ON VIBRATIONAL VORTICES

by

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§ 1. Let be

$$\rho,\ p,\ \vec{v}$$

density, pressure and velocity in a fluid; we assume besides the equations

$$(1) \qquad \rho \frac{d\vec{v}}{dt} = -\operatorname{grad} p, \qquad (2) \qquad \frac{d\rho}{dt} = -\rho \operatorname{div} \vec{v}$$

a so-called state-equation:

$$(3) \qquad p = p(\rho),$$

for incompressible fluids:

$$(3a) \qquad \frac{d\rho}{dt} = 0, \text{ or } \operatorname{div} \vec{v} = 0,$$

for perfect gases:

$$(3b) \qquad p = R\rho T$$

where R is a constant and T the absolute temperature. We must confess that we have to do with ideal cases, when we apply the equations (3a) or (3b); the state equation is without any doubt much more complicated, if the ideal conditions of incompressible fluids or perfect gases are not satisfied.

For motions with great velocities or great accelerations good results have been obtained even in compressible fluids, e. g. in air, in aerodynamics with the assumption that the conditions of incompressibility (3a) can be maintained approximately, so there is a certain probability for the truth of the hypothesis, that in vibrational motions

$$\vec{v} = \vec{v}_0 + \vec{v}_1 \cos(\nu t) + \vec{v}_2 \sin(\nu t) \tag{4}$$

the conditions

$$\begin{cases} \operatorname{div} \vec{v}_1 = 0, \\ \operatorname{div} \vec{v}_2 = 0 \end{cases} \tag{5}$$

will be satisfied with so much greater approximation, the greater the frequency ν is.

For the applications I have in mind the integrals over the space filled with the fluid

$$\frac{1}{2}\int \rho\, v_0^2\, d\tau, \quad \frac{1}{4}\int \rho\, v_1^2\, d\tau, \quad \frac{1}{4}\int \rho\, v_2^2\, d\tau$$

may be comparable with ordinary energies, and we suppose that the first derivatives of

$$\vec{v}_0, \quad \vec{v}_1, \quad \vec{v}_2$$

with respect to the time t, and the first derivatives of $\vec{v}_1$ and $\vec{v}_2$ with respect to x, y, z are negligeable in comparison with $\nu\vec{v}_0$, $\nu\vec{v}_1$, $\nu\vec{v}_2$.

Quite generally the vortex equations

$$\frac{d}{dt}\left(\frac{\operatorname{curl} \vec{v}}{\rho}\right) = \frac{1}{\rho}(\operatorname{curl} \vec{v} \,.\, \operatorname{grad})\, \vec{v} \tag{6}$$

which are consequences of (1), are non-linear differential equations, but in the case of vibrational motions of high frequency they can be reduced to linear equations in a first approximation.

The main terms on the left hand side of the vortex-equations (6) will be:

$$\frac{\nu}{\rho}\left\{-\operatorname{curl}\vec{v}_1 \sin(\nu t) + \operatorname{curl}\vec{v}_2 \cos(\nu t)\right\}.$$

In order to get such large terms on the right hand side, the first derivatives of $\vec{v}_0$ with respect to x, y, z must be very great, so these velocities must have wave character with very small wave-lengths λ, if we have really vibrational vortices, so that the $\operatorname{curl}\vec{v}_1$ and $\operatorname{curl}\vec{v}_2$ are not all zero.

E. g. for waves proceeding from a point (ξ, η, ζ) the vector $\vec{v}_0$ may contain additive terms of the form:

$$\overrightarrow{V_0} \cos \frac{r}{\lambda} 2\pi$$

r being the distance of the variable point (x, y, z) from (ξ, η, ζ) and the first derivatives of $\overrightarrow{V_0}$ not comparable with

$$\frac{1}{\lambda}\overrightarrow{V_0},$$

where λ is extremely small. So the first derivatives of this vector may become very large, and nevertheless we can maintain the assumption, that the integral over the space filled with the fluid

$$\frac{1}{2}\int \rho\, v_0^2\, d\tau$$

remains comparable with ordinary energies. The density ρ may also contain additive terms with wave character, waves proceeding from the point (ξ, η, ζ) with the wave length λ.

If we call — let us drop now the symbols of vector-analysis — $\mu; a, b, c$ the mean values of (1)

$$\rho; \quad \frac{1}{\rho}\left(\frac{\partial w_0}{\partial y}-\frac{\partial v_0}{\partial z}\right), \quad \frac{1}{\rho}\left(\frac{\partial u_0}{\partial z}-\frac{\partial w_0}{\partial x}\right), \quad \frac{1}{\rho}\left(\frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y}\right)$$

in small volumes the dimensions of which are of the order λ, then the vortex-equations will give us the relations:

$$\begin{aligned} \frac{\partial w_1}{\partial y}-\frac{\partial v_1}{\partial z} &= -\frac{\mu}{\nu}\left\{\frac{\partial u_2}{\partial x}a+\frac{\partial u_2}{\partial y}b+\frac{\partial u_2}{\partial z}c\right\}, \ldots; \\ \frac{\partial w_2}{\partial y}-\frac{\partial v_2}{\partial z} &= +\frac{\mu}{\nu}\left\{\frac{\partial u_1}{\partial x}a+\frac{\partial u_1}{\partial y}b+\frac{\partial u_1}{\partial z}c\right\}, \ldots, \end{aligned} \tag{7}$$

where a, b, c are very great, because they have the factor $\frac{1}{\lambda}$, and according to (5):

$$\left\{\begin{aligned} \frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z} &= 0, \\ \frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial w_2}{\partial z} &= 0. \end{aligned}\right. \tag{8}$$

We have differential equations for

$$u_1, \; v_1, \; w_1; \; u_2, \; v_2, \; w_2,$$

if the values of

$$\mu; \; a, \; b, \; c$$

can be determined beforehand.

The simplest case is

$$\mu = \text{const.}, \quad a = \text{const.}, \quad b = \text{const.}, \quad c = \text{const.}, \tag{9}$$

and then we come to the well-known problem of the determination of an electromagnetic field

(1) Eventually adding to curl $\bar{v}_0$ the gradient of a function Φ_0 which can be determined by initial conditions.

$$X,\ Y,\ Z;\ L,\ M,\ N$$

in the Maxwell-Hertz theory, respectively proportional with

$$u_1,\ v_1,\ w_1;\ u_2,\ v_2,\ w_2$$

in the case of an electron in (ξ, η, ζ) moving with constant velocities a, b, c.

The solution of more general problems can be found by superposition of these elementary solutions.

§ 2. The only possible mechanical theory of the electromagnetic field, which can be maintained in agreement with all the facts known up to date is based on the assumption of an ether with vibrational motions (4), in which

$$u_1,\ v_1,\ w_1$$

are resp. proportional to the

$$X,\ Y,\ Z$$

of the Maxwell-Hertz theory, and

$$u_2,\ v_2,\ w_2$$

proportional to the

$$L,\ M,\ N$$

of the Maxwell-Hertz theory. There would be no possibility to come to the differential equations

$$\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}=\frac{1}{c}\frac{\partial L}{\partial t},\ \ldots;\quad \frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}=-\frac{1}{c}\frac{\partial X}{\partial t},\ \ldots$$

of the electromagnetic field, if there were full incompressibility of the ether. This difficulty has been removed here by the assumption, that the vectors $\vec{v}_1$ and $\vec{v}_2$ have to satisfy the condition of incompressibility, whilst the vector $\vec{v}_0$ has not to satisfy the condition of incompressibility, and that it will be of wave-character, if we have not to do with stationary fields.

A special feature of this theory is the assumption, that a moving electron which is considered as a pulsating little sphere (pulsation of the sphere means periodic change of its radius) must rotate with a high speed around an axis parallel to the direction of the motion, the speed of rotation being proportional to the velocity of the electron.

In a stationary electromagnetic field there are no vibrational vortices in the ether; the vector $\vec{v}_1$ produced by electrons at rest is derived from a potential function similar to the velocity potential of a liquid, in which pulsating spheres are imbedded. The vector $\vec{v}_2$ produced by electric currents stationary in closed circuits, that is electrons moving in these circuits, the positive electrons in one direction, the negative electrons moving in the opposite direction, is the vector- potential of these currents. The reason why we have a vibrational vector-potential, is that the rotating electrons are pulsating at the same time; therefore whilst the vector $\vec{v}_0$ produced by stationary electric currents is zero, the vibrational vector-potential produced by stationary currents is produced with the same phase by the positive electrons moving in one direction and the negative electrons in the opposite direction. So notwithstanding the enormous speed of rotation of the moving electrons we avoid the difficulty of enormous energies of the kind

$$\frac{1}{2}\int \rho\, v_0^2\, d\tau.$$

As soon as we have to do with non stationary electromagnetic fields, we must have vibrational vortices for the vectors

$$\vec{v}_1 \text{ rsp. } \vec{v}_2,$$

and for these general cases considerations, as in § 1, are necessary. One important question may be answered here, which will quite naturally arise:

If the density ρ and the vector $\vec{v}_0$ contain additive terms of wave-character with an exceedingly small wave-length λ, it seems according to the vortex-equations, as if also the vectors $\vec{v}_1$ and $\vec{v}_2$ must contain terms of wave-character of the same exceedingly small wave-length, which we have to consider very small in comparison with the wave-lengths of all hitherto-known wave-lengths, and the mean values of these most important terms in small volumes with dimensions of the order λ would be zero.

That this is not necessarily the case, can be seen in our example of the electron moving with constant velocities (a, b, c). In this case — it is true — the vector $\vec{v}_0$ will contain an additive term of wave-character with the wave-length λ

$$\vec{V}_0 \cos \frac{r}{\lambda} 2\pi,$$

but also ρ will have wave-character

$$\rho = \frac{\mu}{1 + k \sin \frac{r}{\lambda} 2\pi} \quad (1),$$

and the mean values of

$$\rho, \frac{1}{\rho} \operatorname{curl} \vec{v}_0 \quad (2)$$

will be

$$\mu; \; a, \; b, \; c,$$

which can be constants in the most simple case.

For more general problems the question can be answered by superposition of these elementary solutions.

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(1) The derivatives of $\vec{v}_0$ k and μ with respect to x, y, z being small in comparison with

$$\frac{1}{\lambda} \vec{v}_0, \; \frac{k}{\lambda}, \; \frac{\mu}{\lambda},$$

(2) V. remark on previous page.