SOBRE EL PROBLEMA DEL ARCO ELASTICO SOMETIDO A PRESIONES CONSTANTES EN EL EXTRADOS Y EN EL INTRADOS

por

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1. He publicado, en el año 1928, en la «Revista de Ciencias», año 31, un trabajo «Sobre el problema del arco elástico encastrado sometido a presiones constantes en el extradós y en el intradós», en el cual he dado fórmulas para las tensiones y presiones máximas en el caso de un arco elástico circular empotrado en dos puntos A_1 y A_2. En dos Notas recientemente presentadas a la Academia de Ciencias de Lima, he completado estas investigaciones y he estudiado también el caso de un arco *apoyado* en dichos dos puntos. Ahora me propongo de ocuparme del caso en el cual en las impostas se cumpla la condición

$$(1) \qquad \int_{R'}^{R} Q(\rho)\,(\rho - r)\,d\rho = 0,$$

siendo r la distancia $OA_1 = OA_2$, que vamos a suponer igual a $\frac{R+R'}{2}$, y siendo Q la tensión tangencial. En lo que se refiere a la bibliografía y a la demostración de las fórmulas utilizadas, ruego al lector de referirse al trabajo citado de la Revista, a las dos Notas en los Actas de la Academia de Lima, 1943, y a un trabajo que ha sido presentado a la Revista de Ciencias y que va a ser publicado también en 1943.

2. Suponiendo las tensiones planas se tienen las tres fórmulas siguientes

$$(2) \qquad P = \frac{a_0}{\rho^2} + 2b_0 + \lambda_0 (2 \log \rho + 1)$$

$$+ \cos\Theta\left(\frac{a_1+\beta_1}{\rho} + 2b_1\rho - 2\alpha_1\rho^{-3}\right) + \operatorname{sen}\Theta\left(\frac{c_1+\delta_1}{\rho} + 2d_1\rho - 2\gamma_1\rho^{-3}\right),$$

$$(3) \qquad Q = -\frac{a_0}{\rho^2} + 2b_0 + \lambda_0 (2 \log \rho + 3)$$

$$+ \cos\Theta \left(6b_1\rho + \frac{2\alpha_1}{\rho^3} + \frac{\beta_1}{\rho}\right) + \operatorname{sen}\Theta \left(6d_1\rho + \frac{2\gamma_1}{\rho^3} + \frac{\delta_1}{\rho}\right),$$

$$(4) \quad U = \frac{\alpha_0}{\rho^2} + \operatorname{sen}\Theta\left(2b_1\rho - \frac{2\alpha_1}{\rho^3} + \frac{\beta_1}{\rho}\right) - \cos\Theta\left(2d_1\rho - \frac{2\gamma_1}{\rho^3} + \frac{\delta_1}{\rho}\right)$$

Para los corrimientos u radial y v tangencial encontramos las fórmulas

$$(5) \qquad u = -\frac{a_0}{2\mu\rho} + \left(\frac{b_0}{\lambda'+\mu} - \frac{c_0}{2\mu}\right)\rho + \frac{c_0}{\lambda'+\mu}\rho \log \rho$$

$$+ \left(\frac{a_1}{4(\lambda'+\mu)} + \frac{\lambda'+2\mu}{2\mu(\lambda'+\mu)}\beta_1\right) \Theta \operatorname{sen}\Theta$$

$$-\left(\frac{c_1}{4(\lambda'+\mu)} + \frac{\lambda'+2\mu}{2\mu(\lambda'+\mu)}\right) \Theta \cos\Theta$$

$$+ \left[\left(\frac{\lambda'+2\mu}{4\mu(\lambda'+\mu)} a_1 + \frac{\beta_1}{2(\lambda'+\mu)}\right)\log\rho + \frac{\mu-\lambda'}{2\mu(\lambda'+\mu)} b_1\rho^2 + \frac{a_1}{2\mu\rho^2} + A\right]\cos\Theta$$

$$+ \left[\left(\frac{\lambda'+2\mu}{4\mu(\lambda'+\mu)} c_1 + \frac{\delta_1}{2(\lambda'+\mu)}\right)\log\rho + \frac{\mu-\lambda'}{2\mu(\lambda'+\mu)} d_1\rho^2 + \frac{\gamma_1}{2\mu\rho^2} + B\right]\operatorname{sen}\Theta,$$

$$(6) \qquad v = C\rho - \frac{\alpha_0}{2\mu\rho} + \frac{\lambda'+2\mu}{\mu(\lambda'+\mu)} c_0\rho\Theta$$

$$+ \left(\frac{a_1}{4(\lambda'+\mu)} + \frac{\lambda'+2\mu}{2\mu(\lambda'+\mu)}\beta_1\right)\Theta\cos\Theta$$

$$+ \left(\frac{c_1}{4(\lambda'+\mu)} + \frac{\lambda'+2\mu}{2\mu(\lambda'+\mu)}\delta_1\right)\Theta \operatorname{sen}\Theta +$$

$$+\left[\left(-\frac{\lambda'+2\mu}{4\mu(\lambda'+\mu)}a_1-\frac{\beta_1}{2(\lambda'+\mu)}\right)\log\rho-\frac{a_1}{4\mu}\right.$$
$$\left.+\frac{3\lambda'+5\mu}{2\mu(\lambda'+\mu)}b_1\rho^2+\frac{\alpha_1}{2\mu\rho^2}-\frac{\beta_1}{2\mu}-A\right]\operatorname{sen}\Theta$$
$$+\left[\left(\frac{\lambda'+2\mu}{4\mu(\lambda'+\mu)}c_1+\frac{\delta_1}{2(\lambda'+\mu)}\right)\log\rho+\frac{c_1}{4\mu}-\frac{3\lambda'+5\mu}{2\mu(\lambda'+\mu)}d_1\rho^2\right.$$
$$\left.-\frac{\gamma_1}{2\mu\rho^2}+\frac{\delta_1}{2\mu}+B\right]\cos\Theta,$$

A, B, C constantes de integración.

3. Las condiciones de simetría y las condiciones en el extradós y en el intradós traen como consecuencia las relaciones

$$a_1=\beta_1=b_1=\alpha_1=0, \qquad \alpha_1=c_1=0,$$

$$a_0=\frac{p'-p-2c_0\log\frac{R}{R'}}{R'^2-R^2}R^2R'^2$$

$$b_0=\frac{-pR^2+p'R'^2-c_0[R^2(2\log R+1)-R'^2(2\log R'+1)]}{2(R^2-R'^2)}.$$

Las condiciones $u=v=0$ en los puntos A_1, A_2 de las impostas, siendo $2\Theta_0$ la apertura del arco son

$$(7)\qquad -\frac{a_0}{2\mu r}+\left(\frac{b_0}{\lambda'+\mu}-\frac{c_0}{2\mu}\right)r+\frac{c_0}{\lambda'+\mu}r\log r\mp$$
$$\mp\frac{\lambda'+2\mu}{2\mu(\lambda'+\mu)}\delta_1\left(\frac{\pi}{2}\mp\Theta_0\right)\operatorname{sen}\Theta_0\pm A\operatorname{sen}\Theta_0$$
$$+\left[\frac{\delta_1}{2(\lambda'+\mu)}\log r+\frac{\mu-\lambda'}{2\mu(\lambda'+\mu)}d_1r^2+\frac{\gamma_1}{2\mu r^2}+B\right]\cos\Theta_0=0,$$

$$(8)\quad Cr+\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}c_0r\left(\frac{\pi}{2}\mp\Theta_0\right)+\frac{\lambda'+2\mu}{2\mu(\lambda'+\mu)}\delta_1\left(\frac{\pi}{2}\mp\Theta_0\right)\cos\Theta_0$$
$$-A\cos\Theta_0\pm\left[\frac{\delta_1}{2(\lambda'+\mu)}\log r-\frac{3\lambda'+5\mu}{2\mu(\lambda'+\mu)}d_1r^2-\frac{\gamma_1}{2\mu r^2}+\right.$$
$$\left.+\frac{\delta_1}{2\mu}+B\right]\operatorname{sen}\Theta_0=0.$$

Sumando y restando y reemplazando a_0, b_0, etc. por las expresiones en las presiones encontramos las 4 relaciones

(9)

$$-\frac{p'-p}{(R'^2-R^2)2\mu r}R^2R'^2+\frac{-pR^2+p'R'^2}{2(R^2-R'^2)}\frac{r}{\lambda'+\mu}+c_0\left\{\frac{\log\frac{R}{R'}R^2R'^2}{\mu r(R'^2-R^2)}\right.$$

$$\left.-\frac{R^2(2\log R+1)-R'^2(2\log R'+1)}{2(R^2-R'^2)}\frac{r}{\lambda'+\mu}+\frac{r\log r}{\lambda'+\mu}-\frac{r}{2\mu}\right\}+$$

$$+d_1\left\{-\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}\Theta_0\operatorname{sen}\Theta_0(R^2+R'^2)\right.$$

$$\left.+\cos\Theta_0\left[-\frac{R^2+R'^2}{\lambda'+\mu}\log r+\frac{\mu-\lambda'}{2\mu(\lambda'+\mu)}r^2-\frac{R^2+R'^2}{2\mu r^2}\right]\right\}+B\cos\Theta_0=0,$$

(10) $$\operatorname{sen}\Theta_0\left\{d_1\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}-\frac{\pi}{2}(R^2+R'^2)+A\right\}=0,$$

(11) $$Cr+\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}c_0r\frac{\pi}{2}-d_1\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}(R^2+R'^2)\frac{\pi}{2}\cos\Theta_0-$$

$$-A\cos\Theta_0=0,$$

(12) $$-c_0\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}r_0\Theta_0+d_1\left\{\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}(R^2+R'^2)\Theta_0\cos\Theta_0+\right.$$

$$+\operatorname{sen}\Theta_0\left[-\frac{\log r}{\lambda'+\mu}(R^2+R'^2)-\frac{3\lambda'+5\mu}{2\mu(\lambda'+\mu)}r^2+\frac{R^2R'^2}{2\mu r^2}-\right.$$

$$\left.\left.-\frac{R^2+R'^2}{\mu}\right]\right\}+B\operatorname{sen}\Theta_0=0.$$

4. Ahora vamos a considerar la condición (1) del momento nulo de las tensiones Q con respecto a los puntos A_1 y A_2.

Tenemos

$$\int_{R'}^{R}\left\{-\frac{a_0}{\rho^2}+2b_0+c_0(2\log\rho+3)\right.$$

$$\left.+\operatorname{sen}\Theta\, d_1\left[6\rho-\frac{2R^2R'^2}{\rho^3}-\frac{2(R^2+R'^2)}{\rho}\right]\right\}d\rho$$

$$= \frac{a_0}{\rho} + 2b_0\rho + c_0\,(2\rho \log \rho + 3\rho - 2\rho) + \operatorname{sen}\Theta\, d_1\,[3\rho^2 + \frac{R^2R'^2}{\rho^2} - 3\,(R^2 + R'^2) \log \rho]\,\Big|_{R'}^{R},$$

$$\int_{R'}^{R} \Big\{ -\frac{a_0}{\rho} + 2b_0\rho + c_0\,(2\rho \log \rho + 3\rho) + \operatorname{sen}\Theta\, d_1\,[6\rho^2 - \frac{2R^2R'^2}{\rho^2} - 2\,(R^2 + R'^2)]\, d\rho = -a_0 \log \rho + b_0\rho^2 + c_0\,(\rho^2 \log \rho + \rho^2) + \operatorname{sen}\Theta\,.\,d_1\,.\,[2\rho^3 + \frac{2R^2R'^2}{\rho} - 2\,(R^2 + R'^2)\,\rho]\,\Big|_{R'}^{R}.$$

Resulta la condición

$$(13)\quad r\left[\frac{a_0}{\rho} + 2b_0\rho + c_0\,(2\rho \log \rho + \rho) + \cos\Theta_0\, d_1\left(3\rho^2 + \frac{R^2R'^2}{\rho^2} - 2 \log \rho\;(R^2 + R'^2)\right)\right] - [-a_0 \log \rho + b_0\rho^2 + c_0\rho^2\,(\log \rho + 1) + \cos\Theta_0\,.\,d_1\,(2\rho^3 + \frac{2R^2R'^2}{\rho} - 2\,(R^2 + R'^2)\,\rho)]\,\Big|_{R'}^{R} = 0,$$

o sea

(14)

$$r\Big\{a_0\,\frac{R'-R}{RR'} + 2b_0\,(R - R') + c_0\,[2R \log R + R - 2R' \log R' - R'] + \cos\Theta_0\, d_1\,[3R^2 - 3R'^2 + R^2 R'^2\;\left(\frac{1}{R^2} - \frac{1}{R'^2}\right) - 2 \log R\,(R^2 + R'^2) + 2 \log R'\;(R^2 + R'^2)]\Big\} - \Big\{-a_0 \log \frac{R}{R'} + b_0\,(R^2 - R'^2) + c_0\,[R^2\,(\log R + 1) - R'^2\,(\log R' + 1)] + \cos\Theta_0\, d_1\,[2\,(R^3 - R'^3) + 2R^2 R'^2\,\frac{R'-R}{RR'} - 2\,(R^2 + R'^2)\,(R - R')]\Big\} = 0,$$

o todavía

$$c_0\left\{-\frac{2\log\frac{R}{R'}}{R'^2-R^2}R^2R'^2\frac{R'-R}{RR'}r-\frac{R^2(2\log R+1)-R'^2(2\log R'+1)}{R^2-R'^2}\right.$$

$$(R-R')r+(2R\log R+R-2R'\log R'-R')r-\frac{2\log\frac{R}{R'}}{R'^2-R^2}R^2R'^2\log\frac{R}{R'}+$$

$$+\frac{R^2(2\log R+1)-R'^2(2\log R'+1)}{2(R^2-R'^2)}(R^2-R'^2)-R^2(\log R+1)+$$

$$\left.R'^2(\log R'+1)\right\}+d_1\cos\Theta_0\left\{r\left[2R^2-2R'^2-2\log\frac{R}{R'}(R^2+R'^2)\right\}\right.$$

$$=-\frac{p'-p}{R'^2-R^2}R^2R'^2\frac{R'-R}{RR'}r-\frac{-pR^2+p'R'^2}{R^2-R'^2}(R-R')r-$$

$$\frac{p'-p}{R'^2-R^2}R^2R'^2\log\frac{R}{R'}+\frac{-pR^2+p'R'^2}{2(R^2-R'^2)}(R^2-R'^2),$$

$$c_0\left\{-2\log\frac{R}{R'}\frac{RR'}{R+R'}r+2(R\log R-R'\log R')r\right.$$

$$\left.-\frac{2(R^2\log R-R'^2\log R')}{R+R'}r-2\log^2\frac{R}{R'}\frac{R^2R'^2}{R^2-R'^2}-\frac{1}{2}(R^2-R'^2)\right\}$$

$$+2d_1\cos\Theta_0\left\{r(R^2-R'^2)-r(R^2+R'^2)\log\frac{R}{R'}\right\}$$

$$=-\frac{p'-p}{R'^2-R^2}R^2R'^2\log\frac{R}{R'}-\frac{pR^2-p'R'^2}{2}$$

$$+\frac{r}{R+R'}(-p'RR'+pRR'+pR^2-p'R'^2).$$

Tenemos

$$(R\log R-R'\log R')(R+R')-(R^2\log R-R'^2\log R')$$
$$-RR'(\log R-\log R')=0$$

y resulta

$$(15)\qquad c_0\left\{-2\frac{R^2R'^2}{R^2-R'^2}\log^2\frac{R}{R'}-\frac{1}{2}(R^2-R'^2)\right\}$$
$$+2d_1r\cos\Theta_0\left\{R^2-R'^2-(R^2+R'^2)\log\frac{R}{R'}\right\}$$

$$= -\frac{d-_{\prime}d}{R'^2-R^2} R^2 R'^2 \log\frac{R}{R'} + r\,(pR - p'R') - \frac{pR^2-p'R'^2}{2}.$$

5. Multiplicando las relaciones (9), (12) por sen Θ_0, $\cos\Theta_0$ y restando, encontramos

$$\text{(16)}\qquad c_0\left\{\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)} r\Theta_0 \cos\Theta_0 + \operatorname{sen}\Theta_0 \frac{1}{2\mu(\lambda'+\mu)(R^2-R'^2)r}\right.$$

$$\left[2\mu R^2 r^2 \log\frac{r}{R} + 2\mu R'^2 r^2 \log\frac{R'}{r} + 2\,(\lambda'+\mu)\,R^2 R'^2 \log\frac{R}{R'} -\right.$$

$$\left.\left. -(\lambda'+2\mu)\,r^2\,(R^2-R'^2)\right]\right\} + d_1\left\{-\frac{\lambda'+2\mu}{\mu(\lambda'+\mu)}\Theta_0\,(R^2+R'^2) +\right.$$

$$\left. + \operatorname{sen}\Theta_0 \cos\Theta_0 \left[\frac{\lambda'+3\mu}{\mu(\lambda'+\mu)} r^2 - \frac{R^2R'^2}{\mu r^2} + \frac{R^2+R'^2}{\mu}\right]\right\}$$

$$= \operatorname{sen}\Theta_0\left\{\frac{(p'-p)R^2R'^2}{(R'^2-R^2)2\mu r} - \frac{-pR^2+p'R'^2}{2(R^2-R'^2)} \quad \frac{r}{\lambda'+\mu}\right\}.$$

En efecto, tenemos

$$\frac{\log\frac{R'}{R} R^2R'^2}{\mu r(R'^2-R^2)} - \frac{R^2(2\log R+1)-R'^2(2\log R'+1)}{2(R^2-R'^2)} \quad \frac{r}{\lambda'+\mu} + \frac{r\log r}{\lambda'+\mu} -$$

$$-\frac{r}{2\mu} = \frac{1}{2\mu(\lambda'+\mu)(R^2-R'^2)r}\left\{2\mu R^2 r^2 \log\frac{r}{R} + 2\mu R'^2 r^2 \log\frac{R'}{r} +\right.$$

$$\left. + 2(\lambda'+\mu)\,R^2 R'^2 \log\frac{R'}{R} - (\lambda'+2\mu)\,r^2\,(R^2-R'^2)\right\}.$$

Las dos ecuaciones (15) y (16) permiten determinar c_0 y d_1 si el determinante es diferente de cero.

Q se determina de (3)

$$Q = -\frac{p'-p}{R'^2-R^2} R^2 R'^2 . \frac{1}{\rho^2} + \frac{-pR^2+p'R'^2}{R^2-R'^2} +$$

$$+ c_0\left\{\frac{2\log\frac{R}{R'} . R^2R'^2}{(R'^2-R^2)\rho^2} - \frac{R^2(2\log R+1)-R'^2(2\log R'+1)}{R^2-R'^2} +\right.$$

$$\left. + 2\log\rho + 3\right\} + \cos\Theta_0 d_1\left(6\rho - \frac{2R^2R'^2}{\rho^3} - \frac{2(R^2+R'^2)}{\rho}\right).$$

6. En el caso particular $\Theta_0 = \frac{\pi}{2}$, se puede inmediatamente calcular c_0

(17)
$$c_0 = \frac{-\frac{p'-p}{R'^2-R^2} R^2 R'^2 \log \frac{R}{R'} + r(pR - p'R') - \frac{pR^2 - p'R'^2}{2}}{-2\log^2 \frac{R}{R'} \cdot \frac{R^2 R'^2}{R'^2 - R^2} - \frac{1}{2}(R^2 - R'^2)},$$

(18)
$$d_1 = \frac{-2\mu(\lambda'+\mu)}{\pi(\lambda'+2\mu)(R^2-R'^2)} \left\{ \left[\frac{(p'-p)R^2R'^2}{(R'^2-R^2)2\mu r} - \right.\right.$$
$$\left. - \frac{-pR^2+p'R'^2}{2(R^2-R'^2)} \; \frac{r}{\lambda'+\mu} \right] - \frac{1}{2\mu(\lambda'+\mu)(R^2+R'^2)r} \cdot$$
$$\cdot [2\mu R^2 r^2 \log \frac{r}{R} + 2\mu R'^2 r^2 \log \frac{R'}{r} + 2(\lambda'+\mu) R^2 R'^2 \log \frac{R'}{R} -$$
$$- (\lambda' + 2\mu) r^2 (R^2 - R'^2)].$$
$$\left. \frac{-\frac{p'-p}{R'^2-R^2} \cdot R^2 R'^2 \log \frac{R}{R'} + r(pR - p'R') - \frac{pR^2-p'R'^2}{2}}{-2\log^2 \frac{R}{R'} \; \frac{R^2 R'^2}{R^2 - R'^2} - \frac{1}{2}(R^2 - R'^2)} \right\}.$$

7. En el caso particular de $p' = 0$ encontramos

(19)
$$Q = -p \frac{R^2(R'^2+\rho^2)}{(R^2-R'^2)\rho^2} + p \frac{R^2R'^2 \log \frac{R}{R'} + (R'^2-R^2)R \left(r - \frac{R}{2}\right)}{-2\log^2 \frac{R}{R'} \cdot R^2R'^2 + \frac{1}{2}(R^2-R'^2)^2} \cdot$$
$$\left\{ 2\log\rho + 3 + \frac{2\log\frac{R}{R'} \cdot R^2R'^2}{(R'^2-R^2)\rho^2} - \frac{R^2(2\log R+1) - R'^2(2\log R'+1)}{R^2-R'^2} \right\}.$$

En el intradós, para $\rho = R'$ tenemos

$$2\log R' + 3 + \frac{2\log\frac{R}{R'} \cdot R^2R'^2}{(R'^2-R^2)R'^2} - \frac{R^2(2\log R+1) - R'^2(2\log R'+1)}{R^2-R'^2} =$$
$$= \frac{1}{R^2-R'^2} \left\{ 2\log R' (R^2 - R'^2) + 2(R^2 - R'^2) - 2R^2(\log R - \right.$$

$$- \log R') - 2R^2 \log R + 2R'^2 \log R' \Big\} =$$

$$= \frac{1}{R^2 - R'^2} \left\{ 2(R^2 - R'^2) + 4R^2 \log \frac{R'}{R} \right\},$$

$$Q = -p \frac{2R^2}{R^2 - R'^2} +$$

$$+ p \frac{R^2 R'^2 \log \frac{R}{R'} + (R'^2 - R^2) R \left(r - \frac{R}{2} \right)}{-2 \log^2 \frac{R}{R'} \cdot R^2 R'^2 + \frac{1}{2} (R^2 - R'^2)^2} \cdot \frac{2(R^2 - R'^2) + 4R^2 \log \frac{R'}{R}}{R^2 - R'^2}.$$

Para $r = \frac{R + R'}{2}$ enncontramos

$$Q = -p \frac{R^2 - R'^2}{2R^2} + p \frac{RR' \left\{ RR' \log \frac{R}{R'} + \frac{1}{2} (R'^2 - R^2) \right\}}{\frac{1}{2} (R^2 - R'^2)^2 - 2R^2 R'^2 \log^2 \frac{R}{R'}} \cdot \tag{20}$$

$$\cdot \frac{2(2R^2 \log \frac{R'}{R} + R^2 - R'^2)}{R^2 - R'^2}.$$

Desarrollando según las potencias de $h = R - R'$ encontramos

$$\log \frac{R}{R'} = \frac{h}{R'} - \frac{h^2}{2R'^2} + \dots$$

$$R^2 - R'^2 - 2R^2 \log \frac{R}{R'} = h(2R' + h) - 2(R' + h)^2 \left(\frac{h}{R'} - \frac{h^2}{2R'^2} + \dots \right)$$

$$= 2R'h + h^2 - 2(R'^2 + 2R'h + h^2) \frac{h}{R'} + (R'^2 + 2R'h + h^2) \cdot \frac{h^2}{R'^2} + \dots$$

$$= h^2 - 4h^2 + h^2 + (h)_3 = -2h^2 + (h)_3,$$

$$RR' \log \frac{R'}{R} - \frac{1}{2} (R^2 - R'^2)$$

$$= R'(h + R') \left(\frac{h}{R'} - \frac{h^2}{2R'^2} + \frac{h^3}{3R'^3} - \dots \right) - \frac{1}{2} h (2R' + h) =$$

$$=h^2-\frac{h^2}{2}-\frac{h^3}{2R'}-\frac{h^2}{2}+\frac{h^3}{3R'}+\ldots=-\frac{h^3}{6R'}+(h)_4$$

$$\frac{1}{2}(R^2-R'^2)-2\log^2\frac{R}{R'}\,.\,R^2R'^2=$$

$$=\frac{1}{2}h^2(2R'+h)^2-2\left(\frac{h}{R'}-\frac{h^2}{2R'^2}+\frac{h^3}{3R'^3}-\ldots\right)^2R'^2(R'+h)^2$$

$$=\frac{h^2}{2}(4R'^2+4R'h+h^2)$$

$$-2\left(\frac{h^2}{R'^2}-\frac{h^3}{R'^3}+\frac{h^4}{4R'^4}+\frac{2h^4}{3R'^4}-\ldots\right).\,R'^2(R'^2+2hR'+h^2)=$$

$$=2h^2R'^2+2R'h^3+\frac{h^4}{2}-2h^2R'^2+2h^3R'-\frac{h^4}{2}-\frac{4}{3}h^4-$$

$$-4h^3R'+4h^4-2h^4=\frac{2}{3}h^4,$$

$$Q=-p\frac{2R^2}{R^2-R'^2}-p\frac{h^3}{6R'}.\,RR'.\,2\,.-\frac{3h^2}{h^4(R^2-R'^2)}.\left\{1+(h)\right\}$$

$$=-p\frac{2R^2}{R^2-R'^2}+\frac{phR}{R^2-R'^2}+(h),$$

con lo que resulta

(21) $$Q=-p\frac{R}{R-R'}+(h).$$