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Meromorphic Functional Calculus and Local Spectral Theory

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1. Introduction

Let X be a complex Banach space, let $T, S \in L(X)$ be commuting continuous linear operators and let $x \in X$. Denoting by $\sigma(x,T)$ the local spectrum of T at x, we have [4, Proposition 1.5] that

$$\sigma(Sx,T) \subset \sigma(x,T). \tag{1}$$

Bartle [1] derived the following relations for $n \in \mathbb{N}$ and $S = (\alpha - T)^n$

$$\sigma(Sx,T) \subset \sigma(x,T) \subset \sigma(Sx,T) \cup \{\alpha\}, \tag{2}$$

where α is a complex number. Similar results have been derived in [2] for S an operator given by the local functional calculus (see [3] for further details).

In this paper we study this problem when S is given by the meromorphic functional calculus. We observe that this operator is closed and unbounded, in general.

As an application we derive some properties of the meromorphic functional calculus such as: relations between the restriction and the coinduced operators of the meromorphic functional calculus, the local spectral mapping theorem (with a different proof from the usual one for the holomorphic functional calculus) and the stability of the Single Valued Extension Property.

Let X be a complex Banach space. We denote by $L(X)$ the class of all (bounded linear) operators on X and by $C(X)$ the class of all closed linear operators with *domain* $D(A)$ and *range* $R(A)$ in X. We say that a closed subspace $Y \subset X$ is an *invariant subspace* under $A \in C(X)$, in symbols $Y \in$

$Inv(T)$, if $A(Y \cap D(A)) \subset Y$. An invariant subspace Y produces two operators: the *restriction* $A|Y$ defined in $D(A) \cap Y$ by $A|Yy = Ay$ and the *coinduced* operator A/Y on the quotient space X/Y, defined by and $(A/Y)(x+Y) = Ax + Y$ on

$$D(A/Y) := \{x + Y \in X/Y : \ (x+Y) \cap D(A) \neq \emptyset\}.$$

Given an operator $A \in C(X)$, a complex number λ belongs to the *resolvent set* $\rho(A)$ of A if there exists $R(\lambda, A) := (\lambda - A)^{-1} \in L(X)$. We denote by $\sigma(A) := \mathbb{C} \backslash \rho(A)$ the *spectrum* of A. The *resolvent map* $R(.,A) : \rho(A) \to L(X)$ is analytic.

Moreover, given an arbitrary closed linear operator $A : D(A) \subset X \longrightarrow X$ and $x \in X$, we say that a complex number λ belongs to the *local resolvent set* of A at x, denoted $\rho(x, A)$, if there exists an analytic function $w : U \subset \mathbb{C} \longrightarrow D(A)$, defined on a neigbourhood U of λ, which satisfies

$$(\mu - A)w(\mu) = x, \tag{3}$$

for every $\mu \in U$. The *local spectrum set* of A at x is $\sigma(x, A) := \mathbb{C} \setminus \rho(x, A)$.

Since w is not necessarily unique, a complementary property is needed to prevent ambiguity. A linear operator A satisfies the *Single Valued Extension Property* (hereafter referred to as SVEP) if for every analytic function $h : U \longrightarrow D(A)$ defined on an open $U \subset \mathbb{C}$, the condition $(\lambda - A)h(\lambda) \equiv 0$ implies $h \equiv 0$. If A satisfies the SVEP, then for every $x \in X$ there exists a unique analytic function $\widehat{x}_A$ defined on $\rho(x, A)$ satisfying (3), which is called the *local resolvent function of A at x*.

We denote also by $\mathbb{C}_\infty$ the one-point compactification of the complex field $\mathbb{C}$. Given $A \in C(X)$ and $x \in X$, we say that $\infty \in \rho_\infty(x, A) := \mathbb{C}_\infty \backslash \sigma_\infty(x, A)$, if there exist an open neighborhood U_∞ and an analytic function $u : U_\infty \to D(A)$ such that $(\mu - A)u(\mu) = x$ for $\mu \in U_\infty \cap \mathbb{C}$.

For every subset $H \subset \mathbb{C}$, $X(A, H) = \{x \in X : \ \sigma(x, A) \subset H\}$ is a linear manifold of X. If $X(A, F)$ is closed for all closed set F, we say that A has *property (C)*. If $T \in L(X)$ satisfies property (C), then T has the SVEP [8, Theorem 2.3].

For $T \in L(X)$, the *holomorphic functional calculus* is defined as follows [9]. Let f be an analytic function defined on an open set $\triangle(f)$ containing $\sigma(T)$. The operator $f(T) \in L(X)$ is defined by the "Cauchy formula"

$$f(T) := \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, T) d\lambda,$$

where Γ is the boundary of a Cauchy domain D such that $\sigma(T) \subset D \subset \overline{D} \subset \triangle(f)$.

The definition of the holomorphic functional calculus was extended to meromorphic functions by Gindler [7]. Let $T \in L(X)$ and let f be a meromorphic function on an open set $\Omega_T(f)$ containing $\sigma(T)$, and let $\alpha_1, \ldots, \alpha_k$ be the poles of f in $\sigma(T)$, with multiplicities $n_1, \ldots, n_k$, respectively. We assume that the poles of f are not eigenvalues of T, and consider the polynomial $p(\lambda) = \prod_{i=1}^{k}(\alpha_i - \lambda)^{n_i}$.

The function $g(\lambda) := f(\lambda)p(\lambda)$ is analytic on a neighborhood of $\sigma(T)$. So we can define the operator $f\{T\}$ of the *meromorphic functional calculus* by

$$f\{T\} := g(T)p(T)^{-1},$$

obtaining a closed operator $f\{T\}$. Obviously, the meromorphic calculus is an extension of the holomorphic calculus.

We denote by $M(T)$, the class of the admisible functions of the meromorphic functional calculus for T.

2. Main results

Before analyzing the relations between $f\{T\}$, $f\{T|Y\}$ and $f\{T/Y\}$, we need the definition of some classes of closed invariant subspaces.

Recall that given $A \in C(X)$, an invariant subspace Y of A is said to be *ν-space* of A if $\sigma(A|Y) \subset \sigma(A)$ [5, Definition 4.1]. And Y is said to be an *A-absorbent* space if for any $y \in Y$ and all $\lambda \in \sigma(A|Y)$, the equation $(\lambda - A)x = y$ has all solutions $x \in Y$ [5, Definition 4.17]. It is clear that an A-absorbent space is a ν-space [5, Proposition 4.18].

THEOREM 1. *Let $T \in L(X)$ and $f \in M(T)$. If Y is a T-absorbent space, then the following properties hold:*

(i) *$f \in M(T|Y)$ and $f\{T\}|Y = f\{T|Y\}$.*

(ii) *$f \in M(T/Y)$ and $f\{T\}/Y = f\{T/Y\}$.*

In order to give relations between the meromorphic functional calculus and the local spectrum, the following result will be useful.

PROPOSITION 1. *Assume $T \in L(X)$ has the SVEP and $f \in M(T)$. Then the following properties hold:*

(i) *If* $S \in L(X)$ *commutes with* T*, then* S *commutes with* $f\{T\}$.

(ii) *If* $x \in D(f\{T\})$*, then* $\sigma(f\{T\}x, T) \subset \sigma(x, T)$.

(iii) *If* $x \in D(f\{T\})$*, then*

$$\sigma(x, T) = \sigma(f\{T\}x, T) \cup Z_x(f, T),$$

where $Z_x(f, T)$ *denotes the set of all zeros of* f *in* $\sigma(x, T)$.

Remark 1. (a) Note that part (ii) and (iii) of the above proposition are similar versions of the equations (1) and (2) respectively for the meromorphic functional calculus.

(b) If we assume in part (ii) that f is an analytic function on $\triangle(f)$ which is identically zero in no component of $\triangle(f) \cap \sigma(T)$, then

$$\sigma(x, T) = \sigma(f\{T\}x, T) \cup \{\alpha \in \sigma(x, T) \cap \sigma_p(T) \ : \ f(\alpha) = 0\},$$

where $\sigma_p(T)$ denotes the set of all eigenvalues of T.

Applying the ideas of [10], it is possible to prove the following theorem for $T \in L(X)$ satisfying the SVEP. However, we give a different proof for $T \in L(X)$ satisfying the property (C) by using Proposition 1.

THEOREM 2. *(Local spectral mapping theorem). Assume* $T \in L(X)$ *has property (C) and let* $f \in M(T)$*. Then for every* $x \in X$,

$$\sigma_\infty(x, f\{T\}) = f(\sigma(x, T)).$$

COROLLARY 1. *Assume* $T \in L(X)$ *has the SVEP,* $f \in M(T)$ *and* Y *is a* T*-absorbent space. Then the following assertions hold:*

(i) $\sigma(y, f\{T\}|Y) = f(\sigma(y, T))$.

(ii) $\sigma((x + Y, f\{T\}/Y) = f(\sigma(x + Y, T/Y)) \subset f(\sigma(x, T))$.

PROPOSITION 2. *Let* $T \in L(X)$ *and let* F *be a closed subset of* $\mathbb{C}$*. If* $f \in M(T)$*, then*

$$X(f\{T\}, F) = X(T, f^{-1}(F)).$$

Using the ideas from [6] and [8], we show that the SVEP is stable under the meromorphic functional calculus.

THEOREM 3. *(Stability of the SVEP). If $T \in L(X)$ satisfies the SVEP, then for each $f \in M(T)$, the operator $f\{T\}$ satisfies the SVEP. Conversely, if $f \in M(T)$ is constant in no component of its domain and $f\{T\}$ satisfies the SVEP, then T satisfies the SVEP.*

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