Lifting Infinitesimal Automorphisms to Higher Order Adapted Frame Bundles

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Abstract: We describe all $F_{\text{ol}}^{m,n}$-natural operators $A$ lifting infinitesimal automorphisms $X$ on foliated $(m+n)$-dimensional manifolds $(M,\mathcal{F})$ with $n$-dimensional foliations $\mathcal{F}$ into vector fields $\mathcal{A}(X)$ on the $r$-th order adapted frame bundle $P^r(M,\mathcal{F})$. Next, we describe all $F_{\text{ol}}^{m,n}$-natural affinors on $P^r(M,\mathcal{F})$.

Key words: foliated manifold, infinitesimal automorphism, natural operator, natural affinor, higher order adapted frame bundle.

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0. Introduction

The present paper is devoted to extend results from our previous papers [4] and [3] to similar results for foliated manifolds instead of manifolds. We modify and joint in respective way the texts of papers [4] and [3]. All manifolds and maps are assumed to be of class $C^\infty$.

The notion on foliated manifolds can be found in many papers, e.g. [5]. Let $F_{\text{ol}}^{m,n}$ denote the category of all $(m+n)$-dimensional foliated manifolds with $n$-dimensional foliations and their foliation respecting local diffeomorphisms. Let $(M,\mathcal{F})$ be a $F_{\text{ol}}^{m,n}$-object. We have the $r$-th order adapted frame bundle

$$P^r(M,\mathcal{F}) = \{ j_0^r \varphi \mid \varphi : (\mathbb{R}^{m+n},\mathcal{F}^{m,n}) \to (M,\mathcal{F}) \text{ is a } F_{\text{ol}}^{m,n}\text{-map} \}$$

over $M$ of $(M,\mathcal{F})$ with the target projection, where $\mathcal{F}^{m,n} = \{ \{ a \} \times \mathbb{R}^n \}_{a \in \mathbb{R}^m}$ is the standard $n$-dimensional foliation on $\mathbb{R}^{m+n}$. Clearly, $P^r(M,\mathcal{F})$ is a principal bundle with the group $\mathcal{G}_{m,n}^{r} = P^r(\mathbb{R}^{m+n},\mathcal{F}^{m,n})_0$ (with the multiplication given by the composition of jets) acting on the right on $P^r(M,\mathcal{F})$ by the composition of jets. Every $F_{\text{ol}}^{m,n}$-map $\psi : (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ can be extended (via composition of jets) into principal bundle (local) isomorphism...
Let \((M, \mathcal{F})\) be a \(\mathcal{F}ol_{m,n}\)-object. A vector field \(X\) on \(M\) is called an infinitesimal automorphism of \((M, \mathcal{F})\) if its flow is formed by local \(\mathcal{F}ol_{m,n}\)-maps \((M, \mathcal{F}) \rightarrow (M, \mathcal{F})\) or (equivalently) if \([X,Y]\) is tangent to \(\mathcal{F}\) for any vector field \(Y\) tangent to \(\mathcal{F}\). The space \(\mathcal{X}(M, \mathcal{F})\) of all infinitesimal automorphisms of \((M, \mathcal{F})\) is a Lie subalgebra in \(\mathcal{X}(M)\).

The general concept of natural operators can be found in [1]. In this paper we need the following partial definition.

**Definition 1.** A \(\mathcal{F}ol_{m,n}\)-natural operator \(\mathcal{A} : T_{\inf \rightarrow \text{Aut}} \hookrightarrow TP^r\) is a family of \(\mathcal{F}ol_{m,n}\)-invariant regular operators (functions)

\[
\mathcal{A} = \mathcal{A}_{(M, \mathcal{F})} : \mathcal{X}(M, \mathcal{F}) \rightarrow \mathcal{X}(P^r(M, \mathcal{F}))
\]

for any \(\mathcal{F}ol_{m,n}\)-object \((M, \mathcal{F})\). (Of course, for some \((M, \mathcal{F})\) one can have \(\mathcal{X}(M, \mathcal{F}) = \emptyset\); then \(\mathcal{A}_{(M, \mathcal{F})} = \emptyset\).) The invariance means that if \(X_1 \in \mathcal{X}(M_1, \mathcal{F}_1)\) and \(X_2 \in \mathcal{X}(M_2, \mathcal{F}_2)\) are related infinitesimal automorphisms of \((M_1, \mathcal{F}_1)\) and \((M_2, \mathcal{F}_2)\) (respectively) by a \(\mathcal{F}ol_{m,n}\)-map \(\psi : (M_1, \mathcal{F}_1) \rightarrow (M_2, \mathcal{F}_2)\) then \(\mathcal{A}_{(M_1, \mathcal{F}_1)}(X_1)\) and \(\mathcal{A}_{(M_2, \mathcal{F}_2)}(X_2)\) are related by \(P^r \psi\). The regularity means that \(\mathcal{A}\) transforms smoothly parametrized families of infinitesimal automorphisms into smoothly parametrized families of vector fields.

A \(\mathcal{F}ol_{m,n}\)-natural operator \(\mathcal{A} : T_{\inf \rightarrow \text{Aut}} \hookrightarrow TP^r\) is said to be of vertical type if \(\mathcal{A}_{(M, \mathcal{F})}(X)\) is a vertical vector field on \(P^r(M, \mathcal{F}) \rightarrow M\) for any infinitesimal automorphism \(X\) of an arbitrary \(\mathcal{F}ol_{m,n}\)-object \((M, \mathcal{F})\).

Let \(k\) be a non-negative integer. A \(\mathcal{F}ol_{m,n}\)-natural operator \(\mathcal{A} : T_{\inf \rightarrow \text{Aut}} \hookrightarrow TP^r\) is said to be of order \(\leq k\) if for any infinitesimal automorphisms \(X_1\) and \(X_2\) of \((M, \mathcal{F})\) and \(x \in M\) the equality of \(k\)-jets \(j^k_x(X_1) = j^k_x(X_2)\) implies \(\mathcal{A}_{(M, \mathcal{F})}(X_1) = \mathcal{A}_{(M, \mathcal{F})}(X_2)\) on the fiber \((P^r(M, \mathcal{F}))_x\) of \(P^r(M, \mathcal{F})\) over \(x\).

**Example 1.** An example of a \(\mathcal{F}ol_{m,n}\)-natural operator \(\mathcal{A} : T_{\inf \rightarrow \text{Aut}} \hookrightarrow TP^r\) of order \(\leq r\) is the flow operator \(P^r\) sending an infinitesimal automorphism \(X\) of a \(\mathcal{F}ol_{m,n}\)-object \((M, \mathcal{F})\) into the complete lift \(P^rX\) of \(X\) to \(P^r(M, \mathcal{F})\). We recall that \(P^rX\) is the vector field on \(P^r(M, \mathcal{F})\) such that if \(\{\Phi_t\}\) is the flow of \(X\) then \(\{P^r(\Phi_t)\}\) is the flow of \(P^rX\). (We observe that to the flow of \(X\) we can apply functor \(P^r\) because the flow is formed by \(\mathcal{F}ol_{m,n}\)-maps.)
Example 2. Let $E \in \mathcal{L}(G_{m,n}^r)$. Let $E^*$ denote the fundamental vector field on $P^r(M,\mathcal{F})$ corresponding to $E$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$. We have the (constant) $\mathcal{F}ol_{m,n}$-natural operator $E^*: T_{\text{Inf} - \text{Aut}} \to TP^r$ defined by $(E^*)(X) = E^*$ for any infinitesimal automorphism $X$ of $(M,\mathcal{F})$. Clearly, the $\mathcal{F}ol_{m,n}$-natural operator $E^*$ is of vertical type.

Definition 2. A $\mathcal{F}ol_{m,n}$-natural affinor on $P^r$ is a $\mathcal{F}ol_{m,n}$-invariant family of tensor fields of type $(1,1)$ (affinors) $B = B(M,\mathcal{F}): TP^r(M,\mathcal{F}) \to TP^r(M,\mathcal{F})$ on $P^r(M,\mathcal{F})$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$. The invariance means that affinors $B_i(M_1,\mathcal{F}_1)$ and $B_i(M_2,\mathcal{F}_2)$ are $P^r\psi$-related for any $\mathcal{F}ol_{m,n}$-map $\psi: (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$.

A $\mathcal{F}ol_{m,n}$-natural affinor $B$ on $P^r$ is said to be of vertical type if $B: TP^r(M,\mathcal{F}) \to VP^r(M,\mathcal{F})$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$, where $VP^r(M,\mathcal{F})$ is the vertical bundle of $P^r(M,\mathcal{F}) \to M$.

Example 3. We have the identity $\mathcal{F}ol_{m,n}$-natural affinor $Id$ on $P^r$ such that $Id: TP^r(M,\mathcal{F}) \to TP^r(M,\mathcal{F})$ is the identity map for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$.

In the present article we solve the following two problems.

Problem 1. To classify all $\mathcal{F}ol_{m,n}$-natural operators $A: T_{\text{Inf} - \text{Aut}} \to TP^r$.

Problem 2. To classify all $\mathcal{F}ol_{m,n}$-natural affinors on $P^r$.

The solution of Problem 1 is given in Theorem 1. We prove that the set of all $\mathcal{F}ol_{m,n}$-natural operators $A: T_{\text{Inf} - \text{Aut}} \to TP^r$ is a free finite-dimensional module over some algebra. We will introduce the module structure and construct explicitly a basis of this module. The solution of Problem 2 is given in Theorem 2.

For $n = 0$, $\mathcal{F}ol_{m,0}$ is the category $Mf_m$ of $m$-dimensional manifolds and their local diffeomorphisms. Thus we reobtain the respective results from [4] and [3]. The part of the present paper concerning Problem 1 (resp. Problem 2) is a respective modification (adaptation) of the paper [4] (resp. [3]).

Natural affinors play a very important role in the differential geometry. They can be applied to study torsions of connections [2]. In our situation
given a $\mathcal{F}ol_{m,n}$-natural affinor $B : TP^r(M, \mathcal{F}) \to TP^r(M, \mathcal{F})$ gives a torsion $\tau_B(\Gamma) = [B, \Gamma]$ of a principal connection $\Gamma : TP^r(M, \mathcal{F}) \to VP^r(M, \mathcal{F})$ on $P^r(M, \mathcal{F})$, where the bracket is the Frolicher-Nijenhuis one. That is why, natural affinors have been studied in many papers.

1. Preliminaries

**Lemma 1.** Let $X, Y \in \mathcal{X}(M, \mathcal{F})$ be infinitesimal automorphisms of $(M, \mathcal{F})$ and $x \in M$ be a point. Suppose that $j^x_0 X = j^x_0 Y$ and $X x$ is not-tangent to $\mathcal{F}$. Then there exists a (locally defined) $\mathcal{F}ol_{m,n}$-map $\psi : (M, \mathcal{F}) \to (M, \mathcal{F})$ such that $j^x_0 X = Y$ near $x$.

**Proof.** A direct modification of the proof of Lemma 42.4 in [1].

**Proposition 1.** Any $\mathcal{F}ol_{m,n}$-natural operator $A : T_{\text{Inf} - \text{Aut}} \to TP^r$ is of order $\leq r$.

**Proof.** A replica of the proof of Proposition 42.5 in [1]. We use Lemma 1 instead of Lemma 42.4 in [1].

The following lemma can be found in some previous Our paper (in printing). For the reader convenience we cite its proof.

**Lemma 2.** Any vector $v \in T_w P^r(M, \mathcal{F})$, $w \in (P^r(M, \mathcal{F}))_x$, $x \in M$ is of the form $P^r X_w$ for some $X \in \mathcal{X}(M, \mathcal{F})$. Moreover $j^x_0 X$ is uniquely determined.

**Proof.** We can assume that $(M, \mathcal{F}) = (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ and $w$ is over 0. Since $P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ is in usual way a sub-principal bundle of $P^r \mathbb{R}^{m+n}$, then by well-known manifold version of the lemma, we find $X \in \mathcal{X}(\mathbb{R}^{m+n})$ such that $v = P^r X_w$ and $j^0_0 X$ is determined uniquely. Any infinitesimal automorphism $Y$ of $(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ gives $P^r Y_w$ which is tangent to $P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$. On the other hand the dimension of $P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ and the dimension of the space of $r$-jets $j^0_0 Y$ of infinitesimal automorphisms $Y$ of $(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ are equal. Then the lemma follows from the dimension argument because the flow operator is linear.

2. The $\mathcal{F}ol_{m,n}$-natural operators $B : T_{\text{Inf} - \text{Aut}} \to T^{(0,0)} P^r$

If (in the definition of $\mathcal{F}ol_{m,n}$-natural operators $A : T_{\text{Inf} - \text{Aut}} \to TP^r$) we replace the space $\mathcal{X}(P^r(M, \mathcal{F}))$ by the space $C^\infty(P^r(M, \mathcal{F}))$ of map-
We have the following general example of \( \mathcal{F}ol_{m,n} \)-natural operators \( B : T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \) lifting infinitesimal automorphisms of \((M,\mathcal{F})\) into maps \( P^r(M,\mathcal{F}) \rightarrow \mathbb{R} \).

**Example 4.** We have the following general example of \( \mathcal{F}ol_{m,n} \)-natural operators \( T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \). Let

\[
\lambda : J_{0}^{-1}(T_{\text{Inf}-\text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow \mathbb{R}
\]

be a map, where \( J_{0}^{-1}(T_{\text{Inf}-\text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \) is the vector space of all \((r-1)\)-jets \( J_{0}^{r-1}X \) at \( 0 \in \mathbb{R}^{m+n} \) of infinitesimal automorphism \( X \in \mathcal{X}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \). Then given an infinitesimal automorphisms \( X \) on a \( \mathcal{F}ol_{m,n} \)-object \((M,\mathcal{F})\) we have \( \mathcal{B}^{<\lambda>}(X) : P^r(M,\mathcal{F}) \rightarrow \mathbb{R} \) given by

\[
\mathcal{B}^{<\lambda>}(X)(\tilde{j}_{0}^{r}(\psi)) = \lambda(\tilde{j}_{0}^{r-1}(\psi^{-1}X))
\]

for all \( \tilde{j}_{0}^{r}(\psi) \in (P^r(M,\mathcal{F}))_{x}, x \in M \), where \( \psi : (\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \rightarrow (M,\mathcal{F}) \) is a \( \mathcal{F}ol_{m,n} \)-map with \( \psi(0) = x \). The correspondence \( \mathcal{B}^{<\lambda>} : T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \) is a \( \mathcal{F}ol_{m,n} \)-natural operator of order \( \leq r-1 \) transforming infinitesimal automorphisms of \((M,\mathcal{F})\) into maps \( P^r(M,\mathcal{F}) \rightarrow \mathbb{R} \).

The set of \( \mathcal{F}ol_{m,n} \)-natural operators \( B : T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \) is (obvious way) an algebra. Actually, given \( \mathcal{F}ol_{m,n} \)-natural operators \( B_1, B_2 : T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \) we have \( \mathcal{F}ol_{m,n} \)-natural operator \( B_1B_2 : T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \) given by

\[
(B_1B_2)(M,\mathcal{F})(X) = (B_1)(M,\mathcal{F})(B_2)(M,\mathcal{F})(X)
\]

for any infinitesimal automorphism \( X \) of a \( \mathcal{F}ol_{m,n} \)-object \((M,\mathcal{F})\), where in the right of the above formula we have the multiplication of real valued functions. Similarly we define the sum \( B_1 + B_2 : T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \).

**Proposition 2.** The map \( \lambda \rightarrow \mathcal{B}^{<\lambda>} \) is an algebra isomorphism from the algebra of smooth maps \( J_{0}^{-1}(T_{\text{Inf}-\text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow \mathbb{R} \) onto the algebra of all \( \mathcal{F}ol_{m,n} \)-natural operators \( T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \).

**Proof.** Clearly, the map \( \lambda \rightarrow \mathcal{B}^{<\lambda>} \) is an algebra monomorphism. Any \( \mathcal{F}ol_{m,n} \)-natural operator \( B : T_{\text{Inf}-\text{Aut}} \rightarrow T^{(0,0)}P^r \) of order \( \leq r-1 \) defines \( \lambda : J_{0}^{-1}(T_{\text{Inf}-\text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow \mathbb{R} \) by

\[
\lambda(\tilde{j}_{0}^{r-1}X) = \mathcal{B}(X)_{\tilde{j}_{0}^{r}(\text{id}_{\mathbb{R}^{m+n}})}.
\]
By the order argument $\lambda$ is well-defined. It is smooth because of the regularity of $B$ (standard argument using the Boman theorem, [1]). Then by the invariance with respect to local trivialization one can easily see that $B = B^{<\lambda>}$. 

Quite similarly as Proposition 1, one can show that any $B$ in question is of order $\leq r - 1$. Then the map $\lambda \to B^{<\lambda>}$ is an isomorphism. 

3. The $\text{Fol}_{m,n}$-natural operators $A : T_{\text{Inf}-\text{Aut}} \leadsto TP^r$

of vertical type

Let $E_\nu \in \mathcal{L}(G^r_{m,n}) (\nu = 1, \ldots, \dim(G^r_{m,n}))$ be a basis of $\mathcal{L}(G^r_{m,n})$. Then the fundamental vector fields $(E_\nu)^*\nu = 1, \ldots, \dim(G^r_{m,n})$ form a basis over $C^\infty(P^r(M, \mathcal{F}))$ of the module of vertical vector fields on $P^r(M, \mathcal{F})$ for any $\text{Fol}_{m,n}$-object $(M, \mathcal{F})$.

The space of all $\text{Fol}_{m,n}$-natural operators $T_{\text{Inf}-\text{Aut}} \leadsto TP^r$ transforming infinitesimal automorphisms of $\text{Fol}_{m,n}$-objects $(M, \mathcal{F})$ into vector fields on $P^r(M, \mathcal{F})$ is (in obvious way) a module over the algebra of $\text{Fol}_{m,n}$-natural operators $T_{\text{Inf}-\text{Aut}} \leadsto T(0,0)^{P^r}$. (Actually, given $\text{Fol}_{m,n}$-natural operators $A : T_{\text{Inf}-\text{Aut}} \leadsto TP^r$ and $B : T_{\text{Inf}-\text{Aut}} \leadsto T(0,0)^{P^r}$ we have $\text{Fol}_{m,n}$-natural operator $BA : T_{\text{Inf}-\text{Aut}} \leadsto TP^r$ given by

$$(BA)_{(M,\mathcal{F})}(X) = B_{(M,\mathcal{F})}(X)A_{(M,\mathcal{F})}(X)$$

for any infinitesimal automorphism $X$ on a $\text{Fol}_{m,n}$-object $(M, \mathcal{F})$, where in right of the above formula is the multiplication of vector fields by real valued functions.) Then by Proposition 2 it is the module over the algebra of all maps $J_0^{-1}(T_{\text{Inf}-\text{Aut}}(\mathcal{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathcal{R}$.

**Proposition 3.** The (sub)module of all vertical type $\text{Fol}_{m,n}$-natural operators $A : T_{\text{Inf}-\text{Aut}} \leadsto TP^r$ is free. The $\text{Fol}_{m,n}$-natural operators $(E_\nu)^*$ in question form a basis over $C^\infty(J_0^{-1}(T_{\text{Inf}-\text{Aut}}(\mathcal{R}^{m+n}, \mathcal{F}^{m,n})))$ of this module.

**Proof.** Since the fundamental vector fields $(E_\nu)^*$ on $P^r(M, \mathcal{F})$ form the basis of the module of vertical vector fields on $P^r(M, \mathcal{F})$, then any $\text{Fol}_{m,n}$-natural operator $A$ (of vertical type) in question is of the form

$$A(X) = \sum \lambda_\nu(X)(E_\nu)^*$$

for some uniquely determined maps $\lambda_\nu(X) : P^r(M, \mathcal{F}) \to \mathcal{R}$, where $X$ is an infinitesimal automorphism of a $\text{Fol}_{m,n}$-object $(M, \mathcal{F})$. Because of the invariance of $A$ with respect to $\text{Fol}_{m,n}$-maps, $\lambda_\nu : T_{\text{Inf}-\text{Aut}} \leadsto T(0,0)^{P^r}$ are $\text{Fol}_{m,n}$-natural operators. [1]
4. A decomposition

**Proposition 4.** Let $\mathcal{A} : T_{\text{Inf-Aut}} \rightarrow T\mathcal{P}^r$ be a $\mathcal{F}\text{O}_{m,n}$-natural operator of order $\leq r$. There is a unique smooth map $\lambda : J_0^{-1}(T_{\text{Inf-Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow \mathbb{R}$ such that $\mathcal{A} - B^{<\lambda>\mathcal{P}^r}$ is of vertical type, where $\mathcal{P}^r : T_{\text{Inf-Aut}} \rightarrow T\mathcal{P}^r$ is the flow operator.

**Proof.** Let $X$ be an infinitesimal automorphism of $(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$. We can write $\mathcal{A}(X)j_0'(id_{\mathbb{R}^{m+n}}) = \mathcal{P}^r \tilde{X}j_0'(id_{\mathbb{R}^{m+n}})$ for some infinitesimal automorphism $\tilde{X}$ (see Lemma 2). Suppose that $\tilde{X}_0 \neq 0$ and $X_0 \neq \mu \tilde{X}_0$ for all $\mu \in \mathbb{R}$. Then there is an $\mathcal{F}\text{O}_{m,n}$-map $\psi : (\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \rightarrow (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ preserving $j_0'(id_{\mathbb{R}^{m+n}})$ such that

$$J^*T\psi(j_0'X) = j_0'X \text{ and } J^*T\psi(j_0'\tilde{X}) \neq j_0'\tilde{X}.$$ 

Then

$$\mathcal{A}(X)j_0'(id_{\mathbb{R}^{m+n}}) = \mathcal{P}^r(\psi_*\tilde{X})j_0'(id_{\mathbb{R}^{m+n}}) \neq \mathcal{P}^r(\tilde{X})j_0'(id_{\mathbb{R}^{m+n}}) = \mathcal{A}(X)j_0'(id_{\mathbb{R}^{m+n}}).$$

This is a contradiction. Consequently, we have

$$(*) \quad T\pi^r \circ \mathcal{A}(X)j_0'(id_{\mathbb{R}^{m+n}}) = \lambda(j_0'^{-1}X)X_0$$

for some (not necessarily unique and not necessarily smooth) map

$$\lambda : J_0^{-1}(T_{\text{Inf-Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow \mathbb{R}$$

and all infinitesimal automorphisms of $(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ with coefficients (with respect to the basis of canonical vector fields on $\mathbb{R}^{m+n}$) being polynomials of degree $\leq r - 1$, where $\pi^r : P^r(\mathbb{R}^{m+m}, \mathcal{F}^{m,n}) \rightarrow \mathbb{R}^{m+m}$ is the usual projection $j_0'^r \psi \rightarrow \psi(0)$.

We are going to show that $\lambda$ can be chosen smooth. Of course (since the left hand side of $(*)$ depends smoothly on $j_0'X$), the map

$$\Phi : J_0^{-1}(T_{\text{Inf-Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow \mathbb{R}$$

given by

$$\Phi(j_0'^{-1}X) = \lambda(j_0'^{-1}X)X^1(0)$$

is smooth and $\Phi(j_0'^{-1}X) = 0$ if $X^1(0) = 0$, where $X_0 = \sum_i X^i(0) \frac{\partial}{\partial x^i}$. Then (this is a known fact from the mathematical analysis) there is a smooth map

$$\Psi : J_0^{-1}(T_{\text{Inf-Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow \mathbb{R}$$
such that $\Phi(j_0^{-1}X) = \Psi(j_0^{-1}X)X^1(0)$. Then we can define new $\lambda = \Psi$. This new $\lambda$ is equal to the old one for $X^1(0) \neq 0$. Then for the new $\lambda$ we have (*) if additionally $X^1(0) \neq 0$. Then we have (*) for all $X$ in question because of the smooth and density arguments.

Then $(A(X) - B^{<\lambda>}(X)P^rX)j_0^r(id_{R^{m+n}})$ is vertical for all infinitesimal automorphisms $X$ of $(R^{m+n}, F^{m,n})$ with coefficients (with respect to the basis of vector fields) being polynomials of degree $\leq r - 1$.

Since the union of orbits with respect to the $fol_{m,n}$-maps preserving $j_0^r(id_{R^{m+n}})$ of all $j_0^rX$ for infinitesimal automorphisms $X$ with coefficients (with respect to the basis as above) being polynomials of degree $\leq r$ is dense in $J_0^r(T_{Inf-Aut}(R^{m+n}, F^{m,n}))$ (see Lemma 1), the vector $(A(X) - B^{<\lambda>}(X)P^rX)j_0^r(id_{R^{m+n}})$ is vertical for all infinitesimal automorphisms $X$ of $(R^{m+n}, F^{m,n})$ with coefficients (with respect to the basis) being polynomials of degree $\leq r$. Then $(A(X) - B^{<\lambda>}(X)P^rX)j_0^r(id_{R^{m+n}})$ is vertical for all infinitesimal automorphisms $X$ of $(R^{m+n}, F^{m,n})$ because of the order argument. Then $A - B^{<\lambda>}\times P^r$ is of vertical type because of the $fol_{m,n}$-invariance and the fact that $P^r$ is a transitive bundle functor (i.e. $P^r(M,F)$ is the $fol_{m,n}$-orbit of $j_0^r(id_{R^{m+n}})$).

5. SOLUTION OF PROBLEM 1

We know that any $fol_{m,n}$-natural operator $A : T_{Inf-Aut} \rightsquigarrow TP^r$ is of order $\leq r$ (see Proposition 1). Then summing up Propositions 3 and 4 we get.

THEOREM 1. All $fol_{m,n}$-natural operators $T_{Inf-Aut} \rightsquigarrow TP^r$ form a free finite-dimensional module over the algebra of all smooth functions

$$J_0^{r-1}(T_{Inf-Aut}(R^{m+n}, F^{m,n})) \rightarrow R.$$ 

The operators $P^r$ and $(E_\nu)^*$ for $\nu = 1, \ldots, \dim(G^r_{m,n})$ form a basis in this module, where $(E_\nu)$ is a basis of $L(G^r_{m,n})$ and given $E \in L(G^r_{m,n})$ the fundamental vector field on $P^r(M,F)$ is denoted by $E^*$.

6. A DECOMPOSITION FOR $fol_{m,n}$-NATURAL AFFINORS

PROPOSITION 5. Let $B$ be a $fol_{m,n}$-natural affinor on $P^r$. There is a unique real number $\lambda$ such that $B - \lambda Id$ is of vertical type.

Proof. Using $B$ we define a linear $fol_{m,n}$-natural operator $A : T_{Inf-Aut} \rightsquigarrow TP^r$ by $A(X) = B(P^rX)$ for any $X \in X(M,F)$ (the linearity means that
\( \mathcal{A}(X) \) is linear in \( X \). By Proposition 4 and the homogeneous function theorem [1], since \( \mathcal{A} \) is linear, there exists a unique real number \( \lambda \) such that \( \mathcal{A} - \lambda \mathcal{P} \) is vertical. Then \( (B - \lambda \text{Id})(\mathcal{P}^r \mathcal{X}_{\sigma}) \) is vertical for any infinitesimal automorphism \( X \in \mathcal{X}(M, \mathcal{F}) \) and \( \sigma \in \mathcal{P}^r(M, \mathcal{F}) \). Then \( (B - \lambda \text{Id})(v) \) is vertical for any \( v \in T\mathcal{P}^r(M, \mathcal{F}) \) because of Lemma 2. Then \( B - \lambda \text{Id} \) is vertical. \[ \square \]

7. An example of \( \mathcal{F}ol_{m,n} \)-natural affinors of vertical type

We have the following example.

**Example 5.** Let
\[
\mathcal{C} : J_0^{-1}(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow (J_0^{-1}(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})))_0
\]
be a linear map, where
\[
J_0^{-1}(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) = \{ j_0^{-1}X \mid X \in \mathcal{X}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \}
\]
and \( (J_0^{-1}(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})))_0 = \{ j_0^{-1}X \mid X \in \mathcal{X}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}), X_0 = 0 \} \).

Define a vertical \( \mathcal{F}ol_{m,n} \)-natural affinor \( B^C : T\mathcal{P}^r(M, \mathcal{F}) \rightarrow V \mathcal{P}^r(M, \mathcal{F}) \) on \( \mathcal{P}^r \) by
\[
B^C(v) = V \mathcal{P}^r \psi((\mathcal{P}^r \tilde{v})\theta), \quad v \in Tj_0^0 \mathcal{P}^r(M, \mathcal{F}), \quad j_0^0 \psi \in \mathcal{P}^r(M, \mathcal{F}),
\]
where \( \theta = j_0^0(id_{\mathbb{R}^{m+n}}) \in \mathcal{P}^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \) is the element and \( \tilde{v} \in \mathcal{X}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \) is an infinitesimal automorphism of \( (\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \) such that \( j_0^0 \tilde{v} = C(j_0^{-1}((\psi^{-1})_s \tau)) \) and \( v = (\mathcal{P}^r \tau)j_0^0 \psi \). One can standardly show that \( B^C(v) \) is well-defined. More precisely (by Lemma 2), \( j_0^0 \tilde{v} \) is uniquely determined by \( v \). Then \( j_0^{-1}((\psi^{-1})_s \tau) \in J_0^{-1}((T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})))_0 \) is determined by \( v \).

Using the linearity of the flow operator, we deduce that \( B^C : T\mathcal{P}^r(M, \mathcal{F}) \rightarrow V \mathcal{P}^r(M, \mathcal{F}) \) is a vertical affinor on \( \mathcal{P}^r(M, \mathcal{F}) \). Clearly the family \( B^C \) is a \( \mathcal{F}ol_{m,n} \)-natural affinor on \( \mathcal{P}^r \).

8. Solution of Problem 2

**Theorem 2.** Any \( \mathcal{F}ol_{m,n} \)-natural affinor on \( \mathcal{P}^r \) is of the form
\[
B = \lambda \text{Id} + B^C
\]
for a unique real number \( \lambda \) and a unique linear map \( C : J_0^{-1}(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \rightarrow (J_0^{-1}(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})))_0 \).
Proof. Because of Proposition 5, we can assume that \( B \) is vertical. Define a linear map

\[
C : J_{r-1}^0(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \to (J^r_0(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})))_0
\]

by \( C(j^{r-1}_0X) = j^r_0\widetilde{X}, \) where \( \widetilde{X} \) is an infinitesimal automorphism of \((\mathbb{R}^{m+n}, \mathcal{F}^{m,n})\) such that \((P^r\widetilde{X})_\theta = B((P^rX)_\theta)\) and \( \widetilde{X} \) is an unique infinitesimal automorphism of \((\mathbb{R}^{m+n}, \mathcal{F}^{m,n})\) such that \( j^{r-1}_0X = j^{r-1}_0\overline{X} \) and \( \overline{X} \) has coefficients with respect to the basis of canonical vector fields \( \frac{\partial}{\partial x_i} \) on \( \mathbb{R}^{m+n} \) being polynomials of degree \( \leq r - 1 \).

Then \( B((P^rX)_\theta) = B^C((P^rX)_\theta) \) for all infinitesimal automorphisms of \((\mathbb{R}^{m+n}, \mathcal{F}^{m,n})\) such that \( X \) has coefficients (with respect to the basis as above) being polynomials of degree \( r - 1 \). Since the union of all orbits with respect to the \( \mathcal{F}o_{l_{m,n}} \)-maps preserving \( \theta \) of jets \( j^r_0X \) of infinitesimal automorphisms \( X \) of \((\mathbb{R}^{m+n}, \mathcal{F}^{m,n})\) with coefficients (with respect to the basis as above) being polynomials of degree \( \leq r - 1 \) is dense in \( J^r_0(T_{\text{Inf} - \text{Aut}}(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \) (see Lemma 1), \( B((P^rX)_\theta) = B^C((P^rX)_\theta) \) for all infinitesimal automorphisms \( X \) of \((\mathbb{R}^{m+n}, \mathcal{F}^{m,n})\). Then \( B(v) = B^C(v) \) for all \( v \in T_\theta P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \) because of Lemma 2. Then \( B = B^C \) because of the \( \mathcal{F}o_{l_{m,n}} \)-invariance and the fact that \( P^r \) is a transitive bundle functor (i.e., \( P^r(M, \mathcal{F}) \) is the \( \mathcal{F}o_{l_{m,n}} \)-orbit of \( \theta \)).

References