Addendum to “Interpolation of Banach Spaces by the $\gamma$-Method”

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Abstract: In this note we correct and simplify an interpolation theorem for radonifying operators in [7].

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1. Introduction

In [7] it was claimed that under B-convexity the radonifying operators $\gamma(H,X)$ form an interpolation scale for the complex method. However the proof given there used the implicit hypothesis of a compact embedding $H_0 \rightarrow H_1$. In this note we give a simpler proof which also eliminates the compactness condition and works in some important situation where $H_0$ is not embedded into $H_1$. Among examples are scales of Sobolev spaces and domains of selfadjoints operators. However the case of general interpolation pairs of Hilbert spaces remains open.

2. Interpolation of $\gamma(H,X)$

Given an interpolation couple $(H_0, H_1)$ of two Hilbert spaces $H_0, H_1$ continuously embedded in some locally convex space $H$ one considers the Hilbert space $H = H_0 \cap H_1$ with the norm $\|f\|_H^2 = \|f\|^2_{H_0} + \|f\|^2_{H_1}$. Since $(x,y)_{H_j} \leq \|x\|_H\|y\|_H$ for $j = 0,1$ the Lax-Milgram theorem gives us positive, selfadjoint operators $B_j$ on $H$ of norm 1 so that

$$(x,y)_{H_j} = (x,B_j y)_H.$$
The interpolation couple \((H_0, H_1)\) of Hilbert spaces is called \textit{commutative} if the two bounded operators \(B_j\) constructed above commute. The point of this definition is the following consequence of the spectral theorem for commuting selfadjoint operators which allows us to replace abstract Hilbert spaces by \(L_2\)-spaces with densities.

**Lemma 2.1.** The interpolation couple \((H_0, H_1)\) is commutative if and only if there is a \(\sigma\)-finite measure space \((\Omega, \mu)\), a unitary map \(U : H_0 \cap H_1 \to L_2(\Omega, \mu)\) and densities \(0 < g_j \leq 1\) so that \(U\) extends to isometries \(U_j : H_j \to L_2(\Omega, g_j d\mu)\).

**Proof.** The spectral theorem (see e.g. [6, p. 246, Problem 4]) gives \((\Omega, \mu), U\) and \(g_j\) so that for \(x, y \in H_0 \cap H_1\)

\[
(x, y)_{H_j} = (x, B_j y)_H = (U x, (UB_j U^{-1}) U y) = \int \overline{f(\omega)} h(\omega) g_j(\omega) d\mu(\omega) = (f, h)_{L_2(\Omega, g_j d\mu)}
\]

with \(f = U(x), h = U(y)\) from a dense subset of \(L_2(\Omega, g_j d\mu)\). Since the operators \(B_j\) have norm one and are positive and injective we obtain \(0 < g_j \leq 1\).

There are some important interpolation couples covered by this situation.

**Examples.**

1. If \(H_0, H_1\) with a continuous dense embedding \(H_0 \cap H_1 = H_0\) and \(B_0 = Id\), so that the pair \((H_0, H_1)\) is commutative.

2. Let \(A\) be a positive injective and selfadjoint operator on a Hilbert space \(G\) and \(H_j = D(A^{\alpha_j})\) with \(\|h\|_j = \|A^{\alpha_j} x\|\) for \(\alpha_j \in \mathbb{R}, j = 0, 1\). Then the pair \((H_0, H_1)\) is commutative. Indeed it is easy to check that \(B_j = A^{2\alpha_j} (A^{2\alpha_0} + A^{2\alpha_1})^{-1}\).

3. As a particular case of (2) we obtain for \(G = L_2(\mathbb{R}^n)\) and \(A = \Delta\) the pairs of Sobolev spaces \(H^2_{\alpha_0}(\mathbb{R}^n), H^2_{\alpha_1}(\mathbb{R}^n)\) for \(\alpha_j \in \mathbb{R}\).

4. An interpolation pair which is not commutative can be obtained as follows:

Let \(H_0 = L_2(\mathbb{R}, \langle t \rangle^{-2} dt)\) with \(\langle t \rangle = (1 + t^2)^{1/2}\) and \(H_1 = H^2(\mathbb{R})\). Then \(L_2(\mathbb{R}) \subset H_0 \cap H_1\). However the multiplication operator \(B_0 f(t) = \langle t \rangle^{-2} f(t)\) and \(B_2 = (I - A)^{-1}\) do not commute.
Theorem 2.1. Let $H_0, H_1$ be separable Hilbert spaces forming a commutative interpolation couple. If $X_0, X_1$ are $\mathcal{B}$-convex, then
\[
(\gamma(H_0, X_0), \gamma(H_1, X_1))[\theta] = \gamma((H_0, H_1)[\theta], (X_0, X_1)[\theta])
\]
with equivalent norms.

Proof. Since it is well known already that for $H = (H_0, H_1)[\theta]$ we have, see [4],
\[
(\gamma(H_0, X_0), \gamma(H_1, X_1))[\theta] = \gamma(H, (X_0, X_1)[\theta])
\]
it only remains to show that
\[
(\gamma(H_0, X_0), \gamma(H_1, X_1))[\theta] = (\gamma(H, X_0), \gamma(H, X_1))[\theta] .
\]

By assumption and the lemma we may assume that $H_0 \cap H_1 = L^2(\Omega, \mu)$ and $H_j = L^2(\Omega, g_j d\mu)$, $j = 0, 1$, for a $\sigma$-finite measure space $(\Omega, \mu)$ and densities $0 < g_j \leq 1$ on $\Omega$. For convenience let us use the abbreviation $\gamma_j(h)$ for $L^2(\Omega, g_j d\mu)$. Note the following continuous inclusion
\[
\gamma(L^2(\Omega, d\mu), X_0 \cap X_1) \subset \gamma_j(g_\theta) \subset \gamma(L^2(\Omega, g_0 \wedge g_1 d\mu), X_0 + X_1)
\]
where $g_\theta = g_0^{1-\theta} g_1^{\theta}$ for $\theta \in [0, 1]$. Hence $(\gamma_0(g_\theta), \gamma_1(g_\theta))$ and $(\gamma_0(g_0), \gamma_1(g_0))$ are interpolation couples and, going back to the definition of the complex interpolation method, we can consider the following [1] Banach space $F$ of all continuous
\[
f : S = \{ z : 0 \leq \Re z \leq 1 \} \to \gamma_0(g_0) + \gamma_1(g_1)
\]
which are all continuous on $S$ and analytic in its interior such that $f(j + it) \in \gamma_j(g_j)$, $j = 0, 1$, with $\|f(j + it)\| \to 0$ for $|t| \to \infty$ and
\[
\|f\|_F = \sup \left\{ \|f(j + it)\|_{\gamma_j(g_j)} : t \in \mathbb{R} \right\} .
\]
Then $(\gamma_0(g_0), \gamma_1(g_1))[\theta] = \{ f(\theta) : f \in F \}$ endowed with the quotient norm
\[
\|T\| = \inf \{ \|f\| : f(\theta) = T \} .
\]
In a similar way one defines $F_\theta$ for the pair $(\gamma_0(g_\theta), \gamma_1(g_\theta))$. For the moment we assume in addition that
\[
\delta < g_j \text{ on } \Omega \text{ for } j = 0, 1 \text{ and some fixed } \delta > 0 . \quad (*)
\]
To prove the inclusion

\[(\gamma_0(g_0), \gamma_1(g_1))|_{\theta} \subset (\gamma_0(g_0), \gamma_1(g_0))|_{\theta},\]

we fix \(\theta \in [0, 1]\) and choose \(f \in \mathcal{F}\) with \(\|f\|_{\mathcal{F}} \leq 2\|f(\theta)\|_{(\gamma_0(g_0), \gamma_1(g_1))|_{\theta}}\).

Next we define a family of operators

\[T(z) : L_2(\Omega, g_0 \, d\mu) \longrightarrow L_2(\Omega, (g_0 \wedge g_1) \, d\mu)\]

with \(z \in S\) as the multiplication operators

\[T(z)f := \left(\frac{g_0}{g_0^z - g_1^z}\right)^{1/2} f.\]

By our assumption we have bounded and invertible operators \(T(z)\) for all \(z\) in the interior of \(S\). However for \(z = j + it\) these operators extend to isometries by [3]

\[T(j + it) : \gamma_j(g_0) \longrightarrow \gamma_j(g_j)\]

and \(T(\theta) = Id\). It follows that \(f \circ T\) belongs to \(\mathcal{F}_\theta\) and \(\|f \circ T\| \leq \|f\|\).

Therefore

\[\|f \circ T(\theta)\| \leq \|f \circ T\| \leq 2\|f(\theta)\|\]

and the inclusion \(\subset\) is continuous. To show the reverse we argue similarly now replacing \(T(z)\) by \(S(z) = T(z)^{-1}\).

It remains to remove the assumption (*). Note that the assumption \(\delta < g_j\) was only used to bound the operators \(T(z)\) and \(T(z)^{-1}\) for \(z\) in the interior of \(S\). For \(z \in \partial S\) and \(z = \theta\) the \(T(z)\) were isometries independent of the condition \(\delta < g_j\), which therefore does not enter the calculation of the norms \(\|f\|\) or \(\|f \circ T\|\). Now, for any \(\delta > 0\), if we consider functions \(f\) which are zero on \(\Omega_{\delta} = \{g_0 \leq \delta\} \cup \{g_1 \leq \delta\}\) the above norm estimate apply with constants independent of \(\delta\). However such functions \(f\) are dense in \(\mathcal{F}\) or \(\mathcal{F}_\theta\).

References


