 that a definitive edition was published of all six of the bamboo texts discovered in Han tomb 247 at Zhangjiashan in December 1983–January 1984, including the *Suan Shu*

Shu. Not only does this edition reproduce photographs of all the bamboo slips, but it transcribes the text (as does Peng Hao's version) indicating where every break between bamboo slips occurs. The reading this collation of the *Suan Shu Shu* gives for Problem 61 is as follows [ZJS 2001, p. 268]:

> [Zhangjiashan 61] From a circle cut a square: in order to turn a circular (piece of) wood into a square (piece of) wood, say its *da si wei* is 2 14/25 *cun*, how large is the square (piece of) wood? (The answer) says: the square is **10 14/105** *cun*. The method says: (obtain) the result by multiplying by 5 as the dividend, [then] **divide by 7, (with) 4 as the divisor**.

Although the Zhangjiashan collation of the text offers no indication of how in general the problem should be interpreted, since the corrections in the data given follow Peng Hao's collation of the text so closely, we may assume that the understanding here is similar, that this is taken to be a problem about the circumference of the circle, which is given as 42 14/25 *cun*, from which the length of the side of the inscribed square is to be found. The corrections made to the original text are virtually the same as Peng Hao's version of this problem, except at the very end where it does not change the "4" to "3," but seems to instruct "let 14/7 be the divisor." Since this makes no sense, we might supply a missing "," suggesting that what is meant is that after multiplying the circumference of the circle by 5 and dividing by 7, to *then* take 4 as the divisor, from which the answer would follow as (14 14/25)(5/7)(1/4), but this gives the wrong answer, 7 3/5 *cun*, according to the revised answer given above, but remarkably, this is the correct answer according to the original problem! We shall consider the significance of this shortly, but for now, consider the version of Problem 61 in the Zhangjiashan version on its own terms. What has clearly happened, since the "correct" answer of 10 14/105 *cun* follows from division by "3" instead of "4" is that the Zhanghiashan collation has forgotten to make the additional change that Peng Hao makes in his reading of the text, and that indeed, the above should read "divide by 7, (with) 3 as the divisor." The objections to this interpretation of the problem, however, are the same as those already made concerning Peng Hao's virtually identical understanding of Problem 61.

As for Problem 62, the Zhangjiashan collation is as follows [ZJS 2001, p. 268]:

> [Zhangjiashan 62] From a square cut a circle: in order to turn a square [piece of wood] into a circular [piece of wood], say the square is 7 3/5 *cun*, how large is the circular (piece of) wood? (The answer) says: 4 *wei* **3 8**/25 *cun*. The method says: one side of the square (piece of) wood is the diameter of the circular (piece of) wood; (obtain) the result by multiplying by **3** as the dividend, and divide by **4** as divisor.

The major difference between this version of the text and Peng Hao's is that the four characters Peng Hao added to the text in the third sentence, "multiply the diameter by itself," are not included here, and so the Zhangjiashan collation offers no hint as to how the area of the circle should be computed. In fact, no interpretation of the text is offered, so it is not immediately clear that this reading in fact assumes that the problem is about computing the area of the inscribed circle from the area of the given square. But the method as given in [ZJS 2001] says to take 3/4 of the data given for the square, and we know from the above discussion that this is almost the formula given in the *Nine Chapters* for computing the area of the inscribed circle from the area of its circumscribed square—but this requires first that the side of the square, i.e. the diameter of the circle be squared, of which 3/4 gives the desired result, the area of the circle.

In fact, the procedure described in the Zhangjiashan collation cannot be right, because it calls for multiplying the 7 3/5 *cun* given for the square by 3, and then dividing by 4, which not only gives the wrong answer (which according to the revised collation above should be 43 8/25 cun^2), but in fact computing as instructed, (7 3/5)(3/4) = 5 7/10 *cun,* the dimension of the answer in *cun* is also wrong for an area, so this too indicates that something is in error. The correct answer according to the emended text does follow in fact from $(7\ 3/5)^2(3/4) = 43\ 8/25\ cun^2$. Again, the Zhangjiashan collation has not followed Peng Hao's interpretation of the text closely enough, and should have included the instruction to "multiply the diameter (i.e. the side of the square) by itself." This would indeed yield the area of the square, 3 /4 of which would then give the area of the inscribed circle.

he same objections apply to this reading of the text as do above to Peng Hao's interpretation of this problem, for there is no evidence within this text itself that makes clear this probTlem is about the relation between the *areas* of a given square and its inscribed circle. And there is too much tampering here with the data of the text to give sufficient confidence that this is a correct reading of the actual intention of the original problem. There is, however, one final set of interpretations of these two problems that we need to consider before venturing to offer some final conclusions of our own.

Duan Yaoyong and Zou Dahai 2003

In 2003 Duan Yaoyong and Zou Dahai published a careful study of all the collations of the two Problems 61 and 62 discussed above, and then offered their own, quite different reading of the two problems, with the advantage as they acknowledged of what they had learned from all of the various solutions they had examined. Duan and Zou begin their analysis of the details of the various approaches that have been taken to Problems 61 and 62 by quoting the assessment of the situation offered by Horng Wann-Sheng and Lin Cang-Yi in their survey of the *Suan Shu Shu* published in 2002. Given the various scholars who had approached the two problems and their lack of agreement, Horng and Lin conclude that: "different collation tactics and different features emerge, but as to which is right and which is wrong, which is good and which is bad, at present there is as yet no final conclusion" [Horng and Lin 2002, quoted from Duan and Zou 2003, p. 171]. Pointing out that Horng and Lin offer no concrete analysis of their own as to what may be reasonable or unreasonable in the various approaches taken as yet to these problems, nor do they offer any new ideas about how to collate the two texts, Duan and Zou venture to offer their own concrete analysis of each of the collations introduced thus far.

Beginning with the Tongxun group and Peng Hao, they note that each understands the term "wei" in both problems as a unit of length. And both understand "Yi Yuan Cai Fang" as knowing the circumference of the circle, to find the length of the side of its inscribed square. But while the Tongxun group takes the ratio 7:5 as mistaken for the ratio of the circumference of the circle to the perimeter of its inscribed square, Peng Hao assumes this is the ratio of the diagonal to the side of a square, which Duan and Zou take to be the more reasonable assumption.

As for Problem 62, Duan and Zou only say that the Tongxun group understands it as a problem involving the length of the side of a square, to find the circumference of its inscribed circle. They note that the ancient mathematicians mistakenly thought the two problems 61 and 62 were converse problems, and therefore, according to the Tongxun group, they gave the methods and data for both problems as inverses of each other, which was wrong. But Duan and Zou do not say whether they agree or disagree with this interpretation. In part 1.1 of their article, Duan and Zou point out that the Tongxun group believed the end of the problem was incomplete, and that there was something missing from the end of the text, which is why the Tongxun group offered no collation for this problem, and perhaps why Duan and Zou offer no further critique of the Tongxun group's presentation of Problem 62.

Instead, they go on to criticize Peng Hao's reading of Problem 62. They rightly point out that he assumes the problem is based on knowing the length of the side of a square, from which one is asked to calculate the area of its inscribed circle. As Duan and Zou note, this requires him to change both the answer and the text describing the method to follow in solving the problem. "But the characters in the original texts of the answers and methods of solution of the two problems are exactly the inverse of each other, and the surviving parts of the texts of the method also have dividends and divisors that are inverses, but according to Peng Hao's collation, it seems that the ancients did not regard these two problems as related, but unfortunately there is clearly a discrepancy between this and the ancients' original understanding of these problems" [Duan and Zou 2003, p. 171].

Almost in passing (in part 1.2 of their article) Duan and Zou point out that the [Zhangjiashan 2001] collation of the text basically adopts Peng Hao's reading of the two problems, but fails to change the final "4" in Problem 61 to a "3," which leads them to admit "we don't know if this is a printing error or not" [Dian and Zou 2003, p. 170]. Indeed, offering virtually no commentary at all for either of these problems, the Zhangjiashan group does not make clear what it may take these problems to mean; nor do Duan and Zhou say anything about the even more serious problem in the collation of Problem 62 given in [Zhangjiashan 2001, p. 268]. The omission there of the step required to calculate the area of the inscribed circle cannot have been a printer's error, but at best, an omission of the editorial group in working from Peng Hao's text, which offers at least a completely satisfactory reading of the problem, even if he is required to make "very many changes in the characters in the text," [Duan and Zou 2003, p. 170].

The last part of their detailed critique of their predecessors is devoted to the interpretations of Problems 61 and 62 by the "two Guos's," Guo Shirong and Guo Shuchun, both of whom assume that the problems in question are concerned with the differences in length between the diameter of a circle and the lengths of the side of its inscribed or circumscribed square [Duan and Zou 2003, p. 171]. The only real difference is in their treatment of the data; whereas Guo Shuchun only changes the data for the answer in Problem 61, and based upon the answer of Problem 62, only changes the data for working out the problem, Guo Shirong by contrast changes the given data as well as the answers and the details of the solution procedures for both problems. As Duan and Zou conclude:

> Moreover, Guo Shuchun understands *ling qi er yi si* as meaning "divide 14 by 7," whereas Guo Shirong changes this to *ling qi cheng, er shi si cheng yi* ("multiply by 7, then divide by 14"). And although both Guo's say that in Problem 61 *da si wei* ("size four *wei*") should be understood as referring to the difference between the diameter of the circle and the length of the side of its inscribed square, which is quite strange, in Problem 62 *si wei* is taken to mean the difference between the diagonal of the square and the diameter of its inscribed circle, which is difficult to comprehend [Duan and Zou 2003, p. 171].

Indeed, earlier they had already said in part 1.3 of their paper concerning Guo Shuchun's analysis, that: "[t]his way of understanding [the problems] is somewhat peculiar, and the collator does not offer any precedents for this point of view," [Duan and Zou 2003, p. 170].

Having thus considered all the variations of their predecessors, Duan and Zou turn to their own interpretations of Problems 61 and 62. Given the inverse nature of the answers and the computational procedures, including what survives of the texts of the methodological procedures of the two problems, they assume that the ancients regarded these two problems as inversely related. Although they maintain we now know this is not the case, they nevertheless say that simply on the basis of the computations, the data of the two problems should not be changed. They point out that the Tongxun group's interpretation did not require any changes in the data of the answer or the set-up of the problem, and thus their interpretation of the problem is very appealing. Consequently, Duan and Zou accept the Tongxun 2000 collation of Problem 61, and their interpolation of two characters, *er yi* (divide by), at the end of the problem, and conclude that this means there must be a missing slip to account for the incompleteness of Problem 61, although it is possible to restore what must have been on the missing part of this text, namely the additional "divide by 4." This also accords nicely, as they point out, with the inverse instruction in Problem 62 to "multiply by 4," [Duan and Zou 2003, 171].

On similar grounds, assuming the reciprocal nature of the two problems, they take the instruction to "divide by 7" in Problem 61 to mean that in Problem 62, "multiply by 7" needs to be included as part of the method [Duan and Zou 2003, p. 172]:

> [Duan and Zou 61] From a circle cut a square: in order to turn a circular (piece of) wood into a square (piece of) wood, say its size is 4 *wei* 2 14/25 *cun*, how large is the square (piece of) wood? (The answer) says: the square is 7 3/5 *cun*. The method says: (obtain) the result by multiplying by 5 as the dividend, and divide by 7, [then] **<u>divide by</u>** 4.
>
> [Duan and Zou 62] From a square cut a circle: in order to turn a square into a circular (piece of) wood, say the square is 7 3/5 *cun*, how large is the circular (piece of) wood? (The answer) says: 4 *wei* 2 14/25 *cun*. The method says: one side of the square (piece of) wood is the diameter of the circular (piece of) wood; (obtain) the result by multiplying by 4, **<u>and then by 7</u>**, as the dividend, and divide by 5.

Based upon the pure economy of corrections, this is a laudable collation of the two texts, but its major flaw is the one Duan and Zou have already noted, namely if the ancients were wrong about these being reciprocal texts, why collate them in the reciprocal format they give above? And what exactly is their final interpretation of these two problems? In the case of Problem 61, if they follow the interpretation of the Tongxun group, do they regard the given length as the circumference of the circle, and the ratio 5/7 as a mistake for the ratio between the circumference and perimeter of the inscribed square, which should have been 21/20? If so, then despite the appearance, these cannot have been reciprocal problems intentionally, but only apparently so, accidentally, by virtue of this mistake. And if the correct ratio were used, then the data in the method and the answer must be completely wrong and require some sort of correction.

On the other hand, earlier in their discussion of the approach Peng Hao takes to this problem, Duan and Zou praise his interpretation of the ratio 5:7 as that of the side:diagonal of the square, and seem to reject the Tongxun group's interpretation of this as a mistaken ratio of the circumference of a circle to the perimeter of its inscribed square. But if the given 42 14/25 *cun* is the circumference of the circle, from which the problem asks the length of the side of its inscribed square, what then is the significance of the method's instruction to multiply by 5, divide by 7, and then divide again by 4? If the 42 14/25 *cun* were taken to be the length of the diameter of the circle, rather than its circumference, then the 5/7 would indeed give the length of the edge of the inscribed square, but then we are still left with the problem of how to interpret the final instruction (in the corrected method) to "divide by 4." Clearly, there is still unfinished business here with respect to the correct interpretation of Problem 61.

What about the collation Duan and Zou offer for Problem 62? Again, for the economy of the changes they make, the award for best solution to date should be their's. But again, what is the overall consequence of their collation of this problem? The only model among the above we have already encountered that they might be prepared to accept is again the Tongxun reading of Problem 62, given the edge of a square, find the circumference of its inscribed circle. Clearly Dian and Zou do not accept either of the Guo's reading of the text, nor Peng Hao's search for the area of the inscribed circle. But if Problem 62 is about finding the circumference of the inscribed circle given the edge of its circumscribed square, do they again regard the application of the ratio 7/5 in the method to be an error for the ratio of the perimeter of the square to the circumference of the inscribed circle (which in this case should now be 4:3)? If so, then the same objections raised earlier concerning the Tongxun group's approach to Problem 62, so far as it goes, all apply here as well. If, on the other hand, they read the problem as given the length of the edge of the square as 7 3/5 *cun*, to find the circumference of the inscribed circle based upon the ratio of the side:diagonal of the square as 5:7, then the correct calculation of this problem is even simpler, and does not really involve the 5:7 ratio at all, since the edge of the circle is the diameter of the inscribed circle, and this multiplied by 3 gives the length of its circumference. On the other hand, if the instruction to "multiply by 7, divide by 5" is applied to the side of the square, this would give the diagonal of the square on the basis of the 7:5 ratio, but this would then be the diameter of the *circumscribed* circle, not the inscribed circle. And there is still a problem with the remaining instruction (sandwiched between the "7" and "1/5"), namely to multiply by 4, the physical significance of which in this case is not at all clear.

Again, we seem to have reached an impasse at the correct interpretation of these twin problems, if they are indeed twins. Is there any way out of this seeming labyrinth? Despite the admirable attempts described above to wring consistency out of the data and methods for these two problems, no one as yet has found a solution that meets with everyone's satisfaction. Is there yet another alternative we might consider?

Two additional approaches to Problems 61 and 62 of the *Suan Shu Shu*

Like Duan Yaoyong and Zou Dahai. I must likewise acknowledge all of the help I have received from studying the interpretations of my colleagues discussed here, the entire Tongxun group for their initial comments on the *Suan Shu Shu*, and the subsequent very insightful and illuminating commentaries by Guo Shuchun, Guo Shirong, Peng Hao, as well as the discussion of these to problems by Duan and Zou. I must also acknowledge the very useful results of a *Suan Shu Shu* discussion group that met over the past two years at the Graduate Center of the City University of New York, which coincided with Andrea Breard's appointment there as a Visiting Research Scholar, and which included Xu Yibao, more recently, Sun Litian, and briefly this summer, Kim Taylor. Our line-by-line reading of the *Suan Shu Shu*, along with all various collations at our disposal, has helped immeasurably to clarify and sharpen my own thinking regarding the *Suan Shu Shu* in general, and the two problems before us now, Problems 61 and 62, in particular.

To begin with, the numbers alone are very seductive, so symmetrical as to leave almost no doubt that at least in terms of the mathematics, these two problems must somehow be inversely related. So the first hypothesis to consider for now is that they were indeed meant to be companion problems, the one the inverse of the other. But if so, what could the inverse nature of the problems be, since given all of the above disagreement and sometimes radical changes to the numbers given, the answer to this question is by no means immediately obvious. Fortunately, there are several clues within the data given for the two problems that taken together suggest the following two schemes for working out the inverse relation between the two problems. Examining for now just the numbers alone, consider the following:

[Problem 61]: $42\ 14/25 \times 5 \times 1/7 \times 1/4 = 7\ 3/5$;

[Problem 62]: 7 3/5 × 4 × 1/5 × 7 = 42 14/25.

Aside from the given 42 14/25 and 7 3/5, the rest of the numbers that must be considered are 5, 7, 4 and 1/4. Assuming along with most of the above commentators that the 5 and 7 concern the ratio of the side:diagonal of the square, then we are left to deal with the 4 and 1/4. These clearly suggest that we are dealing with numbers involved with the perimeters of squares. In the case of Problem 61, thus far every collation and commentary on the text has assumed that this problem assumes as given the circumference of a circle, but on the numbers alone, we must consider the possibility that what is given is the perimeter of a square, 1/4 of which will give the length of its side. On this interpretation solely of the numbers of the problem, the remaining 5/7 would then give the side of a square, not a diagonal. If, on the other hand, the problem were about a given square and its inscribed circle, we are again in trouble because the side of the square is also the diameter of the inscribed circle, from which the circumference should follow as 3 times the edge of the circumscribed square. On the other hand, if we consider the circumscribed circle, its diameter should be 7/5 of the edge, not 5/7, so this too offers no solution to the numbers of the problem as given.

We might abandon this approach at this point altogether, and consider whether or not the given 42 14/25 might represent the dimensions of a circle according to the prevailing interpretation of Problem 61. But to get from the circumference to the diameter of the circle from which to reference the rest of the calculations of the problem, we would need a division by 3 somewhere in the method. On the other hand, if we assume the 42 14/25 is not the circumference but the diameter of the circle, then 5/7 of this would indeed give the length of the side of the inscribed square. But now we are again faced with the "hanging 4" at the end of the statement of the method. If we emend the sentence as many of our colleagues cited above have done, to read "divide by 4," what would be the physical significance of this division in the context of Problem 61? Why take 1/4 of the length of the edge of the inscribed square? We could opt for the interpolation of several characters at this point to instruct *multiplication* by 4, instead of division, which would indeed give the perimeter of the inscribed square, but this would require a change in the method of the problem and a corresponding change in the answer to 121 3/8, i.e. 42 14/25 × 5 × 1/7 × 4 = 121 3/8. And there is something very unsatisfying mathematically about a problem that begins with the diameter of a circle as the given, from which one is asked to find the perimeter of its inscribed square. More satisfying would be given the circumference of the circle, to find the perimeter of its inscribed square, or from the diameter of the circle, to find the edge or perhaps better still, the diagonal, of its inscribed square.

However, before abandoning our original hypothesis too quickly, let us return to the possibility that Problem 61 is indeed about a given square, from which we have computed 1/4 of its perimeter as the length of its side, and we are now to take 5/7 of this length, which would correspond to determining the side of a square with diagonal equal to the side of the square. There is a clue as to how we should proceed at this point, in Problem 62, which reminds us that the side of the square is the same as the diameter of its inscribed circle. If we now take the side of the square as the diagonal of the inscribed circle, this also becomes the diagonal of the square inscribed in that circle, and 5/7 of that will give us the side of the square inscribed in the circle inscribed in the square whose perimeter was given at the outset of the problem. The diagram for this solution is as follows:

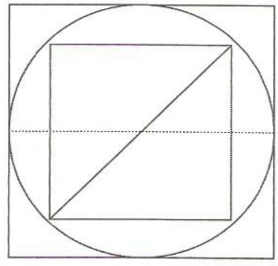

Figure 61d

If this approach "saves the phenomena" for Problem 61, what can we make with this in mind for Problem 62? Here we begin with the given side of the inscribed square, 7 3/5; 7/5 of this will give the diagonal of the corresponding square, which in turn is also the diameter of the circumscribed circle. Multiplying this by 4 gives the perimeter of the square circumscribed around the circle, i.e in all:

[Problem 62]: $7\ 3/5 \times 4 \times 1/5 \times 7 = 42\ 14/25$.

There are, however, several problems to this reconstruction of the reasoning of Problems 61 and 62, assuming they are indeed inverses, namely that the order of the operations in the methods is not exactly what we might expect from the actual workings of the problems, i.e. in Problem 61 given the perimeter, the method should begin with the division by 4 to give the side of the square before determining the length of the square inscribed in the inscribed circle. And in Problem 62, the method should conversely begin by computing from the edge of the inscribed square the diagonal of its circumscribed circle, from which the edge of the circumscribed square follows immediately, after which the multiplication by four should be indicated. But there may be a solution to this difficulty as well, and it is inspired in part by a brief section early in the *Nine Chapters* were the relations between circles and squares are considered.

The passage is not in the *Nine Chapters per se*, but in the commentary on this work by Liu Hui. It comes in a lengthy note that appears in the first chapter devoted to field measurement, namely the first of several rules given for calculating the areas of circular fields, a rule which instructs: "half the circumference and half the diameter multiplied together give the area (of the circle in square) *bu*" [Qian Baocong 1963, vol. 1, p. 103]. Amid his commentary on how the value of π may be determined much more accurately than 3 by inscribing regular polygons of increasingly many sides in the circle and calculating their perimeters, for which Liu Hui provides an algorithmic procedure, he makes an aside in passing about the general relations of squares to circles:

> The ratio of the square to the circle is clearly important for the things near to us, but also for things far away. Thus it may be said that its uses are diverse [Qian Baocong 1963, vol. 1, p. 104].

This is the only place in the *Nine Chapters* where the ratio of the circle inscribed in a square is discussed. Although Liu Hui, in his commentary, does not consider the ratios of their perimeters, he does explain with reference to a now-lost diagram for *hu tian* (bow-shaped fields, i.e. segments of circles, as in Figure 62d below) that the ratio of the area of a circle to the areas of its circumscribed and inscribed squares is 200:157:100 [Qian Baocong 1963, vol. 1, p. 105]. But if the ratios of squares and circles to their inscribed and circumscribed counterparts were indeed considered trivial in Liu Hui's day (which may explain why no problems like 61 and 62 are to be found in the *Nine Chapters*), some 500 years earlier, when none of the diverse consequences that follow from appreciation of the properties of right triangles had yet been studied, to judge by their absence from the *Suan Shu Shu*, there is no reason not to think that the mathematical relations between squares and circles might have been of great interest, and their full exploration indeed a topic of eager discussion among pre-Qin mathematicians. If so, it is not unreasonable to suppose that they would have explored such possibilities as the relations of the perimeters of the squares inscribed and circumscribed around the same circle, a result that might indeed well have impressed them since this ratio is also 5:7, a consequence that follows directly from the fact that the diagonal of the inscribed square is in fact the side of the circumscribed square.

This in turn would explain the order in which the methods of the two problems are given. Consider Problem 61. Given the perimeter of the circumscribed square, we know that the perimeter of the inscribed square will be 5/7 of that; if we then want to compute the length of the side of the inscribed square, it is simply a matter of taking 1/4 of that. The same reasoning applies to Problem 62. Given the length of the side of the inscribed square, we know its perimeter will be 4 times that; since the perimeter of the circumscribed square is 7/5 of the perimeter of the inscribed square, this computation gives the required solution to the problem.

If this interpretation has the benefit of accepting all of the numbers in the given texts of Problems 61 and 62 as they are given, without requiring any changes in the data and only minor additions to repair missing parts of the texts, we are still left with having to reconcile this reconstruction with the actual problems as given on the bamboo slips, and here we may be less successful in remaining true to the actual texts as we currently understand them. A closer look at the text of Problem 61 suggests the following re-translation from what was offered at the beginning of this discussion on p. 2:

[JWD 61] **Using a circle, draw/inscribe**[10] **squares:** Beginning with a circle, inscribe (draw)square figures (i.e., inscribe and circumscribe squares); say the length of the larger (the perimeter of the circumscribed square) is 42 14/25 *cun*, how much is the (length of the edge of the inscribed) square? (The answer) says: the square is 7 3/5 *cun*. The method says: multiply by 5 as the dividend; divide by 7, (**divide by**) 4.

Even allowing for the liberties taken above in translating Problem 61 to fit the possibility that the problem is about squares inscribed and circumscribed around a circle, there seems to be no way of dealing with the language of Problem 62 in an inverse fashion, where there is explicit reference in the text to answering the question: "how much is the inscribed circle?" How might this be reconciled with the given data if the two Problems 61 and 62 are indeed taken to be inverse problems? At this point we could simply say that the author or copyist got this part of the text wrong, and that it should have read: "Given the edge of a square inscribed in a circle, what is the perimeter of the square circumscribed about the circle?" But this is not what the text says. The only alternative is to accept the text as given, and admit that this must be a different problem from Problem 61. But if so, what sort of problem was it intended to be?

Given our interpretation of Problem 61, if Problem 62 follows a similar pattern, perhaps it was meant to consider the relation between the circumferences of circles inscribed and circumscribed around a given square [see Figure 62d]. Given that the diameter of the inscribed circle and the edge of the square are the same, 3 times the edge of the square will give the circumference of the inscribed circle; 7/5 of this would then give the circumference of the circumscribed circle, requiring an unfortunate but unavoidable change in the answer to this version of the problem, which works out as follows:

Problem 62: $7\ 3/5 \times 3 \times 7/5 = 31\ 23/25$.

This would require a revision of the text more or less along the following lines:

[JWD 62] **Using a square, draw/inscribe circles:** Beginning with a square, inscribe circular figures (i.e., inscribe and circumscribe circles); say (the diameter of the inscribed circle) is 7 3/5 *cun*, how much is the (circumference of the circumscribed) circular figure? (The answer) says: the (circle) is **31 23**/25 *cun*. The method says: (the length of) the side of the inscribed square figure is the same as the diameter of the inscribed circular figure; (**multiply by 3, then**) multiply by **7** as the dividend; divide by 5.

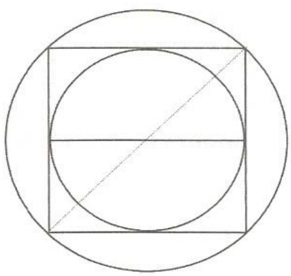

Figure 62d

The American transcendentalist philosopher Ralph Waldo Emerson once wrote that "consistency is the hobgoblin of small minds," and so we should perhaps resist the urge to find total consistency between Problems 61 and 62. Still, I remain haunted by the perfect fit of the two inverse sets of numbers, but see no way to make these fit with the actual language of the two problems as presented, especially with Problem 62. But if we are willing to make a small change in the conceptualization of this problem, it still fits as a companion with its predecessor Problem 61, albeit with a number of changes needed in the data of the problem, but no more than some of our colleagues have also felt compelled to make in trying to reach consistent and satisfying interpretations of these two problems.

[10] Here *cai* is not translated as a noun or adjective meaning wood or wooden, but as a verb meaning to draw or inscribe, drawing in the sense of both inscribing and circumscribing, or as an adjective meaning inscribed or drawn; *fang*/*yuan cai* are not translated literally as square/circular inscription, but as inscribed square/circular figure.

Nevertheless, it must also be admitted that the relatively abstract interpretation of "cai" as "inscribe" rather than "cut," referring to a more general sense of inscribing squares and circles, rather than physically cutting circles from squares and squares from circles, may well be more sophisticated than the authors of the *Suan Shu Shu* were capable. But a literal, concrete interpretation of this problem, especially of Problem 62, encounters insurmountable difficulties as described above.

Conclusion

The treatment of the two problems considered here, the seemingly inverse problems of squares and circles, demonstrates that there may be irreconcilable differences between the numbers presented in a given problem and the actual interpretation or correct understanding of the problems in question. Perhaps the difficulties here are the result of a copyist looking for symmetrical consistency in the computations of what seemed to be inverses. But the actual statements of the two given problems show that although related in considering squares and circles, they are not strictly speaking inverse. And therein lies the problem with these two problems of the *Suan Shu Shu*. Thus we can only conclude that there are two distinct possibilities—if we accept the numbers as given in each of these two problems, then the statements of the problems cannot be correct and would need to be conceived rather differently to coincide with the numbers in question; or, if we accept the statements of the problems as correct, then the numbers given must be wrong, especially in the case of Problem 62, which requires considerable emendation. In either case, what we have here is a very clear example of how interested some Chinese were in approaching problems in ways that were not only of practical application, but had significant purely mathematical dimensions as well. And certainly the investigation of these two problems as inversely related would have naturally stimulated the creative instincts of any true mathematician.

BIBLIOGRAPHY

CHEN Junyue and CHEN Yanping, "The *Suan shu shu* and the *Nine Chapters on Mathematical Procedures*," in *Collected Papers of the Archaeological Society of Hubei Province*, **1** (1987), pp. 220-222.

DAUBEN, Joseph W., "算數書 *Suan shu shu* (A Book on Numbers and Computations). English Translation with Commentary," *Archive for History of Exact Sciences*, **60** (2006), in press.

DU Shiran, "Jianglin Zhangjiashang zhujian Suan Shu Shu chutan" (A Preliminary Study on the Bamboo Strips *Suan Shu Shu* Unearthed from Zhangjiashang, Jianglin), *Zi ran kexueshi yanjiu* (*Studies in the History of Natural Sciences*), 7(3)(1988), pp. 201-204.

DUAN Yaoyong and ZOU Dahai, "*Suan Shu Shu* 'Yi Yuan Cai Fang,' 'Yi Fang Cai Yuan' liang wen jiaozheng" (New Collations of the Two Paragraphs "Yi Yuan Cai Fang" and "Yi Fang Cai Yuan" in the Unearthed Mathematical Book *Suanshu Shu*), *Ziran kexueshi yanjiu* (Studies in the History of Natural Science), **22**(2)(2003), pp. 168-172.

GUO Shuchun, *Huijiao Jiu zhang suan shu* (A Critical Textual Study of the *Nine Chapters on Mathematical Procedures*), Shenyang: Liaoning jiaoyu chubanshe, 1990.

_________, "Theoretical Research Tendency of Mathematics in Pre-Qin Times Viewed from *Suan Shushu*," *Proceedings of the HPM 2000 Conference: History of Mathematics Education, Challenges for a New Millennium. A Satellite Meeting of ICME-9*, HORNG Wann-Sheng and LIN Fou-Lai, eds., Taipei, Taiwan: Department of Mathematics, National Taiwan Normal University, 2000, Vol. I, pp. 35-40.

_________, "*Suan shu shu* jiao kan" (Collation of the *Suan shu shu*), *Zhongguo kexu shiliao* (China Historical Materials of Science and Technology), **22**(3)(2001), pp. 202-219.

_________, "*Shilu Suan Shu Shu de lilun gongxian yu bianzua*" (Some Views on Theoretical Aspects of the *Suan Shu Shu* and its Compilation), *Faguo hanxue* (*Sinologie Française*), vol. 6. Beijing: Zhonghua shuju, 2002, pp. 505-537.

_________, "*Suan Shu Shu Chutan*" (A Preliminary Study on the *Suan Shu Shu*), *Guoxue yanjiu* (*Studies on National Heritages*), **11**(2003), 307-349.

GUO Shuchun and LIU Dun, *Jiaodian Suan jing shi shu* (Collation of the *Ten Mathematical Classics*), Shenyang: Liaoning jiaoyu chubanshe, 1998.

GUO Shirong, "*Suan shu shu* kanwu" (Corrections for the *Suan shu shu*), *Neimenggu Shifan Daxue Xuebao* (Journal of Inner Mongolia Normal University), **30** (3) (2001), pp. 276-285.

HORNG Wann-Sheng, "*Suan Shu Shu Chutan*" (On the *Suan Shu Shu*: A Preliminary Study), *Shida Xuebao. Kexue jiaoyu lei* (Journal of National Taiwan Normal University, Science and Education Edition), **45**(2)(2000a), pp. 77-91.

__________, "*Suan Shu Shu de jizhe lunzheng*" (Several Examples of Logical Reasoning in the *Suan Shu Shu*), *Taiwan lishi xuehui tongxun* (Newsletter of the Taiwan Historical Society), **11**(2000b), pp. 44-52.

__________, "*Guangyu Suan Shu Shu lunzheng de yige beizhu*" (A Note on Several Examples of Logical Reasoning in the *Suan Shu Shu*), *HPM tongxun* (*HPM Newsletter*), **5**(10)(2002), pp. 1-8.

HORNG Wann-Sheng and LIN Cang-yi, "*Suan Shu Shu bufeng timing de zai jiaokan*" (Further Collation on The Names of Certain Sections in the *Suan Shu Shu*), *HPM tongxun* (*HPM Newsletter*), **5**(2)(3)(2002), pp. 6-24.

JI Zhigang, "*Zantan yu yihan: du Peng Hao zhu Zhangjiashan hanjian Suan Shu Shu zhushi*" *(Praise and Regrets: Some Thoughts on Peng Hao's Commentaries on the Suan Shu Shu)*, *Ziran kexueshi yanjiu* (*Studies in the History of Natural Sciences*), **23**(1)(2004), pp. 91-96.

LI Di, *Zhongguo shuxue tongshi* (A General History of Chinese Mathematics), vol. 1, Nanjing: Jiangsu jiaoyu chubanshe, 1997.

LI Xueqing, "*Zhongguo shuxue shi shang de zhongda faxian*" (A Very Important Discovery for the History of Chinese Mathematics), *Wenwu Tiandi* (A Magazine for Relics), **1**(1985), p. 47.

LI Yan and DU Shiran, *A Concise History of Chinese Mathematics*, John N. Crossley and Anthony W.-C. Lun, trans., Oxford: Claredon Press: 1987.

Martzloff, Jean-Claude, *Histoire des mathématiques chinoises*, Paris: Masson, 1987.

________, *A History of Chinese Mathematics*, Stephen S. Wilson, trans., New York: Springer-Verlag,

1997.PENG Hao, "Zhongguo zuigu de shuxuezhe zuo *Suan shu shu*" (The Earliest Arithmetic Book in Chinese), *Wenwu* (Cultural Relics), **9**(2000), pp. 85-90.

__________, *Zhangjiashan hanjian 《Suan shu shu》 zhushi* (Commentaries on the *Book on Calculating with Numbers*, on Bamboo Strips, unearthed from Zhangjiashan*)*, Beijing: Kexue chubanshe, 2001.

QIAN Baocong, ed., *Suan jing shi shu* (Ten Mathematical Classics), Beijing: Zhonghua shuju, 1963.

SHEN Kangshen, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Shen Kangshen, John N. Crossley, and Anthony W.-C. Lun, trans., Oxford: Oxford University Press, 1999.

Tongxun 2001, SU Yiwen, SU Junhong, SU Huiyu, CHEN Fengzhu, LIN Cangyi, HUANG Qingyang, and YE Jihai , "*Suan shu shu* jiaokan" (Collation of the *Suan shu shu*), *HPM tongxun* (HPM Newsletter), **3**(11)(2000), pp. 2-20.

Vogel, Kurt, tr., *Chiu chang suan shu: Neun Bücher arithmetischer Technik, Ein Chinesiches Rechenbuch für den praktischen Gebrauch aus den Frühen Hanzeit*, Braunschweig: Friedrich Vieweg Verlag (Ostwalds Klassiker), 1968.

Wenwu 2000, "Jiangling Zhangjiashan Hanjian zhengli xiaozu" (Study Group for the Collation and Arrangement of the Han Dynasty Bamboo Slips from Zhangjiashan, Jiangling County), ed., *Jiangling Zhangjiashan Hanjian Suan shu shu shiwen* (Transcription of the Bamboo *Suanshushu* or *A Book of Arithmetic* from Jiangling), *Wenwu* (Cultural Relics), **9** (2000), pp. 78-84.

Zhangjiashan 2001, Zhangjiashan ersiqi hao Hanmu zhujian zhengli xiaozu (Study Group for the Collation and Arrangement of the Bamboo Slips from Zhangjiashan Han Dynasty Tomb 247), ed., *Zhangjiashan Hanmu zhujian* (Bamboo Strips from Han Tombs at Zhangjiashan), Beijing: Wenwu chubanshe, 2001, pp. 83-96; 249-272.

ZOU Dahai, "*Chu tu Suan shu shu chu tan*" (Preliminary Research on the Unearthed Mathematics Book *Suan shu shu*), *Ziran kexueshi yanjiu* (Studies in the History of Natural Sciences), **20**(3)(2001), pp. 193-205.