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A NEW TRAVELLING WAVE SOLUTION OF THE MIKHAILOV–NOVIKOV–WANG SYSTEM USING THE EXTENDED TANH METHOD

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Abstract. In this paper, we consider the new fifth-order integrable Mikhailov–Novikov–Wang (MNW) system. By using the extended tanh method, travelling wave solutions of the (MNW) system are presented. As a particular case, new exact solutions of the Kaup–Kupershmid (KK) are obtained.

Key words and phrases. Mikhailov–Novikov–Wang system; travelling wave solution; tanh method; extended tanh method.

Resumen. En este artículo consideramos el nuevo sistema Mikhailov–Novikov–Wang (MNW) integrable de quinto orden. Utilizando el método extendido tanh, se presentan soluciones de onda viajera del sistema (MNW). Como un paso particular, se obtienen nuevas soluciones exactas de Kaup–Kupershmid (KK).

Palabras claves. Sistema Mikhailov–Novikov–Wang, soluciones de onda viajera, método tanh, método tanh extendido.

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1. INTRODUCTION

The Mikhailov–Novikov–Wang system [1],[2] reads

$$\begin{cases} u_t = u_{xxxxx} - 20uu_{xxx} - 50u_x u_{xx} + 80u^2 u_x + w_x \\ w_t = -6wu_{xxx} - 2u_{xx} w_x + 96wuu_x + 16w_x u^2, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $w = w(x,t)$ are differentiable functions. The (MNW) system was derived using the symmetry approach. We refer to [1] for more details about this system. In the particular case that $w(x,t) = 0$, (1.1) reduces to the well-known Kaup–Kupershmidt (KK) equation [1], [2], [3], [4], [5]

$$u_t = u_{xxxxx} - 20uu_{xxx} - 50u_x u_{xx} + 80u^2 u_x. \tag{1.2}$$

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A plenty of exact solutions for the Kaup–Kupershmidt equation can be found using the inverse scattering transform for the corresponding spectral problem [5]. In [2], Sergyeyev presents a *zero curvature representation* **ZCR** for the M–N–W system. Knowing a **ZCR** involving an essential (spectral) parameter enables one to use the inverse scattering transform and construct plenty of exact solutions for the system in question, including multisoliton and finite-gap solutions. The searching of exact solutions of nonlinear partial differential equations is of great importance for many researches. A variety of powerful methods such as tanh method [6], general projective Riccati equation method [7], Hirota method [8], and many other methods have been developed in this direction. There is no unified method that can be used to handle all types. In this work, we will use the extended tanh method [9] to construct a new exact periodic and soliton solution for the (MNW) system, and we found new exact solutions for the KK equation as a special case. This paper is organized as follows: In Sec. 2, we will review briefly the extended tanh method. In Sec. 3, we give the mathematical framework to search exact solutions for the system. In Sec. 4, we obtain exact solutions of the MNW system. In Sec. 5, exact solutions of the KK equations are obtained as a particular case. Finally some conclusions are given.

2. The extended tanh method

The extended tanh method can be summarized as follows. For given a nonlinear equation that does not explicitly involve independent variables

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{2.1}$$

we use the wave transformation

$$u(x,t) = v(\xi), \quad \xi = x + \lambda t, \tag{2.2}$$

where λ is a constant.
Under the transformation (2.2), (2.1) reduces to the ordinary differential equation in the function $v(\xi)$

$$P(v, v', v'', \ldots) = 0. \tag{2.3}$$

The next crucial step is to introduce a new variable $\phi(\xi)$ which is a solution of the Riccati equation

$$\phi'(\xi) = \mu(\phi(\xi)^2 + k), \tag{2.4}$$

where k, μ are constants.

The solutions of the equation (2.4) are given by

$$\phi(\xi)=\begin{cases}-\frac{1}{\mu\xi}, & k=0\\ \sqrt{k}\tan(\mu\sqrt{k}\xi) & k>0\\ -\sqrt{k}\cot(\mu\sqrt{k}\xi) & k>0\\ -\sqrt{-k}\tanh(\mu\sqrt{-k}\xi) & k<0\\ -\sqrt{-k}\coth(\mu\sqrt{-k}\xi) & k<0.\end{cases} \tag{2.5}$$

We seek solutions of (2.3) in the form

$$\sum_{i=0}^{M} a_i\phi(\xi)^i+\sum_{M+1}^{2M} a_i\phi(\xi)^{M-i}, \tag{2.6}$$

where $\phi(\xi)$ satisfies the Riccati equation (2.4), and a_i are unknown constants. The integer M can be determined by balancing the highest derivative term with nonlinear terms in (2.3), before the a_i can be computed. Substituting (2.6) along with (2.4) into (2.3) and collecting all terms with the same power $\phi(\xi)^i$, we get a polynomial in the variable $\phi(\xi)$. Equaling the coefficients of this polynomial to zero, we can obtain a system of algebraic equations, from which the constants a_i, λ, μ, $(i=1,2,\ldots,M)$ are obtained explicitly. Lastly, we found solutions for (2.1) in the original variables.

3. Mikhailov–Novikov–Wang system

Mikhailov–Novikov–Wang system [1][2] reads

$$\begin{cases}u_t=u_{xxxxx}-20uu_{xxx}-50u_xu_{xx}+80u^2u_x+w_x\\ w_t=-6wu_{xxx}-2u_{xx}w_x+96wuu_x+16w_xu^2.\end{cases} \tag{3.1}$$

We search for exact solutions of system (3.1) in the form

$$\begin{cases}u(x,t)=v(\xi)\\ w(x,t)=w(\xi)\\ \xi=x+\lambda t.\end{cases} \tag{3.2}$$

As a result system (3.1) reduces to the nonlinear ordinary differential system

$$\begin{cases}\lambda v'(\xi)-v^{(5)}(\xi)+20v(\xi)v^{(3)}(\xi)+50v'(\xi)v''(\xi)-80v^2(\xi)v'(\xi)-w'(\xi)=0\\ \lambda w'(\xi)+6w(\xi)v'''(\xi)+2v''(\xi)w'(\xi)-96w(\xi)v(\xi)v'(\xi)-16w'(\xi)v^2(\xi)=0.\end{cases} \tag{3.3}$$

We obtain

$$w'(\xi)=\lambda v'(\xi)-v^{(5)}(\xi)+20v(\xi)v^{(3)}(\xi)+50v'(\xi)v''(\xi)-80v^2(\xi)v'(\xi). \tag{3.4}$$

The first equation of (3.3) can be written as follows:

$$(\lambda v(\xi)-v^{(4)}(\xi)+20v(\xi)v''(\xi)+15((v'(\xi))^2)-\frac{80}{3}(v^3(\xi))-w(\xi))'=0. \tag{3.5}$$

Integrating (3.5) once with respect to ξ we get

$$\lambda v(\xi) - v^{(4)}(\xi) + 20v(\xi)v''(\xi) + 15(v'(\xi))^2 - \frac{80}{3}v^3(\xi) - w(\xi) = c, \tag{3.6}$$

where c is integration constant. As we look for the exact solutions of special form, we set $c = 0$ for simplicity. Therefore

$$w(\xi) = \lambda v(\xi) - v^{(4)}(\xi) + 20v(\xi)v''(\xi) + 15(v'(\xi))^2 - \frac{80}{3}v^3(\xi). \tag{3.7}$$

Substituting (3.4) and (3.7) into the second equation of (3.3), we obtain after simplifications the following ordinary differential equation:

$$\begin{cases} \lambda^2 v'(\xi) - \lambda v^{(5)} + 26\lambda v(\xi)v'''(\xi) + 52\lambda v'(\xi)v''(\xi) - 192\lambda v^2(\xi)v'(\xi) - 2v''(\xi)v^{(5)}(\xi) + \\ 160v(\xi)v''(\xi)v'''(\xi) + 100v'(\xi)(v''(\xi))^2 - 2880v^2(\xi)v'(\xi)v''(\xi) + \\ 16v^2(\xi)v^{(5)} - 480v^3(\xi)v'''(\xi) + 3840v^4(\xi)v'(\xi) - 6v'''(\xi)v^{(4)} + 90(v'(\xi))^2v'''(\xi) + \\ 96v(\xi)v'(\xi)v^{(4)} - 1440v(\xi)(v'(\xi))^3 = 0. \end{cases} \tag{3.8}$$

4. Exact solutions of Mikhailov–Novikov–Wang system

Acording to the extended tanh method, and after balancing, we seek solutions of (3.8) in the form

$$\sum_{i=0}^{2} a_i\phi(\xi)^i + \sum_{3}^{4} a_i\phi(\xi)^{2-i}, \tag{4.1}$$

where $\phi(\xi)$ satisfies (2.5).
Substituting (4.1) into (3.8) and using (2.4), we obtain a polynomial in $\phi(\xi)$. Equaling the coefficients of this polinomial to zero, we obtain an algebraic system. Upon solving the algebraic system with respect to the unknowns k, μ, λ, a_0, a_1, a_2, a_3, a_4 we obtain the following set of solutions:

1. $a_0 = \pm\frac{\sqrt{\lambda}}{2}$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ $a_4 = \frac{3\lambda}{4\mu^2}$, $k = \pm\frac{\sqrt{\lambda}}{\mu^2}$
2. $a_0 = \pm\frac{\sqrt{\lambda}}{2}$, $a_1 = 0$, $a_2 = \frac{3\mu^2}{4}$, $a_3 = 0$ $a_4 = 0$, $k = \pm\frac{\sqrt{\lambda}}{\mu^2}$
3. $a_0 = \pm\frac{\sqrt{\lambda}}{8}$, $a_1 = 0$, $a_2 = a_2 = \frac{3\mu^2}{4}$, $a_3 = 0$ $a_4 = \frac{3\lambda}{64\mu^2}$, $k = \pm\frac{\sqrt{\lambda}}{4\mu^2}$.

(The other solutions of the algebraic system are given by $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, k, λ, a_0 arbitrary constants. But this case is not interesting).
According to (4.1), (2.5) and taking into account the solutions of the algebraic system, after simplifications we restrict ourselves to considering only the following set of solutions for (3.8) :

- For $\lambda > 0$, $\mu > 0$, $k > 0$
  $v_1(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\cot^2(\lambda^{\frac{1}{4}}\xi)$.

  $v_2(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\tan^2(\lambda^{\frac{1}{4}}\xi)$.

$$v_3(\xi) = \frac{\sqrt{\lambda}}{8} + \frac{3\sqrt{\lambda}}{16}\tan^2(\tfrac{1}{2}\lambda^{\frac{1}{4}}\xi) + \frac{3\sqrt{\lambda}}{16}\cot^2(\tfrac{1}{2}\lambda^{\frac{1}{4}}\xi).$$

- For $\lambda > 0$, $\mu > 0$, $k < 0$

$$v_4(\xi) = -\frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\coth^2(\lambda^{\frac{1}{4}}\xi).$$

$$v_5(\xi) = -\frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\tanh^2(\lambda^{\frac{1}{4}}\xi).$$

$$v_6(\xi) = -\frac{\sqrt{\lambda}}{8} + \frac{3\sqrt{\lambda}}{16}\tanh^2(\tfrac{1}{2}\lambda^{\frac{1}{4}}\xi) + \frac{3\sqrt{\lambda}}{16}\coth^2(\tfrac{1}{2}\lambda^{\frac{1}{4}}\xi).$$

We obtain solutions for system (3.1). By (2.2) and (3.7), the solutions for $u(x,t)$ and $w(x,t)$ are given by:

- For $\lambda > 0$, $\mu > 0$, $k > 0$

$$\begin{cases} u_1(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\cot^2(\lambda^{\frac{1}{4}}(x+\lambda t)) \\ w_1(x,t) = \frac{\lambda^{\frac{3}{2}}}{6} \end{cases} \tag{4.2}$$

$$\begin{cases} u_2(x,t) = \frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\tan^2(\lambda^{\frac{1}{4}}(x+\lambda t)) \\ w_2(x,t) = \frac{\lambda^{\frac{3}{2}}}{6} \end{cases} \tag{4.3}$$

$$\begin{cases} u_3(x,t) = \frac{\sqrt{\lambda}}{8} + \frac{3\sqrt{\lambda}}{16}\tan^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t)) + \frac{3\sqrt{\lambda}}{16}\cot^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t)) \\ w_3(x,t) = \frac{\lambda^{\frac{3}{2}}}{6} \end{cases} \tag{4.4}$$

- For $\lambda > 0$, $\mu > 0$, $k < 0$

$$\begin{cases} u_4(x,t) = -\frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\coth^2(\lambda^{\frac{1}{4}}(x+\lambda t)) \\ w_4(x,t) = -\frac{\lambda^{\frac{3}{2}}}{6} \end{cases} \tag{4.5}$$

$$\begin{cases} u_5(x,t) = -\frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\tanh^2(\lambda^{\frac{1}{4}}(x+\lambda t)) \\ w_5(x,t) = -\frac{\lambda^{\frac{3}{2}}}{6} \end{cases} \tag{4.6}$$

$$\begin{cases} u_6(x,t) = -\frac{\sqrt{\lambda}}{8} + \frac{3\sqrt{\lambda}}{16}\tanh^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t)) + \frac{3\sqrt{\lambda}}{16}\coth^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t)) \\ w_6(x,t) = -\frac{\lambda^{\frac{3}{2}}}{6} \end{cases} \tag{4.7}$$

5. Exact solutions of the KK equation

If we take $w(x,t) = 0$ in the MNW system (3.1), we obtain the KK equation

$$u_t = u_{xxxxx} - 20uu_{xxx} - 50u_xu_{xx} + 80u^2u_x. \tag{5.1}$$

In this case, the set of solutions of KK equation are given by:

- $u_1(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\cot^2(\lambda^{\frac{1}{4}}(x+\lambda t)).$
- $u_2(x,t) = \frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\tan^2(\lambda^{\frac{1}{4}}(x+\lambda t))$

- $u_3(x,t) = \frac{\sqrt{\lambda}}{8} + \frac{3\sqrt{\lambda}}{16}\tan^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t)) + \frac{3\sqrt{\lambda}}{16}\cot^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t))$
- $u_4(x,t) = -\frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\coth^2(\lambda^{\frac{1}{4}}(x+\lambda t))$
- $u_5(x,t) = -\frac{\sqrt{\lambda}}{2} + \frac{3\sqrt{\lambda}}{4}\tanh^2(\lambda^{\frac{1}{4}}(x+\lambda t))$
- $u_6(x,t) = -\frac{\sqrt{\lambda}}{8} + \frac{3\sqrt{\lambda}}{16}\tanh^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t)) + \frac{3\sqrt{\lambda}}{16}\coth^2(\frac{1}{2}\lambda^{\frac{1}{4}}(x+\lambda t))$.

Remark 1: One also can consider solutions with $\lambda < 0$, $\mu < 0$.

6. Conclusions

In other work of the author [10], exact solutions of the MNW system were obtained by the tanh method. In this paper, we have used *the extended tanh method* to derive a new exact solution (u_3, w_3) of the MNW system. In the same form, u_3, u_6 are new exact solutions of the KK equation.

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Referencias

[1] A.V. Mikhailov & V. Novikov & J.P.Wang, On clasification of integrable non-evolutionary equation, *Prepint nlin.*SI/0601046 (arXiv.org)

[2] A. Sergyeyev, Zero curvature representation for a new fifth-order integrable system , *to appear in Fund. Prink. Mat.* **12**2006 (in Russian); English translation to appear in J. Math. Sci.; prepint version (in English): *nlin.SI*/0604064(arXiv.org)

[3] D. Baldwin, U. Goktas, W. Hereman, L. Hong, R.S. Martino & J.C.Miller, Symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDFs, *J. Symbolic Compt.* **37**(2004), no. 6, 669-705; Prepint version: nlin.SI/0201008(arXiv.org)

[4] N.A. Kudryashov ,Nonlinear differential equations with exact solutions expressed via the Weierstrass function, *Z. Naturforsch. A*, **59** (2004), no. 7-8, 443-454. Prepint version: nlin.CD/0312035 (arXiv.org).

[5] D.J. Kaup, On the inverse scattering problem for cubic eingevalue problems of the class $\phi_{xxx} + 6q\phi_x + 6r\phi = \lambda\phi$, *Stud. Appl. Math.* **62**(1980), 189-216.

[6] E. Fan & Y.C. Hon, Generalized tanh Method Extended to Special Types of Nonlinear Equations, *Z. Naturforsch. A*, **57**(2002), no. 8, 692-700.

[7] Z. Yan, The Riccati equation with variable coefficients expansion algorithm to find more exact solutions of nonlinear differential equation, *Comput. Phys. Comm.* **152**(2003), no. 1, 1-8. Prepint version available at **http://www.mmrc.iss.ac.cn/pub/mm22.pdf/20.pdf**

[8] M. Remoissenet, *Waves Called Solitons*, Springer, Dijon (1999), p.277

[9] A. M Wazwaz, The extended tanh method for abundant solitary wave solutions of nonlinear wave equations, *Applied Mathematics and Computation***2**(2006).

[10] C. A. Gómez, *Exact Solutions For a New Fifth-Order Integrable System* , Submitted to Revista Colombiana de Matematicas. **2**(2006).

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