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ABOUT LACON'S FOUNDATIONS OF GEOMETRY IN 1881: AN UNKNOWN ATTEMPT BEFORE HILBERT

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RESUMEN

Este trabajo analiza los Fundamentos de Geometría *(1881) del matemático griego Vassilios Lacon (1830-1900) en el contexto de la formación de la comunidad matemática griega. Rector de la Universidad de Atenas y Director del Departamento de Filosofía, sus textos para la enseñanza secundaria, su labor como docente universitario y su profundo conocimiento de las principales líneas de avance de las matemáticas de su tiempo, especialmente en geometría, se sitúan en el origen del desarrollo de las matemáticas en la Grecia moderna y marcan el inicio de la investigación en la primera escuela matemática griega.*

ABSTRACT

This paper studies Lacon's Foundations of Geometry (1881) within the framework of the formation of the Greek mathematical community. Vassilios Lacon (1830-1900), Dean of the Department of Philosophy and Rector of the Athenian University, Professor of Mathematics, author of textbooks for secondary education, with his profound knowledge of the main trends of mathematical development, especially in Geometry, insured the formation of the new generations and the start of serious mathematical research in modern Greece.

Palabras clave: Matemáticas, Geometría, Grecia, Siglos XIX-XX, Vassilios Lacon (1830-1900).

Keywords: Mathematics, Geometry, Greece, 19th-20th Centuries, Vassilios Lacon (1830-1900).

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I. Introduction

When we started our research on the first mathematical schools[1] in Greece, we already knew the important work in algebra and in geometry of Kyparissos Stephanos (1857-1917)[2], a student of Hermite in Paris and a very active researcher[3], as well as that of Ioannis Hadjidakis (1844-1921)[4] in differential geometry, a part of which Blaschke and Reichardt referred to in their book[5]. From 1869 to 1873 Hadjidakis attended Kronecker's, Kummer's and Weierstrass's courses in Berlin and D. Hilbert favorably reviewed one of his first papers[6].

Also remarkable was the mathematical contribution of their students[7], who formed the second mathematical school in Greece: the multi-faced Nikolaos Hadjidakis (1873-1941)[8] contribution to differential geometry[9]. He attended the lectures of Darboux, Klein, Hilbert and Schwarz; the prolific Georges Remoundos'[10] (1878-1928) work in analysis, theory of functions and differential equations. Remoundos studied at the École Normale Supérieure and was a postgraduate student of Picard. Finally, there was the philosophic Panayiotis Zervos (1878-1952)[11], who followed the lectures of Hadamard, Poincaré and Picard and whose paper on Monge's problem[12] remains until today a classical reference.

During our difficult research in the Archives of the Athenian University[13], we were persuaded that in the origin of the development of mathematics in Greece a wide-ranging mathematician, a methodic teacher, a profoundly cultivated person was «hidden». And indeed our conviction was justified. The «hidden» personality who —after the first heroic years of the University— insured the formation of the new generations with his profound knowledge and who contributed with his interventions to preserve the calm climate[14] as Dean of the Department of Philosophy[15] and as Rector of the University was Vassilios Lacon (1830-1900), professor of mathematics, whose name —as well as all his scientific activities— remains even today in the background.

II. Short biographical sketch

Originating from the island of Kea (one of the Cycladics islands nearest to Athens), he graduated in the Philosophy Department with honours, received the first doctorate in mathematics in 1850[16] (number 1 according to the Register of the University) and the prize of 100 drachmas (an important sum for that epoch). Lacon, who was really a good scientist[17], after his postgraduate studies in Paris (Sorbonne, 1851-1854) with Chasles[18] and Duhamel[19], returned to Athens with Serret[20] and became professor of mathematics in 1868 and taught there until the last years of his life.

During his academic life, he showed evidence of his qualifications as a methodical and advanced teacher (he taught non-Euclidean geometry in the academic years of 1894-95 and 1895-96)[21], as author of many didactic books —especially for secondary education[22]— and, foremost, as a gifted mathematician. The other side of his polyvalent personality was that of a lover of literature[23], whose papers in the journal Athena[24] mostly contributed to the elucidation of several delicate points of the ancient Greek classics and are still a challenge for an appropriate specialist to research.

His books[25] are very difficult to find, particularly his contribution to the Third Olympiad (1875)[26], where he presented the bibliography of physical and mathematical sciences of Greece, which seems to be lost. The only work from which we can judge Lacon as a mathematician[27] remains his inaugural speech as Rector of the University in 1881, published the same year that the *Foundations of Geometry*. This work, composed of 58 pages, was not an exact transcription of the speech; in all probability, he limited himself to the first pages, which contain the historical part and the contemporary concepts of geometry.

At the end of nineteenth century a trend of axiomatization of mathematical structures appeared, having as an incomparable model —even with criticism from the earliest commentators Proclus and Pappus to Clavius, to Grassmann etc.— Euclid's Elements. The existence and the development of non Euclidean geometry provoked the revision of the first axiomatic system[28]. So, mathematicians reexamined the nature of its axioms and worked on the rectification of Euclid's construction. Several mathematicians concentrated their efforts to put Euclid's Geometry on a foundation as solid as possible[29].

All efforts started with Moritz Pasch, who from 1873 lectured at Giessen University[30], and his *Vorlesungen über neue Geometrie* (published in 1882), a first exhibition of geometry, conformed to the rigorous rules of synthetic methods, and taking its highest form with the famous attempt of David Hilbert[31], whose lectures on Euclidean Geometry in Göttingen gave birth to his *Grundlagen der Geometrie* (1899)[32].

Meanwhile, in Athens, Vassilios Lacon influenced by the development of the non-Euclidean Geometry from Chasles' lectures and from Helmholtz's idea[33], presented his own *Foundations of Geometry* in 1881, the fruit of his research of many years.

III. Lacon's Historical Exposition

At the beginning of his work he presented the definitions of the fundamental concepts, so he started to establish the notions of principles.

Mathematics, as is every demonstrative science, noted Lacon, is a system of knowledge arising from proofs. But as a proof is reasoning or a system of reasoning, using the classic Aristotelian definition[34], he concludes that obviously every demonstrative science originates from some unproved data.

In Geometry, Lacon remarks that the first notions are those of extension, boundary of a solid, surface, straight line, point[35], part and whole, continuity, position. Therefore, the unproved propositions are axioms[36] or common notions in Euclid and postulates. The second step for Lacon is to focus his attention on the clear distinction between axioms[37],[38] and postulates based on Geminus' and Proclus' conceptions, but the distinguishing of these delicate notions was not unanimously accepted, as Euclid's «manuscripts» had been copied several times.

Finally he adopted Geminus' concept in which an axiom is an obvious and unproved knowledge and postulate[39], and started his exposition to present the foundations of geometry. Now he was able to research the nature and the number of the principles of Geometry, taking into consideration that the principles must be as simple and clear[40] as possible, none of them must be omitted and all of them must be included in the proofs.

Lacon emphasizes that Euclid's Elements[41] is an admirable construction[42], but lacks certain principles. Indeed, Euclid's Elements started with a series of definitions, which are not in reality principles as they do not express the essence of the things, they do not correspond to the synthesis of intelligible elements[43].

The definitions[44] of straight line and that of a plane are vague[45]. The concepts of equality or inequality are not defined as first notions, and some axioms, as we are going to prove —continued Lacon—, are considered by Euclid in his proofs without clearly explaining if they are axioms or postulates, but probably all this is due to his many revisers[46]. It is quite sure that even in antiquity Euclid's Elements[47] was not the only treatise on the Foundations of Geometry. Lacon continued that Apollonius of Perga wrote a treatise on axioms[48] and it is known that he proved the axiom: things that are equal to the same thing are also equal to one another. In this way Apollonius calls equal only things which can coincide. Following Proclus, Heron minimizes the number of axioms to three, while Pappus maximizes Euclidean axioms. Therefore, Lacon stressed that many inaccurate works had been written on the Foundations of Geometry.

Continuing the historical framework of his paper[49], Lacon referred to the 5th Euclidean postulate and its consequences. First of all[50], he mentions Lobatchevskii, who in 1829[51], as Lacon quotes, discovered a new geometry, the imaginary[52] one «which Gauss defined as non-Euclidean and Bolyai as absolute». Besides, Lacon revealed to his audience —the elite of the Greek intellectual community and the authorities of the country headed by King George the First—, probably for the first time in Greece, Riemann's important achievement[53] on the Foundations of Geometry[54], as well as Helmholtz's contribution[55].

IV. Lacon's criticism on the books on Geometry

Until that time, Lacon mentioned that the books of Geometry were incomplete, even Legendre's book, which was one of the most important and in great use in Greece and in other countries[56]. In the beginning of the book there are five axioms from which only one, the fourth, is a real axiom or rather a postulate. Lacon said that Blanchet, reviser of Legendre's Geometry, understood it well, and omitted the other four axioms, but in this way he committed the opposite error presenting very few, because the rest are hidden in the definitions or are omitted.

For many years, Lacon declared that he studied many books and worked to conceive and write a complete system on the Foundations of Geometry, so as to prove the existence of the plane, whose definition, according to Duhamel, contained infinity of conditions probably incompatible. About the proof of the plane, and apart from Bolyai and Lobatchevskii, whose works were not included in the teaching books, Lacon quotes some well-known mathematicians who had studied this question such as Crelle[57], Gerling[58], Baltzer[59], and Duhamel[60].

V. Lacon's axiomatic system

After all these interesting and recent references[61], which revealed Lacon's erudition on the subject, he started to develop his ideas on the Foundations of Geometry and stated the first notions, whose usual definitions are more explications than real definitions, and 17 axioms and postulates —not including those which are logical axioms—. Lacon constructed his own system, starting, of course, from Euclid, whose comprehension on the new geometrical ideas and philosophical conceptions he «corrects». Here we can see the relation of these constructions to the ideas of ancient Greek philosophy.

Lacon starts from the notion of solid. Then, a surface is a boundary of a solid[62], a line a boundary of a surface[63], and a point the extremity of a line[64] that has no part[65]. Here we can see Lacon's affinity with Lobatchevskii, who constructed his *Foundations of Geometry* (1829) starting with the propositions of geometrical solids, which he considered as the abstractions of solids which move and put in his Foundations the solid's notion, the contact's relation of solid and those of their sections: «The contact, says Lobatchevskii[66], is the characteristic property of solids and gives them the title of geometrical beings when we bear in mind this quality of solids without taking into consideration other qualities, essential or accidental». This initial conception is not determined by others simpler; it is not the consequence of others simpler and we «directly received it in nature by our senses»; this conception «is not derivative of another. This is why it must not be interpreted». From these first notions, Lobatchevskii determined the notions of surface, line and point.

This approach, which appeared in Lacon's text, exists naturally in the background of Helmholtz's ideas, which were one of the starting points of his exposition.

Lacon gives an interesting definition, that of geometric figure[67], in two steps (definition 7 and 8), as a distant echo of a Platonic one in his dialogue *Menon*, where a geometrical figure is defined on the concept of boundary considering surfaces, lines and points as themselves independently of solids, surfaces with boundaries[68].

Now we present his text of axioms, postulates and definitions, without exposing the proofs. The translation is, sometimes, somewhat free. The comments are ours.

Axiom I

Any figure can change place without changing itself and in a way that a point of it can take the place of any other point[69].

Comment: Lacon's approach reminds very intensively of Helmholtz's ideas in his foundations of geometrical system, in which he poses the set of movements in the space. Axiomatizing the known qualities of these movements, Helmholtz received the expressions of the elements of length and consequently of the distance between two points. This significant idea was strictly developed by Sophus Lie (Leipzig *Berichte* 1886) and was only indicated by Lacon.

Definition I

Two figures are called equal if they can take such a place that any point of the one can become a point of the other.

If the figure A is part or equal to a part of a figure B, A is called smaller than B and B greater than A.

Axiom 2

The figures cannot be, at the same time, equal and unequal, i.e. than in one way of superposition the one can apply on the other and in another way, not.

Axiom 3

All the positions that a figure can take can all take when we consider it as connected with another figure and forming with it a new figure.

Comment: Silently Lacon accepts that this figure is not deformed. From Riemann the transposition of a figure without deformation satisfying it, which later Helmholtz named axiom of liberal mobility.

Axiom 4

A line connecting a point lying in a solid with a point out of it necessary meets the surface of the solid.

Comment: It is one of the first tentative of separation, the axiom of division of the space by close surfaces in two domains (interior-exterior). We must stress J. de Tilly's attempt (*Mémoires de la Société des Sciences physiques et naturelles de Bordeaux, 3, 2ème série,* 1897, p. 107).

Postulate I

Given any two points of a figure, arbitrarily chosen but then fixed, the figure can move in a way that after taking always new positions, it will finally come to its first one. The positions that it had taken are the only ones that, when the above mentioned two points remain fixed, this solid can take.

Comment: As it follows from the commentary of this postulate given by Lacon himself, this concerns the free mobility of solid figures without any change of the shape in all parts of the space. One of the most principal restrictions of Helmholtz, which the German scholar put himself connecting to the invariance of Riemann expression[70] if $ds^2 = \sum_{i,j} g_{ij} dx_i dx_j$ (where $g_{ij} = g_{ji}$ are continuous) or this one of the rectangular coordinates[71] $ds = \sqrt{dx^2}$ is also request ds^2.

Postulate 2

From all the lines that can relate two given points A and B, there is one, and only one, that when a figure containing this line turns, with the points A and B fixed, remains immovable. This line is called **straight line**.

Postulate 3

The points through which can pass a straight line AB (segment) when it turns around A are the points of a solid. This solid is called a **sphere (**ball), A is called its **center**.

Axiom 5

If two unequal straight lines are given, there are always multiples of the smaller that are greater than the greater[72].

Postulate 4

There is a straight line depassing the straight line, which joints two arbitrary points of a finite figure.

Axiom 6

If two angles $A\hat{B}C$ $D\hat{E}F$ apply the one on the other, with the side ED applying on the side BA and EF on BC, they can apply in other ways, EF on BA and ED on BC.

Postulate 5

If an angle turns around one of its sides, the other side describes a surface. This surface is without gaps, i.e. its ending in a single closed line.

Postulate 6

If an angle turns around the one of its sides, each point of the other side describes a line.

Postulate 7

Two surfaces without gaps, which cannot apply the one on the other, if they can be connected along the same closed line they form the boundary of a solid.

Postulate 8

A closed line on the surface of a sphere divides it in two parts, each of which has this line as a boundary.

Comment: As in the case of the axiom 4 (see above), we can consider this postulate as one of the first tentative of separation of the proposition about the division of the surface by close curves in two domains' (interior-exterior).

Postulate 9

Euclid's on the parallels.

Comment: Lacon remains faithful to the 5th Euclidean postulate.

Postulate 10

If a magnitude becomes greater, by extensions, but has some upper bound, there is a limit of it.

Comment: This proposition of Lacon shows the possibility of analytical development of this topic, which he mentioned in his commentary of the first postulate (Riemann's expression).

Postulate 11

By adding equal surfaces or solids, one cannot create two figures, the one of which will be part of the other.

In Lacon's work there are some theorems with their demonstrations.

VI. Final Remarks

In the chain of research concerning the Foundations of Geometry in the last third of the 19th century which was marked by Helmholtz (1868), Klein (1872), Pasch (1882), Lie (1886), Poincaré (1887), Peano (1889), and which was crowned by the famous Hilbertian *Grundlagen,* Lacon's appointment as professor at the University of Athens (1881), a boundary of cultural Europe, had a modest style. Nevertheless, it is remarkable that his activities belong to the principal direction in the development of mathematics, although he probably did not know the Erlanger Program (we doubt if he could find it in Athens). The famous 1872 Klein's Erlanger Program[73] became internationally known much later[74], when it was translated into French, English and Italian in the early 1890's. He had correctly evaluated the perspective of the road chosen by Helmholtz and he tried to follow it, though faintly, as he had a vague conception in his mind of group theory. Already Liouville, during the years 1844 to 1846, wrote on Galois Theory (see f. ex. *Exercices d' Analyse et de Physique Mathématique*, 1844, pp. 151-252). Serret also lectured on it at the Sorbonne. It is quite strange that Lacon did not profit from this theory during his stay in Paris, from which the axiomatization developed (we doubt if he knew group theory, although it became slowly very *à la mode* in the mathematical centers of Europe) and also, as we say today, the topological character of certain geometrical propositions. Lacon seems to follow

Klein's Erlanger Program, which argued that the geometry of position must be found before the geometry of metrics, and focused his attention on the geometry of position. Lacon is one of the first who distinguished the axiom of division of the space by close surfaces in two domains (interior-exterior).

All these presentiments —we can say— would be realized during the two next decades of the 19th century in the work of Lie, Poincaré, Peano, Enriques, Pierri, and lastly, Hilbert. Lacon's attempt reveals, on the one hand, the objective character of the development of mathematical ideas: he was a mathematician who could not have permanent connections with the most important mathematical centers, who could not often visit the well-established scientific libraries, and who had not the mathematical talent of Klein and Lie, but who correctly guessed the principal guide line for the development of these questions. On the other hand, Lacon's contribution stressed the start of serious mathematical research which would be marked in the near future by the results of mathematicians already known in Europe such as K. Stephanos (who favorably mentions Lacon's work in his Rector speech) and I. Hadjidakis.

Lacon was an able and learned mathematician possessing all the necessary capacity to understand important things found by other mathematicians; at the same time, he was a philologist conducting new research and critical study and, in conjunction, a man capable of having sufficiently deep (but revolutionary, as did Euclid and Hilbert) thoughts on the Foundations of Geometry. His titles of glory are that, although not a great creator mathematician like Stephanos and Hadjidakis, he saw all the fundamental importance of the works on non-Euclidean geometry, stressed in his inaugural Rector's speech their importance and scope and, with all this understanding, tried to formulate a complete system of axioms for Euclidean Geometry[75].

NOTES

1. On the initiative of professors S. Demidov and M. Hormigón, who organized a symposium in 1993 on the formation of mathematical schools in the 19th and 20th centuries, during the 19th International Congress of History of Science in Zaragoza. For the proceedings of this Symposium, see *Historic Math. Issledov, special issue, 2e série,* Moscow, 1997, 1-137.
2. Stephanos defended his Thesis, *Sur la théorie des formes binaires et sur l' élimination* having as president of the jury Charles Hermite. With it, and with his long memoir on «Substitutions», Stephanos became internationally known. Thus, he belonged, in research, to what could be called the «avant-guarde» of «Modern Algebra» of that time. However, many mathematicians admired him for his theorems in pure geometry

(see F. Klein's *Vorlesungen über höhere Geometrie*, N.Y., reprint 1949). The young D. Hilbert had many references to Stephanos, and Poincaré sent to him reprints of his works. Stephanos' name appeared in the honorary committees of international conferences and journals. There are references to Stephanos in contemporary works in pure mathematics. See my papers PHILI, Ch. (1999) «Kyparissos Stephanos and his paper on quaternions». *Acta historiae rerum naturalium necnon technicarum, 3 (New Series)*, Prague, 35-46; PHILI, Ch. (1997) «Sur le développement des Mathématiques en Grèce durant la période 1850-1950. Les fondateurs». *Ist. Math. Issl.*, *special issue, 2e série,* Moscow, 80-102.

3. For more details see my paper «Kyparissos Stephanos (1857-1917): Un mathématicien grec d'envergure entre deux pays et deux siècles» (to appear in *Revue d'Histoire des Mathématiques*, Société Mathématique de France).
4. Besides his important and advanced researches in Differential Geometry, Analysis and Algebra, he wrote textbooks, in Greek, covering the mathematical needs of that epoch from Primary School to the University. Especially, his books on Differential and Integral Calculus were excellent fundamental textbooks for many generations of Greek mathematicians. For more details see my paper: PHILI, Ch. (2000) «On Some Aspects of the Scientific Society in Athens at the End of XIX Century. Mathematics and Mathematicians». *Archives Internationales d'Histoire des Sciences*, *50 (145),* 302-320.
5. BLASCHKE, W & REICHARDT, H. (1960) *Einführung in die Differentialgeometrie zweite Auflage*. Springer, Berlin-Göttingen-Heidelberg, p. 57.
6. «Ueber einige Eigenschaften der Flächen mit constantem Krümmungsmass», *Jahrb. Der Math. Bd. XI*. (1881), 527-528.
7. Stephanos' and J. Hadjidakis' successors at the University of Athens were Nicolaos J. Hadjidakis —son of J. Hadjidakis—, G. Remoundos and P. Zervos. Their books have been superseded only by *The Infinitesimal Calculus* of the last one (1929), which introduced in Greece Set Theory, Dedekind cuts, and the fully «arithmetized» Calculus; this book was written with the strictest accuracy in the style of «dialogue», whose colloquial style obliged the reader to think. One of the ardent partisants of this book was the later, great Greek topologist C. D. Papakyriakopoulos (Athens, 1913-Princeton, 1976; Veblen prize 1964).
8. N. Hadjidakis was a poet, an excellent translator (mainly of Scandinavian literature) and a polyglot (he spoke more than twelve European languages fluently).
9. See for example: «Trois formules très générales relatives aux courbes dans l'espace», *C.R.Ac.Sci. Paris,* 1899; «Displacements Depending on One, Two, k Parameters in a Space of n Dimensions», *Amer. Journ. of Mathematics,* 1900; «Bemerkung zur Aufsatze von Herrn Kommerell: Ein Satz über geodätische Linien», *Arch. Math. und Physik*, 1902.
10. Remoundos' mathematical contribution appeared in many books of his epoch. See for example: BLUMENTHAL, O. (1907) *Principes de la théorie des fonctions entières d'ordre infini*, Paris; ZORETTI, L. (1910) *Leçons sur le prolongement analytique*. «Collection de monographies sur la théorie des fonctions publiée sur la direction de M. E. Borel». Paris, Gauthier-Villars; MONTEL, P. (1927) *Leçons sur les familles normales de fonctions analytiques et leurs applications*. «Collection Borel». Paris.

11. His philosophical papers remain even today an attractive object of research.
12. ZERVOS, P. (1932) *Sur le problème de Monge, Mémorial des Sciences Mathématiques*, Paris, Gauthier-Villars. See also my paper: PHILI, Ch. (2003) «Reflexions on Picard theorem and Monge's Problem on two Greek mathematicians: G. Remoundos and P. Zervos». *Acta Historiae rerum naturalium neucon Technicarum*, Prague, 95-117 (International Conference for J. Folta).
13. Almost all the documents, manuscripts, reports, correspondence, books, letters of introduction, of the early nineteenth century seem to be lost. We wish to express here our warmest thanks to the staff and to the director, Mr. P. Kimourtzis, for their invaluable help during our research in the proceedings of the sessions of the Philosophy Department as well as those of the University Senate.
14. The second half of the 19th century was an agitated epoch for Greek history.
15. From the foundation of the University in 1837 until 1904, the Mathematics Department was an «unimportant» part of the Department of Philosophy.
16. Unfortunately, we have been unable to find his thesis.
17. He must have been a successful teacher of mathematics, since he was invited to teach the subject to the young members of the royal family of Greece.
18. «In 1846 a chair of higher geometry was established at the Sorbonne for him» [KOLMOGOROV, A.N. & YOUSCHKEVITCH, A.P. (eds.) (1996) *Mathematics of the 19^th^ Century, Geometry and Analytic Functions*, Birkhäuser, p. 38].
19. Probably the courses of Michael Chasles and Jean Duhamel —both very thoughtful mathematicians— deeply influenced Lacon's career, Chasles having a very rare knowledge and understanding of the evolution of Geometry from ancient times to his days and Duhamel being a born philosopher on mathematics. For a man with a philosophical turn of mind, like Lacon, and a solid background in Euclidean Geometry and basic mathematics, this was really a blessing and an initiation for research.
20. Few years before, Auguste Bravais (1811-1863), a physicist and mineralogist, studied groups of motion to determine the possible structures of crystals; see *Journal de Math. 14* (1849), 141-180. This work impressed Jordan, who undertook research on analytic representation of groups (representation theory of groups in our modern terminology) and on infinite groups (see «Mémoire sur le group de mouvements», *Annali di Mat. (ii), 2* (1868/9), 167-215 and 322-345). It is quite impossible that Lacon, during his stay in Paris, did not know these papers. Nevertheless, he never uses group theory and that separates him from the modern theories of his time.
21. See the Report of the Rector of the University of Athens professor Ioannis Hadjidakis (Athens, 1896). See also my paper: PHILI, Ch. (2000) «Partisans and Deniers of non-Euclidean Geometry, The Case of A. Karagiannides». In: *Proceedings of the International Congress on European Scientific Unification in the17^th^ and 18^th^ centuries*. Athens (to appear in Greek).
22. He wrote textbooks on Geometry, Algebra, Arithmetic, Trigonometry, Cosmography; also a collection of problems in theoretical Arithmetic and a book on Physics.
23. It seems that his son George Lacon (1878-1955), who studied philology, inherited his talent and, under the pseudonym of Georges Karthaios, became a well-known poet.

24. For more details about this journal see my paper: PHILI, Ch. (2000) «Some Aspects from the History of Scientific Society in Athens at the end of XIX Century: Mathematics and Mathematicians». *Archives Internationales d'Histoire des Sciences, 50 (145),* 302-320.
25. In spite of our research in the National Library of Greece, we could not find them.
26. Evangelis Zappas (1800-1865), an ardent patriot and a great benefactor, was the first to conceive the idea for the revival of the Olympic ideal in 1856, before its realization by Baron Pierre de Coubertin (1863-1937) in 1896. He even offered the Minister of Foreign Affairs, A. Ragavis, the architectural plans for the buildings for the Olympics and those for the Stadium. The main goal of the Committee was firstly to present every four years the achievements in industry and agriculture of the Greek state, and the intellectual progress of the country as well. By the law of 19 August, 1858, article 13, a purely intellectual Olympiad was established to take place every four years to which the University contributed with professors from each department who were elected to participate by the University Senate. In this national festival, they gave their reports on the intellectual work and progress of the nation in the sciences for the last four years. The first day of the Olympiad started with a Te Deum, and the second day was dedicated to the academic festival in which the prize established by Zappas was awarded to the best contribution.
27. In 1910, Kyparissos Stephanos, as Rector of the University of Athens, in his speech *On the Development and the Significance of Sciences*, referred to Lacons' work.
28. Cantor's and Dedekind's efforts concerning the axiomatization of real numbers appeared at almost the same time.
29. For more details, see FREUNDENTHAL, H. (1957) «Zur Geschichte der Grundlagen der Geometrie». *Nieuw Archief voor Wiskunde (4) 5*, 105-142. Nevertheless, we must mention that Bolzano's *Betrachtungen über einige Gegenstände der Elementargeometrie* (Prague, 1804) laid the foundations of elementary geometry. Points, lines and planes are undefined elements and Bolzano tried to develop this theory analyzing the geometrical concepts logically (see also BOLZANO, B. (1948) *Oeuvres.* Prague, vol. V, pp. 8-181).
30. Pasch delivered these lectures «in the winter semester of 1873-74 at Klein's suggestion» [TOEPELL, M. (1986) «On the Origins of David Hilbert's *Grundlagen der Geometrie*». *Arch. for the History of Exact Sciences, 35,* p. 333].
31. See CONTRO, W.S. (1976) «Von Pasch zu Hilbert». *Archives for the History of Exact Sciences, 15*, 283-295.
32. Festschrift zur Feier der Enthülung des Gauss-Weber Denkmals in Göttingen, Leipzig, 1899; Zweite durch zusätze vermehrte und mit fünf Anhängen versehene Auflage, Leipzig, 1903. For more details see TOEPELL, M (1986) «On the Origins of David Hilbert's *Grundlagen der Geometrie*». *Arch. for the History of Exact Sciences, 35*, 329-344; POINCARÉ, H. (1902) «Les Fondements de la Géométrie, par M. Hilbert», *Journal des Savants*, *Bull. Sci. Math. XXVI* (2e série).

The famous French writer R. Queneau was a lover of mathematics (his contribution was presented by Prof. A. Lichnerowicz of the French Academy of Sciences, see *C.*

R. Ac. Sci., 6 V., Paris, 1968). Inspired by Hilbert's *Grundlagen der Geometrie,* he presented the *Foundations of Literature* replacing in Hilbert's propositions the words «points», «lines», «planes» by «words», «phrases», «paragraphs» and really this new axiomatic system still functions and sets literature axiomatics (see *Bibliothèque Oulipienne, 3*; PHILI, Ch. (2002) «R. Queneau, the workshop of Potential Literature and the foundations of Geometry. In: *Proceedings of the 19th Congress of Mathematical Education.* Komotini 2002, 905-920.

33. See TILLY, J. de (1879) «Essai sur les principes fondamentaux de la Géométrie et de la Mécanique». *Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux, 2e série t. III*.
34. «A syllogism is a form of words in which, when certain assumptions are made, something other than what has been assumed necessarily follows from the fact that the assumptions are such» *Pr. Anal. 24b*, 19-21. For more details see MAIER, H. *Die Syllogistik der Aristoteles.* I. Band, 1896; II. Band, 1900.
35. Cf. Steiner's statement: «Die in der Geometrie erforderlichen Grundvostellung sind: der Raum, die Ebene, die Gerade und der Punkt». *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten voneinander*. Berlin, 1832; *Ges. Werke I,* Berlin, 1881, reprint 1971, p. 237.
36. Pasch, in his *Vorlesungen,* notes that the axioms are by no means self-evident truths but just assumptions designed to yield the theorems of any particular geometry (2nd ed. revised by Max Dehn, 1926, p. 90).
37. For more details see HELMHOLTZ, H. (1956) «On the Origin and Significance of Geometrical Axioms». In: James Newman (ed.) *The World of Mathematics*. N.Y, vol. I, 646-648.
38. Heron from Alexandria's quotation about the meaning of axiom is still very interesting: «It is an axiom, according to Aristotle, when it is both familiar to him who hears it and believable in itself, as those that are equal to the same are themselves equal; and when one has no convincing personal notion, but is convinced by someone else and accepts it, this is a hypothesis; because that the circle is such a figure is not usually conceived by someone before he is taught, but when he hears it, he accepts it without proof» Heron Alexandrinis, ed. I. L. Heiberg, 1899, def. IV pp. 12, 15-23 pp. xxiii-xxiv in E. Stamatis Euclides I Elementa I-IV Post. I. L. Heiberg Teubner, 1969.
39. Lacon exposed the different position of Aristotle: «postulate [is] any provable proposition that is assumed and used without being proved» [Aristotle op. cit., 75^{β} 33-35]. He also exposed the different position of Aristotle [Posterior Analytics Book I]. See, for example, Heath's opinion: «Although Aristotle gives clear idea of what he understood by a postulate, he does not give any instances from geometry; still less has he any allusion recalling the particular postulates found in Euclid» [HEATH, T.L. (1913) *The Thirteen Books of Euclid's Elements*. New York, p. 202]. «What is a postulate? When either the statement is unknown or it is taken without (preceding) acceptance by who learns it, then it is called postulate, as that all the right angles are equal» [Heron Alexandrinus, op. cit. IV, p. 158, 150-160, 7. In: E. Stamatis, p. xxiv].

40. In his first publication «On the Foundations of Geometry» [*Kazan Bull.* 1829-30], Lobatchevskii declared: «The first concepts with which any science begins must be clear and reduced to the smallest possible number» [LOBATCHEVSKII, N.I. (1946) *Collected Works.* Moscow, vol. I, p. 186 (in Russian)].
41. A remarkable study of geometry and of Euclid's Elements is contained in S.P. Zervos' paper, On the Development of Mathematical Intuition; on the Genesis of Geometry; further remarks. (dedicated to prof. A. Kawaguchi) Tensor N. S. Vol. 26, 1972, 397-467.
42. Brunschwicg considered that Euclid was more a professor of logic than a professor of geometry. BRUNSCHWICG, L. (1972) *Les Etapes de la Philosophie Mathématique*, Nouv. Ed., p. 84; cf. also Leibniz's statement where he stresses the affinity between logic and geometry: «ce sont les propositions universelles, c'est-à-dire les définitions, les axiomes, et les théorèmes déjà démontrés; qui font le raisonnement et le contiendraient quand la figure n'y serait pas… la logique des géomètres ou les manières d'argumenter qu'Euclide a expliquées et établies en parlant des propositions, sont une extension ou promotion de la logique générale» [*Nouveaux Essais*, Livre IV, Ch. II § 9].
43. Ibidem.
44. In Euclid we do not find the true definition which according to Leibniz's classic expression «fait voir la possibilité du défini» [*Nouveaux Essais*, Livre III, Ch. III § 19].
45. Brunschwicg considered that these definitions could pass for unsolved enigmas or as a wonder of profundity [Op. cit., 86]. P. Tannery offered the interpretation that these two definitions are closely related to the constructions. Cf. also Leibniz's opinion: «Euclide, faute d'une idée distinctement exprimée, c'est à dire d'une définition de la ligne droite (car celle qu'il donne en attendant est obscure, et ne lui sert point dans les démonstrations)» [*Nouveaux Essais*, Livre IV, Ch. 12 § 6]. In 1882, Pasch, in his *Vorlesungen,* noted also that Euclid's common notions, such as point and line, were not really defined. Thus point, line, plane might be chosen as the undefined terms. Peano, in his *Principi* (1881), marked that the basic elements are undefined. We must note that G. Peano gave a set of axioms for Euclidean Geometry, in his well-known book *I Principi di Geometria* (1889), using the notions of point, segment and motion. A few years later his pupil M. Pieri adopted the point and the motion as undefined concepts [PIERI, M. (1899) «Della geometria elementare comme sistema ipotetico deduttivo». *Memoire della R. Ac. di Torino S. II IL*; PIERI, M. (1898) «I Principi della geometria di posizione composti in sistema logico-deductivo». *Ac. R. di Torino*]. G. Veronese, continuing this Italian School of Geometry, uses the line, the segment and the congruence of segments as undefined elements [*Fondamenti di Geometria*, 1891]; in «Les postulats de la géométrie dans l'enseignement» [*Comptes Rendu du deuxième Congrès International des Mathématiques à Paris*, 1900, p. 449] he introduced another postulate: «Any segment, even variable, contains a point distinct of its extremitees».
46. TANNERY, P. (1884) «Sur l'authenticité des axiomes d'Euclide». *Bull. des Sciences Mathématiques, VIII.*
47. TANNERY, P. (1887) *La Géométrie grecque*. Paris.
48. PROCLUS (1873) *Commentaire sur le premier livre d' Euclide*, éd. Friedlein, p. 194.

49. An interesting exposition of the major developments of geometry is contained in SHING-SHEN CHERN (1990) «What is Geometry?» *Am. Math. Monthly, 97 (8)*, 679-686.
50. He also mentions Legendre's attempt to transform it into a theorem [LEGENDRE, A.M. (1794) *Eléments de Géométrie*, I[e] éd; LEGENDRE, A.M. (1833) «Refléxions sur différentes manières de démontrer la théorie des parallèles ou le théorème sur la somme de trois angles du triangle». *Mém. Ac. Sci. Paris T. XIII*].
51. Actually Lobatchevskii's research started earlier. In 1815 he worked on the theory of parallels. His first geometric work having the title *Geometry*, written in 1823, was published posthumously and in 1826 he concluded that the fifth postulate is independent of the other axioms of Euclidean Geometry. The fruit of his research appeared in his paper of 1826: «An Abbreviated Exposition of the Foundations of Geometry with a Rigorous Proof of the Theorem on Parallels», read to the *Physical-Mathematical Department of Kazan University*. Later this paper was included in Lobatchevskii's first publication of non-Euclidean Geometry, entitled on the *Foundations of Geometry*, 1829-30.
52. *The Imaginary Geometry* appeared in 1835 and *An Application of Imaginary geometry to certain integrals* in 1836.
53. KLEIN, F. (1894) «Riemann und seine Bedeutung in der Entwicklung der modernen Mathematik». *Jahrber. d. Deutsch. Math, Ver. Bd. IV*, 72-82.
54. *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*, presented on the 10 June of 1854 as his *Probevorlesung* at Göttingen University, in an audience not composed uniquely by mathematicians. Riemann's work was posthumously published in 1867 in Göttingen, *Abh. XIII*, under the editorship of Dedekind. Afterwards it was translated into French by J. Houël, *Ann. di. Mat. T. III*, 1870; Riemanns *Werke I Aufl.,* 1876, 254-312. W. K. Clifford presented it in English, *Math. Vol. VII*, 1873; W. Dickstein translated it into Polish, *Comm. Acad. Litt. Cracov, Vol IX*, 187) and finally D. Sintsov translated it into Russian in 1893, *Mem. of the Physical Math. Society of Kazan's University*.
55. «Ueber die thatsächlichen Grundlagen der Geometrie». Heidelberg, *Verh. der naturwiss. und Vereins. Bd IV*, 197-202, *Bd. V.*, 31-32, 1869; *Wiss. Abhandlungen von H. Helmholtz, Bd. II*, 610-617, Leipzig, 1883; French transl. by J. Houël in *Mém. de la Société des Sci. Phys. et Nat. Bordeaux t. V* (1868). «Ueber die thasachen, die der Geometrie zum Grunde liegen». *Gött. Nach. Bd. XV*, 193-221, 1868; *Wiss. Abh. Bd. II*, 618-639. «The Axioms of Geometry», *The Academy Vol., I*, 123-181, 1870; *Revue des cours scientifiques t. VII*, 498-501, 1870. «Ueber die Axiome der Geometrie. Populäre wissenschaftliche Vorträge». *Heft 3*, 21-54, Brunswick, 1876; English transl. *Mind Vol. I*, 301-321; Fr. *Revue Scientifique de la France et de l'Etranger (2) T. XII*, 1197-1207, 1877. «Ueber den Ursprung, Sinn und Bedeutung der geometrischen Sätze». *Wiss. Abh. Bd. II,* 640-660; English transl. *Mind, Vol II*, 212-224, 1878.

In his paper *Uber die Thatsächen, die der Geometrie zu Grunde liegen*, an obvious echo of Riemann's paper of 1854, first among the restrictions, Helmholtz placed the requirement of free mobility of solid figures without modification of the shape in all parts of space, i.e. the presence of motions that brings congruent figures into coincidence. Helmholtz based his research on the following «hypotheses»: I) The space of n dimensions is an n-fold extended manifold. II) The existence of movable but invariant (solid) bodies or systems of points is assumed. III) Completely free mobility of solid bodies is assumed. IV) If a solid body revolves about n-1 points chosen so that the position of the body depends on only one independent variable, then a revolution without reverse rotation will eventually bring the body back to the initial position from which it departed. V) Space has three dimensions. VI) Space is infinite in extent» (pp. 368-382).

In conclusion Helmholtz wrote that: «Riemann's research and mine, taken together, thereby show that the postulates given above […] constitute a sufficient foundation for developing the study of space» (p. 382) [KOLMOGOROV, A.N. & YOUSHKEVICH A.P. (eds.) (1996) *Mathematics of the 19th Century. Geometry and Analytic Functions*. Birkhäuser Verlag (transl. from Russian by R. Cooke), pp. 108-109].

56. Already in 1829, Ioannis Carandinos worked on this purpose to bring the euphoric mathematical climate which reigned in France and translated Legendre's *Elements of Geometry,* the 12th ed. of 1823. For more details see my paper: PHILI Ch. (1996) «La reconstruction des mathématiques en Grèce: l'apport de Jean Carandinos, 1784-1834». *L'Europe Mathématique*, Edition de la Maison des Sciences de l'Homme, Paris, 305-319. But Carandinos' translation is not the only one. Professor Damaskinos also translated it. See e.g. its 4th edition of 1874, Athens.

57. Lacon referred to CRELLE, A.L. (1853) «Zur Theorie der Ebene». *Journ. für die reine und angew, Math. 45*, 15-34.

58. GERLING, Ch.L. (1839) «Fragment über die Begründung des Begriffes der Ebene». *Journ für die reine und angew. Math.. 20*, 332-336. The role of Gerling on the theory of parallels is significant. Gerling corresponded with Gauss on this topics (*Werke Bd. VII*, 167-169), and in 1819 he sent to him Schweikart's work *Astralgeometrie astralische Grössenlehre*. Also through Gerling, Gauss was informed about the text written by the son of W. Bolyai (*Werke Bd. VII*, 220).

59. *Elemente der Mathematik Bd. II,* 5 Auflage, Leipzig, 1878. Baltzer was the first to recognize in public the works of Lobatchevskii and Bolyai (father and son).

60. *Des Méthodes dans les Sciences de Raisonnement,* 5 vols., 1866-72.

61. It is remarkable that Lacon mentioned many geometers who were involved in divulging non-Euclidean geometry. He referred to the German Ch. L. Gerling (1788-1864), R. Baltzer (1818-1887) but he did not mention Fr. Schmidt or the French J. Houël, or the Italians G. Battaglini, E. Beltrami and A. Forti.

62. Definition 3.

63. Definition 5.

64. He covers the Euclidean definition, where it makes clear that in length a line is finite.

65. Definition 6.

66. Lobatchevskii *Works,* t. II, p. 168 (in Russian).

67. We must only stress that the figure's concept was important; as consequence, in Greek something which has no figure is ugly.

68. «And now you can learn what I call figure. Because, concerning every figure I say this, that at what a solid ends, this is a figure; and when I have grasped this I can say that

it is the figure of the solid» [*Menon, 76 a*]. Cf. also Euclid's definition 13: «A boundary is what is the end of something».

69. Lacon considers the movement of the figure without deformation, see also below, Axiom 3.
70. RIEMANN, B. (1953) *Gesammelte mathematische Werke*. 2nd ed. Dover, pp. 272-287, 391-404, 308.
71. Idem, 310.
72. Dropping Archimedes axiom results geometry, which is called non-Archimedean; in it there are segments such that the multiple of one by any whole number, however large, need not exceed another. Veronese, in his *Fondamenti di Geometria*, constructed such a geometry. Dehn, omitting the Archimedean axiom, obtained many interesting theorems [*Math. An. 53* (1900), 404-439].
73. *Vergleichende Betrachtungen über neure Geometrische Forschungen*.
74. Reprinted in *Math. An. XIV* (1893), 63-100; see also *Annales de l'Ecole Normale Supérieure*, 1891, 87-102 and 172-210; *New York Math. Soc. Bull.*, 1893, 215-249, *Ann. di Mat. (2) t. XVII* (1899), 301-343.
75. Although Lacon maintained scientific connections abroad, his work remained completely unknown and left no traces in the European mathematical literature.