CONTRIBUCIONES CIENTÍFICAS
EN HONOR DE MIRIAN ANDRÉS GÓMEZ
(Laureano Lambán, Ana Romero y Julio Rubio, editores),
Servicio de Publicaciones, Universidad de La Rioja,
Logroño, Spain, 2010.

LU EQUIVALENCE AND SIGNATURES OF BEZOUTIANS AND HANKEL MATRICES

GEMA M. DIAZ TOCA AND NADIA BEN ATTI

Dedicado a la memoria de Mirian Andrés Gómez

RESUMEN. En este artículo introducimos una equivalencia LU de matrices de Bézout y de Hankel. La presentación de aplicaciones de esta equivalencia nos lleva a hacer una revisión del significado y el cálculo de la signatura de las matrices de Bézout y Hankel asociadas a dos polinomios de coeficientes reales. Concluimos el artículo discutiendo el caso que aparece cuando sus coeficientes dependen de parámetros reales.

ABSTRACT. In this paper we introduce a LU equivalence of Bezoutians and Hankel matrices. The presentation of applications of this equivalence leads us to make a survey of the meaning and computation of the signature of Bezout and Hankel matrices associated to two real polynomials. We conclude the paper discussing the case which arises when their coefficients depend on real parameters.

1. INTRODUCTION

Probably two of the best known matrices associated to two polynomials over a field are the Bezout matrix and the Hankel matrix.

Concerning the Bezout Matrix, although the resultant of two univariate polynomials is defined as the determinant of their Sylvester matrix, the original definition was given by the determinant of Bezout matrix, introduced by Bézout in 1748. As for the Hankel matrix, Kronecker already investigated this matrix obtaining the first occurrence of Subresultant polynomials (see [17]).

More precisely, let $\mathbb{F}$ be a field and $u(x)$, $v(x) \in \mathbb{F}[x]$. We will denote the degree of a polynomial by the greek letter δ. Assume that $n = \delta(u(x)) > \delta(v(x))$. Then, Bezout and Hankel Matrices associated to $u(x)$ and $v(x)$, which will be denoted respectively by $\mathrm{Bez}(u, v)$ and $\mathrm{H}(u, v)$, are symmetric matrices of order n with a lot of properties in common.

For example, they are highly related to the greatest common divisor of $u(x)$ and $v(x)$. It is well known that

$$\mathrm{rank}(\mathrm{Bez}(u, v)) = \mathrm{rank}(\mathrm{H}(u, v)) = n - \delta(\gcd(u, v)),$$

Key words and phrases. Block LU factorization, Hankel matrix, Bezout matrix, Euclidean algorithm, Subresultants.

Partially supported by the Spanish "Ministerio de Ciencia e Innovacion" under the Project MTM2008-04699-C03-01.

and their minors provide all the coefficients of Subresultant polynomials. Moreover, both matrices represent the same linear map with respect to different bases (for more details, see [4]).

This paper introduces some applications of the block LU factorization of $\mathrm{Bez}(u, v)$ and $\mathrm{H}(u, v)$ introduced in [2], as the real root counting problem.

In some sense, this paper links several results which have independently appeared in [1], [2], [4], [10], [14] and [15].

We organize the paper as follows. In Section 2 we introduce a block LU factorization of Bezout and Hankel matrices associated to two polynomials. In Section 3, we recall the definitions of inertia and signature of a matrix and describe some applications of Bezout and Hankel matrices related to their signatures. We conclude in Section 4 with the relation between signed subresultants and bezoutians together with the discussion of the parametric case.

2. Block LU factorization of $\mathrm{Bez}(u, v)$ and $\mathrm{H}(u, v)$

First of all, recall the definitions of Bezout and Hankel matrices associated to $u(x)$ and $v(x)$.

2.1. Definitions. Let $R(x)$ be the power series expansion of the function $v(x)/u(x)$ at the infinity

$$R(x) = \frac{v(x)}{u(x)} = \sum_{i=1}^{\infty} h_i x^{-i}.$$

This power series defines the $n \times n$ Hankel matrix, $\mathrm{H}(u, v)$, whose (i, j) entry is h_{i+j-1} $(i, j \in \{1, \ldots, n\})$

$$\mathrm{H}(u, v) = \begin{pmatrix} h_1 & h_2 & \cdots & h_n \\ h_2 & h_3 & \cdots & h_{n+1} \\ \vdots & \vdots & & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n-1} \end{pmatrix}.$$

The Bezout Matrix (or Bezoutian) associated to $u(x)$ and $v(x)$ is the symmetric matrix

$$\mathrm{Bez}(u, v) = \begin{pmatrix} c_{0,0} & \cdots & c_{0,n-1} \\ \vdots & & \vdots \\ c_{n-1,0} & \cdots & c_{n-1,n-1} \end{pmatrix},$$

where the $c_{i,j}$ are defined by the Cayley expression:

$$\frac{u(x)v(y) - u(y)v(x)}{x - y} = \sum_{i,j=0}^{n-1} c_{i,j} x^i y^j.$$

For example, if $u(x) = u_n x^n + u_{n-1}x^{n-1} + \ldots + u_0$, then the matrix $\mathrm{Bez}(u, 1)$ is the following upper Hankel triangular matrix

$$\mathrm{Bez}(u,1) = \begin{pmatrix} u_1 & \ldots & u_n \\ \vdots & ⋰ & \\ u_n & & \end{pmatrix}.$$

2.2. Block LU Factorization. In [11] (1983) and [12] (1984), it is stated that any Hankel matrix h admits a block LU factorization. More concretely,

> There is an upper unitriangular matrix $\mathcal{A}$ such that
>
> $$\mathcal{A}^t h \mathcal{A} = \mathcal{D} \tag{1}$$
>
> where $\mathcal{D}$ is a block diagonal matrix and each block is a lower Hankel triangular matrix. We will say that $\mathcal{D}$ and h are LU-equivalent.

We extend the definition of LU-equivalent saying that a block diagonal matrix D and a matrix B are LU–equivalent when there exists an upper triangular matrix A such that $A^t\, B\, A = D$.

In [1], we present a new algorithm for the block factorization of Hankel matrices different from the classical one. In [2], we apply such an algorithm to the matrices $\mathrm{Bez}(u, v)$ and $\mathrm{H}(u, v)$, coming to the following conclusions.

1. Let J denote the backward identity matrix. If $u(x)$ and $v(x)$ are coprime, then $H(u, v)$ is LU-equivalent to the block diagonal matrix D_h,

   $$D_h = \mathrm{Diag}(J\mathrm{Bez}(q_1, 1)J, J\mathrm{Bez}(q_2, 1)J, \ldots, J\mathrm{Bez}(q_t, 1)J, \mathrm{H}(r_{t-1}, r_t)), \tag{2}$$

   and $J\mathrm{Bez}(u, v)J$ is LU-equivalent to the block diagonal matrix D_b,

   $$D_b = \mathrm{Diag}(J\mathrm{Bez}(q_1, 1)\, J, J\mathrm{Bez}(q_2, 1)J, \ldots, J\mathrm{Bez}(q_t, 1)J, J\mathrm{Bez}(r_{t-1}, r_t)J), \tag{3}$$

   where $q_1, q_2, \ldots,$ and q_t are the successive quotients and r_t is the last nonzero remainder which appear in the signed Euclidean Algorithm applied to the pair $(u(x), v(x))$.

   For signed Euclidean Algorithm we understand the Euclidean algorithm applied in the following way:

   $$r_{-1}(x) = u(x), r_0 = v(x), r_{i-2}(x) = r_{i-1}(x)q_i(x) - r_i(x).$$

   Observe that in this case, $r_t \in \mathbb{F}$. Hereafter, we will say the signed quotient sequence and the signed remainder sequence associated to $(u(x), v(x))$.
2. If $u(x)$ and $v(x)$ are not coprime, then both matrices are LU-equivalent to the block diagonal matrix

   $$\mathrm{Diag}(J\,\mathrm{Bez}(q_1, 1)\, J, J\,\mathrm{Bez}(q_2, 1)\, J, \ldots, J\,\mathrm{Bez}(q_t, 1)\, J, D_z) \tag{4}$$

   where $\{q_1, q_2, \ldots, q_t\}$ is the signed quotient sequence of $(u(x), v(x))$ and D_z is the zero matrix of order equal to the degree of the greatest common divisor of $u(x)$ and $v(x)$.

Example 2.1.

$u(x) = 42\,x^{10}+197\,x^9+443\,x^8+1507\,x^7+1204\,x^6+1851\,x^5+458\,x^4-4114\,x^3-1049\,x^2+145\,x,$

and

$v(x) = 14\,x^9 + 85\,x^7 + 389\,x^6 - 122\,x^5 + 764\,x^4 + 47\,x^8 - 871\,x^3 - 266\,x^2 + 55\,x.$

In this case,

$\gcd(u, v) = 14x^2+5x, q_1 = 3x+4, q_2 = -x^3-5x+1, q_3 = x^2+4$ *and* $q_4 = -x^2-3x+4.$

As expected, both $H(u, v)$ *and* $\mathrm{Bez}(u, v)$ *are LU-equivalent to the following block diagonal matrix defined by Bezoutians associated to each quotient and 1, multiplied by the backward matrix J (see Equation (4)),*

$$\begin{pmatrix}
3 & & & & & & & & & \\
 & 0 & 0 & -1 & & & & & & \\
 & 0 & -1 & 0 & & & & & & \\
 & -1 & 0 & -5 & & & & & & \\
 & & & & 0 & 1 & & & & \\
 & & & & 1 & 0 & & & & \\
 & & & & & & 0 & -1 & & \\
 & & & & & & -1 & -3 & & \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & 0 & 0
\end{pmatrix}$$

3. Signatures and Inertias

Recall that the inertia of a square matrix $A \in \mathbb{C}^{n\times n}$, written In A, is the triple of integers

$$\mathrm{In}\, A = \{\pi(A), \nu(A), \delta(A)\} \tag{5}$$

where $\pi(A), \nu(A), \delta(A)$ denote the number of eigenvalues of A, counted with their algebraic multiplicities, lying in the open right half-plane, in the open left half-plane, and on the imaginary axis respectively.

If A is hermitian, $\pi(A)$ (respectively $\nu(A)$) is the number of positive (respectively negative) eigenvalues and the signature of A, written sig A, is defined as

$$\mathrm{sig}\, A = \pi(A) - \nu(A). \tag{6}$$

Given $u(x)$ and $v(x)$ in $\mathbb{R}[x]$, with $\gcd(u, v) = 1$ and $\delta(u) \geq \delta(v)$, the matrices $\mathrm{Bez}(u, v)$ and $\mathrm{H}(u, v)$ are congruent because

$$\mathrm{Bez}(u, v) = \mathrm{Bez}(u, 1)\mathrm{H}(u, v)\mathrm{Bez}(u, 1).$$

Thus, by the Sylvester's law of inertia, it holds that

$$\mathrm{In}\,\mathrm{Bez}(u, v) = \mathrm{In}\,\mathrm{H}(u, v),$$

and by Eqs. (2) and (3) of Section 2 we have

$$\mathrm{In}\,\mathrm{H}(u, v) = \mathrm{In}\, D_h = \mathrm{In}\,\mathrm{Bez}(u, v) = \mathrm{In}\, D_b$$

On the other hand, the inertia of a block diagonal matrix $\mathfrak{D} = \mathrm{Diag}(\mathfrak{D}_{11}, \ldots, \mathfrak{D}_{LL})$ is equal to $\mathrm{In}\,\mathfrak{D} = \sum_{i=1}^{L} \mathrm{In}\,(\mathfrak{D}_{ii})$. Thus, since the blocks $D_{h_{ii}}$ and $D_{b_{ii}}$ are equal to $J\mathrm{Bez}(q_i, 1)J$ except for the last ones, according to the Iohvidov's rule we have

$$\pi(D_{h_{ii}}) = \pi(D_{b_{ii}}) = \begin{cases} \dfrac{\delta(q_i)}{2}, & \delta(q_i) \text{ even;} \\ \\ \dfrac{\delta(q_i) + \mathrm{sign}(\mathrm{lc}(q_i))}{2}, & \delta(q_i) \text{ odd.} \end{cases}$$

$$\nu(D_{h_{ii}}) = \nu(D_{b_{ii}}) = \begin{cases} \dfrac{\delta(q_i)}{2}, & \delta(q_i) \text{ even;} \\ \\ \dfrac{\delta(q_i) - \mathrm{sign}(\mathrm{lc}(q_i))}{2}, & \delta(q_i) \text{ odd.} \end{cases}$$

As for the last blocks, in accord with the notation used in Eqs. (2) and (3), we would have $L = t + 1$,

$$D_{h_{LL}} = \mathrm{H}(r_{t-1}, r_t) \quad D_{b_{LL}} = J\mathrm{Bez}(r_{t-1}, r_t)J \text{ with } r_t \in \mathbb{R},$$

and

$$\pi(D_{LL}) = \nu(D_{b_{LL}}) = \begin{cases} \dfrac{\delta(r_{t-1})}{2}, & \delta(r_{t-1}) \text{ even;} \\ \\ \dfrac{\delta(r_{t-1}) + \mathrm{sign}(\mathrm{lc}(r_{t-1})\, r_t)}{2}, & \delta(r_{t-1}) \text{ odd.} \end{cases}$$

$$\nu(D_{LL}) = \nu(D_{b_{LL}}) = \begin{cases} \dfrac{\delta(r_{t-1})}{2}, & \delta(r_{t-1}) \text{ even;} \\ \\ \dfrac{\delta(r_{t-1}) - \mathrm{sign}(\mathrm{lc}(r_{t-1})\, r_t)}{2}, & \delta(r_{t-1}) \text{ odd.} \end{cases}$$

Note that in the Euclidean algorithm, $q_{t+1} = \frac{r_{t-1}}{r_t}$. Therefore, the signed quotient sequence provides the inertia and furthermore

$$\mathrm{sig}\,\mathrm{Bez}(u, v) = \mathrm{sig}\,\mathrm{H}(u, v) = \sum_{i=1}^{L} \begin{cases} 0, & \delta(q_i) \text{ even;} \\ \\ \mathrm{sign}(\mathrm{lc}(q_i)), & \delta(q_i) \text{ odd.} \end{cases} \tag{7}$$

Our conclusions can be used in order to simplify a lot of proofs of results concerning the signature of Hankel matrices. For example, in [7], L. Gemignani proves Identity (7) but not in a direct way from the block LU factorization but with an elaborated proof. In [14], Theorem 3.4 turns out to be a corollary of Theorem 3.1. And in [1], we present a simple proof for the Frobenius Theorem which characterizes the signature of a Hankel matrix by the sign of its nonzero principal leading minors.

Next we present some interesting applications of Bezout and Hankel matrices related to their signatures.

3.1. Cauchy Index and Real Root Counting. Given two real polynomials $u(x)$ and $v(x)$ and $a < b$ in $\mathbb{R} \cup \{-\infty, \infty\}$, the Cauchy index of the real rational function $\frac{v(x)}{u(x)}$ on (a, b) is the number of jumps of the function from $-\infty$ to ∞ minus the number of jumps of the function from ∞ to $-\infty$. It is denoted by $I_a^b \frac{v(x)}{u(x)}$.

Assume that $\delta(v(x)) < \delta(u(x))$ (otherwise, we redefine $v(x)$ as $\operatorname{rem}(v(x), u(x))$). Denote by $V(a)$ the number of sign changes of the signed remainder sequence of $u(x)$ and $v(x)$ evaluated at a.

Summarizing several results of classical Theory of Matrices (see [6] and [13]) and of Real Algebraic Geometry (see [3]), we have the following general result.

$$I_{-\infty}^{\infty} \frac{v(x)}{u(x)} = \operatorname{sig}(\operatorname{Bez}(u, v)) = \operatorname{sig}(\mathrm{H}(u, v)) = V(-\infty) - V(\infty)$$

As particular cases of Cauchy indexes,

$$I_{-\infty}^{\infty} \frac{v(x)u'(x)}{u(x)} = \sharp\{\alpha \in \mathbb{R} | u(\alpha) = 0 \wedge v(\alpha) > 0\} - \sharp\{\alpha \in \mathbb{R} | u(\alpha) = 0 \wedge v(\alpha) < 0\},$$

and so

$$I_{-\infty}^{\infty} \frac{u'(x)}{u(x)} = \operatorname{sig}(\operatorname{Bez}(u, u')) = \operatorname{sig}(\mathrm{H}(u, u')) = \sharp\{\alpha \in \mathbb{R} | u(\alpha) = 0\},$$

$$\nu(\operatorname{Bez}(u, u')) = \sharp \text{ pairs of complex conjugate zeros}$$

3.2. Stability problems. An $n \times n$ matrix is said to be a stable matrix if all eigenvalues have negative real part, that is, its inertia is equal to $(0, n, 0)$. Stable matrices are of particular interest in the study of differential equations: "The matrix A is stable if and only if for every solution vector $x(t)$ of $\dot{x} = Ax$, we have $x(t)- > 0$ as $t- > \infty$ ".

Moreover, given a polynomial $p(x) = a_n x^n + \cdots + a_0$, the inertia $\operatorname{In}(p)$ of $p(x)$ is defined as the triple of nonnegative integers

$$\{\pi(p), \nu(p), \delta(p)\}$$

where $\pi(p)$ ($\nu(p)$ and $\delta(p)$) denotes the number of zeros of p(x) counting multiplicities with positive (negative and zero) real parts. If $\pi(p) = \delta(p) = 0$, $p(x)$ is said to be stable.

Therefore, a matrix A is stable if and only if its characteristic polynomial is stable. Then, studying if a matrix is stable or not leads us to determine the distribution of zeros relative to the imaginary axis. This problem is known as the Routh–Hurwitz problem.

The solution of this problem for real polynomials is highly related to Bezout and Hankel matrices as follows.

Theorem 3.1 (The Liénard-Chipart Criterion)**.** *A real polynomial $u(x) = h(x^2) + xg(x^2)$ is stable if and only if the coefficients of $h(x)$ have the same sign as $\operatorname{lc}(u(x))$ and $\operatorname{Bez}(h, g)$ is positive definite.*

Theorem 3.2 (The Markov criterion). *A real polynomial* $u(x) = h(x^2) + xg(x^2)$ *is stable if and only if the coefficients of* $h(x)$ *have the same sign as* $\mathrm{lc}(u(x))$ *and* $\mathrm{H}(h, g)$ *is positive definite.*

For proofs see [6] and [13].

Thus, in view of (7), $\mathrm{Bez}(h, g)$ is positive definite if and only if the quotients of $h(x)$ and $g(x)$ have degree 1 and positive leading coefficients.

4. Signed Subresultants and Bezout Matrices

As it is mentioned in [14], the use of Euclidean Algorithm in order to compute the signature is not the best choice if we want to control the size of intermediate computations or if the coefficients of the polynomials depend on parameters. Another method derives from subresultant polynomials.

In [8] and [9] the authors introduce the Signed Subresultant Polynomials.

Suppose $\mathrm{degree}(u) = n$ and $\mathrm{degree}(v) = m$. Following the notation that appears in [3], if $\mathrm{Sres}_j(u, v)$ denotes the classical j-th Subresultant polynomial, then the j-th signed subresultant Polynomial, denoted by $\mathrm{SR}_j(u, v)$, is equal to $(-1)^{(n-j)(n-j-1)/2}\mathrm{Sres}_j(u, v)$, for $0 \le j \le m$. By convention,

$$\mathrm{SR}_n(u, v) = \mathrm{sign}(\mathrm{lc}(u)^{n-m-1})u(x),$$

$$\mathrm{SR}_{n-1}(u, v) = \mathrm{sign}(\mathrm{lc}(u)^{n-m+1})v(x),\ \mathrm{SR}_j = 0, m < j.$$

The j-th signed principal subresultant, denoted by $\mathrm{sr}_j(u, v)$, is defined as the coefficient of x^j in $\mathrm{SR}_j(u, v)$, for $j < n$. By convention $\mathrm{sr}_n(u, v) = \mathrm{sign}(\mathrm{lc}(u)^{p-q})$.

Obviously, the signed subresultants have the same properties as the classical ones. However in [10] the authors introduce a new theorem Structure Theorem for Subresultants which defines a new algorithm for computing them with better complexity than the classical one.

Theorem 4.1 (New Structure Theorem).
Let $0 \le j < i \le n$. *Suppose that* $\mathrm{SR}_{i-1}(u, v)$ *is non-zero and of degree* j.

1. *If* $\mathrm{SR}_{j-1}(u, v)$ *is zero, then* $\mathrm{SR}_{i-1}(u, v) = \gcd(u, v)$ *and* $\mathrm{SR}_l(u, v) = 0$, $l \le j - 1$.
2. *If* $\mathrm{SR}_{j-1}(u, v) \neq 0$ *has degree* k *then*

$$\mathrm{sr}_j(u, v)\,\mathrm{lc}(\mathrm{SR}_{i-1}(u, v))\,\mathrm{SR}_{k-1}(u, v) =$$

$$= -\mathrm{Rem}(\mathrm{sr}_k(u, v)\,\mathrm{lc}(\mathrm{SR}_{j-1}(u, v))\,\mathrm{SR}_{i-1}(u, v),\ \mathrm{SR}_{j-1}v).$$

Moreover if $j \le m, k < j - 1, \mathrm{SR}_k(u, v)$ *is proportional to* $\mathrm{SR}_{j-1}(u, v)$.

a) $\mathrm{SR}_{j-2}(u, v) = \cdots = \mathrm{SR}_{k+1}(u, v) = 0$

b) $\mathrm{sr}_k(u, v) = (-1)^{(j-k)(j-k-1)/2}\,\dfrac{(\mathrm{lc}(\mathrm{SR}_{j-1}(u, v)))^{j-k}}{\mathrm{sr}_j(u, v)^{j-k-1}}$,

c) $\mathrm{lc}(\mathrm{SR}_{j-1}(u, v))\,\mathrm{SR}_k(u, v) = \mathrm{sr}_k(u, v)\,\mathrm{SR}_{j-1}(u, v)$

4.1. Signed Subresultants and Cauchy Index. Due to the closed relation between the remainders and Subresultants, it is possible to compute the Cauchy index by using only the signed principal subresultants.

Let $s = s_p, \ldots, s_0$ be is a finite list of elements in an ordered field such that $s_p \neq 0, s_{p-1} = \ldots = s_{q+1} = 0$ and $s_q \neq 0$. Let $s' = s_q, \ldots, s_0$ (if q+1=0, s' is the empty list). Then $D(s)$ is inductively defined as follows:

$$D(s) = \begin{cases} 0, & \text{if } s' = \emptyset; \\ D(s') + (-1)^{(p-q)(p-q-1)/2}\operatorname{sign}(s_p s_q), & \text{if p-q odd;} \\ D(s'), & \text{if p-q even.} \end{cases}$$

Then, if $\operatorname{sr}(u, v)$ is the principal signed subresultant sequence, then

$$D(\operatorname{sr}(u, v)) = I_{-\infty}^{\infty} \frac{v(x)}{u(x)}. \tag{8}$$

For proof, see [3], Chapter 9.

In [4] we prove that (classic) Subresultants of $u(x)$ and $v(x)$ can be obtained through minors of $\operatorname{Bez}(u, v)$. Hence, if $\mathbf{B}_j$ denotes the $j \times j$ principal minor extracted from the last j columns and rows of $\operatorname{Bez}(u, v)$ (that is, the j leading principal minor of $J\operatorname{Bez}(u, v)J$), it holds that

$$\operatorname{lc}(u)^{n-m} \operatorname{sr}_j(u, v) = \mathbf{B}_{n-j}, \text{ for } j < m$$

As a consequence, we can also compute $I_{-\infty}^{\infty} \frac{v(x)}{u(x)}$ using minors of Bezout matrices. In fact, we obtain a version of the Frobenius Theorem for Bezout matrices.

Corollary 4.1. *Let* $\operatorname{Bez}(u, v)$ *be the Bezout matrix associated to* $u(x)$ *and* $v(x)$*. Then*

$$D(\boldsymbol{B}) = I_{-\infty}^{\infty} \frac{v(x)}{u(x)},$$

with $\boldsymbol{B}$ *equal to*

$$\operatorname{sign}(\operatorname{lc}(u)^{n-m}), \operatorname{sign}(\operatorname{lc}(u)^{n-m+1})\operatorname{lc}(v), \underbrace{0, \ldots, 0}_{n-m-1}, \operatorname{sign}(\operatorname{lc}(u)^{n-m})\mathbf{B}_{n-m}, \ldots$$
$$\ldots, \operatorname{sign}(\operatorname{lc}(u)^{n-m})\mathbf{B}_n$$

Thus we can use either (7) or (8) in order to obtain $I_{-\infty}^{\infty} \frac{v(x)}{u(x)}$, depending on the sizes of coefficients.

As far as the complexity of computing the signed subresultants is concerned, it is described in [14] a fast algorithm for integer input polynomials based on the exact signed Euclidean division announced in Theorem 4.1.

However, an important case arises when coefficients depend on parameters. The more coefficients we have, the less efficient is the computation of quotients and exact remainders.

In [4], we compare four algorithms for computing the sequence of the principal subresultants. One of them was the algorithm derived from Theorem 4.1. In order to compute the leading principal minors of $J\mathrm{Bez}(u,v)J$, we propose the use of the Samuelson-Berkowitz algorithm.

This algorithm computes the characteristic polynomial of a square matrix such that, in the process, it computes all the characteristic polynomials of leading principal submatrices in increasing order. Since leading principal minors are the independent coefficients of the computed characteristic polynomials, this algorithm applied to $J\mathrm{Bez}(u,v)J$ allows us to compute the wanted sequence.

Using this algorithm, there are no divisions and the arithmetic operations performed are products of the entries of the Bezout matrix and of the coefficients of characteristic polynomials of submatrices.

The number of arithmetic operations used in this computation is higher than in the computation of Subresultant polynomials, however the cost of the arithmetic operations is smaller and when the coefficients depend on parameters, the minors of the Bezout matrix are usually computed faster.

Example 4.1. *The analysis of the equilibrium points of a perturbed four dimensional harmonic oscillator in 1-1-1-1 resonance is reduced to study the double real roots of the following real polynomial*

$$u(x) = 10n^2x^3 - 9n(2y^2+3nz)x^2 + \left(4y^4 + 12n\left(y^2z+h\right) + n^2\left(8n^2+16y^2+9z^2\right)\right)x$$
$$-2\left(2h+8n^3\right)y^2 - 6hnz,$$

with n, y, z and h in $\mathbb{R}$ and $n > 0$. The double roots determines the critical values of the energy momentum map. So we are interested in studying the conditions for the existence and the multiplicity of real roots (for more details, see [5]*).*

Let us add that this example does not have the purpose to show a polynomial whose coefficients depend on a large enough number of parameters to make inefficient the use of Theorem 4.1 in order to compute Subresultants, but to study a problem which appears in a real physique context.

Then, following the above notation with $v(x) = u'(x)$,

$$\mathbf{B}_3 = 10n^2\mathrm{sr}_0(u,u') = 100\,n^6\left(-20480\,n^{12} - 22464\,z^2n^{10} + \ldots + 23616\,y^{10}zn + 2624\,y^{12}\right)$$

$$\mathbf{B}_2 = 10n^2\mathrm{sr}_1(u,u') = -600\,n^6\left(80\,n^4 - 153\,n^2z^2 + 160\,n^2y^2 + 120\,hn - 204\,ny^2z - 68\,y^4\right)$$

$$\mathbf{B}_1 = 10n^2\mathrm{sr}_2(u,u') = 10n^2\mathrm{lc}(u') = 300n^4$$

$$\mathrm{sr}_3(u,u') = 1$$

Since $n > 0$, let

$$b_3 = -20480\,n^{12} - 22464\,z^2n^{10} + \ldots + 23616\,y^{10}zn + 2624\,y^{12}$$

$$b_2 = -80\,n^4 + 153\,n^2z^2 - 160\,n^2y^2 - 120\,hn + 204\,ny^2z + 68\,y^4,$$

$$b_1 = 1$$

$$b_0 = 1$$

Thus, the following table shows the number of different real roots of u in the various cases corresponding to all the possible signs for b_2 and b_3.

$$
\begin{array}{c|ccccccccc}
b_0 & + & + & + & + & + & + & + & + & + \\
b_1 & + & + & + & + & + & + & + & + & + \\
b_2 & + & - & + & - & 0 & 0 & + & - & 0 \\
b_3 & + & - & - & + & + & - & 0 & 0 & 0 \\
\hline
nb. & 3 & 1 & 1 & -1 & 1 & 1 & 2 & 0 & 1
\end{array}
$$

In fact, the sign conditions with

- $b_2 < 0$ *and* $b_3 > 0$,
- $b_2 < 0$ *and* $b_3 = 0$,
- $b_2 = 0$ *and* $b_3 > 0$

have empty realizations.

As far as the multiplicity is concerned:

Existence of triple root
Since $\mathrm{degree}(u) = 3$, *our polynomial has a triple root when the degree of the gcd of* $u(x)$ *and* $u'(x)$ *is equal to 2, that is, the rank of* $\mathrm{Bez}(u, u') = 1$ *and*

$$rk(\mathrm{Bez}(u, u')) = 1 \Leftrightarrow b_3 = b_2 = 0, b_1 \neq 0$$

Existence of double root
Since $\mathrm{degree}(u) = 3$, *our polynomial has a double root when the degree of the gcd of* $u(x)$ *and* $u'(x)$ *is equal to 1, that is, the rank of* $\mathrm{Bez}(u, u') = 2$ *and*

$$rk(\mathrm{Bez}(u, u')) = 2 \Leftrightarrow b_3 = 0,\ b_2 \neq 0$$

References

[1] N. Ben Atti, G. M. Diaz–Toca. Block diagonalization and LU-equivalence of Hankel matrices. *Linear Algebra and its applications* **412**, 247–269, 2006.

[2] N. Ben Atti, G. M. Diaz–Toca. Block LU factorization of Hankel and Bezout Matrices and Euclidean Algorithm. *Int J. Computer Math* **86**, 135–149, 2009.

[3] S. Basu, R. Pollack, M. F. Roy. *Algorithms in Real Algebraic Geometry.* 2nd edition, Springer, 2006.

[4] G. M. Diaz–Toca, L. Gonzalez–Vega. Various New Expressions for Subresultants and Their Applications. *Applicable Algebra in Engineering, communication and Computing* **15**(3-4), 233–266, 2004.

[5] J. Egea. *Reducciones toroidales en hamiltonianos promediados por la resonancia 1-1-1-1 semisimple.* PhD Thesis, Universidad de Murcia, 2007.

[6] F. Gantmacher. *Théorie des matrices, Tome 1, (Théorie Générale)*, Dunod, Paris, 1966.

[7] L. Gemignani. Computationally Efficient Applications of the Euclidean algorithm to Zero Location. *Linear Algebra and Applications* **249**, 79–91, 1996.

[8] L. Gonzalez-Vega, H. Lombardi, T. Recio, M-F. Roy. Spécialisation de la suite de Sturm et sous-resultant I. *Informatique théorique et applications* **24**, 561–588, 1990.

[9] L. Gonzalez-Vega, H. Lombardi, T. Recio, M-F. Roy. Spécialisation de la suite de Sturm et sous-resultant II. *Informatique théorique et applications* **28**, 1–24, 1994.

[10] L. Gonzalez-Vega, H. Lombardi, T. Recio, M-F. Roy. Sturm-Habitcht sequence, determinants and real roots of univariate polynomials. *Quantifier Elimination and Cylindrical Algebraic Decomposition.* Bob Caviness and Jeremy Johnson Eds., pp. 300–316. Springer-Verlag, Wien, New York, 1998.

[11] W. B. Gragg, A. Lindquist. On the partial realization problem. *Linear Algebra Appl.* **50**, 277–319, 1983.

[12] G. Heinig, K. Rost. *Algebraic Methods for Toeplitz-like Matrices and Operators.* Birkhäuser Verlag, Basel, 1984.

[13] P. Lancaster, M. Tismenetsky. The Theory of Matrices With Applications. 2nd edition, Academic Press, Orlando, 1985.

[14] T. Lickteig, M. F. Roy. Sylvester–Habicht Sequences and Fast Cauchy Index Computation. *J. Symbolic Computation* **31**, 315–341, 2001.

[15] M. F. Roy, S. M. Sedjelmaci. Iterative accelerated euclidean algorithm. To appear in *Journal of Symbolic Computation.*

[16] J. von zur Gathen, J. Gerhard. *Modern Computer Algebra.* 2nd edition, Cambridge University Press, 2003.

[17] J. von zur Gathen, T. Lücking. Subresultants revisited. *Theoretical Computer Science* **297**, 199–239, 2003.

Departamento de Matemática Aplicada. Universidad de Murcia, Spain
E-mail address: `gemadiaz@um.es`

Ecole nationale des sciences de l'informatique, Université de la Manouba, Tunisia
E-mail address: `nadia.benatti@ensi.rnu.tn`