Boletín de Matemáticas
Nueva Serie, Volumen XIV No. 1 (2007), pp. 64–67

BANACH THEOREM FOR APPROXIMATIVELY SURJECTIVE MAPPINGS

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Abstract. We prove an analogous of Banach Theorem for approximatively surjective mappings on Schwartz b-spaces associated to Fréchet spaces.

Key Words and Phrases. Approximatively surjective mapping, Schwartz b-space.

Resumen. Se prueba un resultado análogo al Teorema de Banach para funciones aproximadamente sobreyectivas sobre b-espacios de Schwartz asociados a espacios de Fréchet.

Palabras claves. Función aproximadamente sobreyectiva, b-espacios de Schwartz

2000 Mathematics Subject Classification: 46M05, 46M15, 46M40.

1. Introduction and notations

On the category of Fréchet spaces (i.e. complete and metrizable locally convex spaces) and continuous linear mappings **Fré**, the projective limit is a left exact functor which is not necessary right exact. But Mittag-Leffler Lemma gave a sufficient condition in order that the projective limit functor be exact ([7], p. 23). In this Lemma, the morphisms of the projective system considered are mappings with a dense range. We will use the bornological property, and we call mappings which satisfy such condition approximatively surjective mappings. In the Banach case (and in the Fréchet case) it is clear that a mapping is

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approximatively surjective if and only if it has a dense range. Let us remark that Hörmander ([3], p. 44) used such Lemma in the Remark after Corollary 10.6.10.

On the other hand, the Banach Theorem says that if E and F are Fréchet spaces and $u : E \longrightarrow F$ is a surjective continuous linear mapping, then for every open subset V of F, there exists an open subset U of E such that $u(U) = V$. It is a topological result. It will be very interesting to have a bornological Theorem. In this sense, Buchwalter [1] proved that if E is a b-space with a countable base, then the surjective bounded linear mapping u from E into a b-space F is bornologically surjective i.e. for each bounded subset C of F, there exists a bounded subset B of E such $u(B) = C$. But in [2], Grothendieck established that there exist two Fréchet spaces E and F and a surjective continuous linear mapping $u : E \longrightarrow F$ such that the bounded linear mapping $u : E_b \longrightarrow F_b$ is not bornologically surjective where E_b is the space E with its von Neumann boundedness (i.e. a subset B is bounded in E if it is absorbed by all neighborhoods of the origin for the topology of E).

In this paper, we replace the class of bornologically surjective mappings by the class of approximatively surjective mappings. We will prove a similar version of Banach Theorem for the class of Schwartz b-spaces associated to Fréchet spaces. More precisely, if E and F are Fréchet spaces and $u : E \longrightarrow F$ is a continuous linear mapping such that $u(E)$ is dense in F, we will show that the bounded linear mapping $u : E_c \longrightarrow F_c$ is approximatively surjective where E_c is the space E with its Schwartz boundedness. As a consequence, we will obtain an analogous of Banach Theorem if we replace the Fréchet spaces E and F by b-spaces.

To prove our results, we need to recall the definition of the category of b-spaces in the sense of Waelbroeck [6]

A b-space (E, β) is a vector space E with a bounded structure β such that $E = \cup_{B\in\beta} B$, with $B \in \beta$ if $B \subset B_1 \cup B_2$ whenever $B_1, B_2 \in \beta$, without any non-null vector subspace of E belonging to β, and in which for every $B \in \beta$ there exists a $B_1 \in \beta$ with $B \subset B_1$, B_1 absolutely convex, and E_{B_1}, the subspace absorbed by B_1 with the norm-gauge associated to B_1, is a Banach space.

Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u : E \longrightarrow F$ is bounded, if it maps bounded subsets of E into bounded subsets of F.

2. Main result

We start by defining approximatively surjective mappings. If (E, β_E) and (F, β_F) are two b-spaces, a bounded linear mapping $u : E \longrightarrow F$ is approximatively surjective if for every completant bounded subset B of F, there exists a completant bounded subset B_1 of F with $B \subset B_1$, and there exists a completant bounded subset A of E such that $u(E_A)$ is dense in F_{B_1}. It is clear that a bounded linear mapping $u : E \longrightarrow F$ between Banach spaces is approximatively surjective if and only if $u(E)$ is dense in F.

If E is a b-space, a subset C of E is compact if there exists a bounded completant subset B of E such that C is compact in the Banach space E_B. We denote by $\mathcal{C}_E$, the family of all compact subsets of E, and by E_c the space E endowed with the boundedness $\mathcal{C}_E$. It is a Schwartz b-space i.e. a b-space in which every bounded completant subset B of E_c is included in a bounded completant subset A of E_c such that the inclusion mapping $i_{AB} : (E_c)_B \longrightarrow (E_c)_A$ is compact.

The next result is devoted to Banach Theorem for Schwartz b-spaces associated to Fréchet spaces.

Theorem 2.1. *Let u be a continuous linear mapping between Fréchet spaces E and F, with dense range in F. Then for every absolutely convex closed precompact subset M of F there are a precompact closed subset C in E and a precompact closed subset D of F such that $M \subset D$ and $u(E_c)$ is dense in F.*

Proof. Let M be a precompact closed subset of F. Since $u(E)$ is dense in the Fréchet space F, we can apply a result of A. and W. Robertson ([5], Theorem 41.4 (3), p.184) or Köthe [4] to get a sequence $(y_n)_n$ tending to the origin in F, contained in $u(E)$ such that every $x \in M$ can be written as $x = \sum \lambda_n y_n$ with $\sum |\lambda_n| \leq 1$. Select, for each n, $x_n \in E$ with $u(x_n) = y_n$. Since E is metrizable, there is a sequence (α_n) of scalars such that $(\alpha_n x_n)_n$ tends to the origin in E. The closed absolutely convex hull C of $(\alpha_n x_n)_n$ is closed and precompact in E, and the closed absolutely convex hull of $(y_n)_n$ is precompact and closed in F. Clearly $M \subset D$. It is enough to show that $D \subset u(E_c) + (\frac{1}{2})D$, from where the conclusion follows easily by induction. Fix $x \in D$, $x = \sum \lambda_n y_n$, $\sum |\lambda_n| \leq 1$. Select $n(0)$ with $\sum_{n>n(0)} |\lambda_n| < \frac{1}{2}$. Then $z = \sum_{n>n(0)} \lambda_n y_n \in \frac{1}{2}D$, and $x - z = \sum_{n=1 n(0)} \lambda_n x_n) \in u(E_c)$. The proof is complete.

Whenever E and F are b-spaces, we obtain the following consequence:

Corollary 2.2. *Let $u : E \longrightarrow F$ be an approximatively surjective bounded linear mapping between b-spaces. Then the bounded linear mapping $u : E_c \longrightarrow F_c$ is approximatively surjective.*

Proof. Let K_1 be a compact of F, then it is included in a Banach space F_B for some completant bounded subset B of F. Also, there exists a completant bounded subset B_1 of F with $B \subset B_1$, and there exists a completant bounded subset D of E such that $u(E_D)$ is dense in F_{B_1}.

Now, by Theorem 2.1, there exists a compact K_2 of F, which contains K_1 and there exists a compact K_3 in E_D such that for every $\varepsilon > 0$, we have

$$K_2 \subset \varepsilon K_2 + \cup_M Mu\,(K_3).$$

This shows the result.

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RECIBIDO: Marzo de 2007. ACEPTADO PARA PUBLICACIÓN: Junio de 2007