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A Note on Zemanian Spaces†

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In [8,9] Zemanian extended the (non-modified) Hankel transformation to distributions. An essential ingredient of the construction consisted in finding an appropriate testing function space on which Hankel transform acts as an automorphism. What is now called the Zemanian space, and what we will denote by $\mathcal{Z}_\nu$, consists of all C^∞ functions ψ on $\mathbf{R}_+$ for which the quantities

$$\gamma_{n,k}(\psi) = \sup_{x>0} \left| x^n \left(\frac{1}{x} \frac{d}{dx} \right)^k (x^{-\nu-1/2}\psi(x)) \right|, \quad n,k = 0,1,2,\ldots$$

are finite. It was shown by Zemanian [8,9] that $\mathcal{Z}_\nu$ with the topology $\mathcal{T}_\nu(\gamma)$ induced by the family of semi-norms $\gamma_{n,k}$ is a Frechet space and Hankel transform is an automorphism of $(\mathcal{Z}_\nu, \mathcal{T}_\nu(\gamma))$.

This note suggests an alternative approach to this issue. We first solve a similar problem in the modified Hankel transform setting. As one can expect the space of even Schwartz–class functions does the job, cf. Proposition 2. Then, by using natural connections linking both transforms, we translate obtained testing space to the setting of the Hankel transform. This shows that $\mathcal{S}_e(\mathbf{R})$ skewed by the factor $x^{\nu+1/2}$ is appropriate for the Hankel transform, see Corollary 3. Finally, we check that the resulting space coincides with the original Zemanian space and this is the content of Proposition 4.

We would like to stress the fact that we do not pretend to claim the main results of this note to be new (see the papers of van Eijndhoven and de Graaf [4] and van Eijndhoven and van Berkel [5]). We find it reasonable, however,

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to present here straightforward proofs of these results. Also, we take an opportunity to point out the fact that considering the Zemanian space as the skewed space of even Schwartz functions is pretty effective in several situations. Throughout this note $\nu \geq -1/2$ is assumed to be fixed and $\mathbf{R}_+$ denotes the half-line $(0,\infty)$. Given f in $L^1(\mathbf{R}_+, x^{2\nu+1}dx)$ its modified Hankel transform $H_\nu f$ is defined by

$$H_\nu f(x) = \int_0^\infty \frac{J_\nu(xy)}{(xy)^\nu} f(y) y^{2\nu+1} dy, \quad x > 0.$$

Here $J_\nu(x)$ denotes the Bessel function of the first kind of order ν. When $\nu = (n-2)/2$, $n \geq 2$, the modified Hankel transform $H_\nu f$ replaces the Fourier transform of the radial function $f(||x||)$ on $\mathbf{R}^n$, see [7, p.155]. Let $\mathcal{S}_e(\mathbf{R})$ denote the space of all even Schwartz functions on $\mathbf{R}$. We will frequently tacitly identify elements of $\mathcal{S}_e(\mathbf{R})$ with their restrictions to $\mathbf{R}_+$.

We start with the following elementary result.

LEMMA 1. *A C^∞ function φ on $\mathbf{R}_+$ belongs to $\mathcal{S}_e(\mathbf{R})$ if and only if*

$$||\varphi||_{n,k} = \sup_{x>0} \left| x^n \left(\frac{1}{x}\frac{d}{dx}\right)^k \varphi(x) \right| < \infty \tag{1}$$

for all $n, k = 0, 1, 2, \ldots$.

Proof. An elementary argument based on succesive application on Taylor's formula shows that if f is an even C^∞ function on $\mathbf{R}$ then the function

$$F(x) = \begin{cases} f'(x)/x & , \ x > 0 \\ f''(0) & , \ x = 0 \\ f'(-x)/(-x) & , \ x < 0 \end{cases}$$

is also an even C^∞ function on $\mathbf{R}$. Let $\varphi \in \mathcal{S}_e(\mathbf{R})$. Then the Mean Value Theorem shows that $|\varphi'(x)| \leq Cx, x > 0$, where $C = \sup_{t>0} |\varphi''(t)|$. Hence the inequality

$$\sup_{x>0} \left| x^n \left(\frac{1}{x}\frac{d}{dx}\right)^k \varphi(x) \right| < \infty$$

follows for $n = 0$ and $k = 1$. For the general case with arbitrary $n, k = 0, 1, 2 \ldots$, we then apply the preceding remark. To show the opposite implication we first note that the conditions

$$\sup_{x>0} \left| \left(\frac{1}{x}\frac{d}{dx}\right)^k f(x) \right| < \infty, \quad k = 0, 1, 2, \ldots,$$

imposed on a C^∞ function f on $\mathbf{R}_+$ imply $f^{(2k)}(x) = O(1)$ and $f^{(2k+1)}(x) = O(x)$ for $k = 0, 1, 2, \ldots$ as $x \to 0^+$. This easily follows by induction on k once we remark that

$$\left(\frac{d}{dx}\right)^{2k} = \sum_{j=0}^{k} A_{kj} x^{2(k-j)} \left(\frac{1}{x}\frac{d}{dx}\right)^{2k-j}, \quad k = 1, 2, \ldots, \tag{2}$$

and

$$\left(\frac{d}{dx}\right)^{2k+1} = \sum_{j=0}^{k} B_{kj} x^{2(k-j)+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{2k+1-j}, \quad k = 0, 1, \ldots, \tag{3}$$

where the A's and B's denote constants. Let φ satisfy (1). We will show that the function

$$G(x) = \begin{cases} \varphi(x) & , \ x > 0 \\ \lim_{t\to 0} \varphi(t) & , \ x = 0 \\ \varphi(-x) & , \ x < 0 \end{cases}$$

has the derivatives of any order at zero, hence it is an even Schwartz function on $\mathbf{R}$. Existence of $\lim_{t\to 0} \varphi(t)$ follows from

$$\varphi(x_1) - \varphi(x_2) = \varphi'(\theta_{x_1,x_2})(x_1 - x_2), \quad 0 < x_1 < \theta_{x_1,x_2} < x_2,$$

and the estimate $\varphi'(x) = O(1)$, $x \to 0$. Hence $G(x)$ is continuous at $x = 0$. To show that $G'(0)$ exists and equals zero take an $\epsilon > 0$ and choose a positive δ such that $|\varphi'(x)| \le \epsilon/2$ for $0 < x < \delta$. Let $0 < h < \delta$. Choosing $0 < h_1 < \delta$ such that $|\varphi(h_1) - \lim_{t\to 0^+} \varphi(t)| \le \epsilon h/2$ we find

$$\begin{aligned} \left|\frac{G(h) - G(0)}{h}\right| &\le \left|\frac{\varphi(h) - \varphi(h_1)}{h}\right| + \epsilon/2 \\ &\le |\varphi'(\theta_{h,h_1})| \cdot \frac{|h - h_1|}{h} + \epsilon/2 \le \epsilon. \end{aligned}$$

Continuity of $G'(x)$ at $x = 0$ now follows from the estimate $\varphi'(x) = O(x)$, $x \to 0^+$. Existence of $\lim_{t\to 0^+} \varphi^{(2k)}(t)$, $G^{(2k)}(0)$ and $G^{(2k+1)}(0)$ and the identities $G^{(2k)}(0) = \lim_{t\to 0^+} \varphi^{(2k)}(t)$, $G^{(2k+1)}(0) = 0 = \lim_{t\to 0^+} \varphi^{(2k+1)}(t)$, $k = 1, 2, \ldots$, now proceed by induction on k by using an argument analogous to that given above. This finishes the proof of the opposite implication, hence the lemma. ∎

We now equip $\mathcal{S}_e(\mathbf{R})$ with a topology $\mathcal{T}$ generated by the family of seminorms defined in (1). By [10, Lemma 5.2.2, p.131] it can be easily checked that $(\mathcal{S}_e(\mathbf{R}), \mathcal{T})$ is a Frechet space. The inversion formula $H_\nu H_\nu \varphi = \varphi$ holds for φ in $\mathcal{S}_e(\mathbf{R})$. In fact, A.L. Schwartz [6] established validity of such a formula under much less restrictive assumptions on a function φ.

PROPOSITION 2. *The modified Hankel transform H_ν acts as an automorphism on $(\mathcal{S}_e(\mathbf{R}), \mathcal{T})$.*

Proof. For $\mu \geq -1/2$, φ in $\mathcal{S}_e(\mathbf{R})$ and $k = 0, 1, 2, \ldots$ there hold

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k H_\mu \varphi = (-1)^k H_{\mu+k} \varphi \tag{4}$$

and

$$H_{\mu+k}\left(\left(\frac{1}{y}\frac{d}{dy}\right)^k \varphi\right) = (-1)^k H_\mu \varphi. \tag{5}$$

(4) and (5) are implied, for $k = 1$, by the identities

$$\frac{1}{t}\frac{d}{dt}\left(\frac{J_\mu(t)}{t^\mu}\right) = -\frac{J_{\mu+1}(t)}{t^{\mu+1}}, \quad \frac{1}{t}\frac{d}{dt}(t^{\mu+1} J_{\mu+1}(t)) = t^\mu J_\mu(t)$$

and the general case then follows. Using both (4) and (5) gives for $n, k = 0, 1, \ldots$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k H_\nu \varphi(x) = (-1)^{n+k} H_{\nu+k+n}\left(\left(\frac{1}{y}\frac{d}{dy}\right)^n \varphi\right)(x).$$

Then, multiplying the above by x^n, $x > 0$, gives the bound

$$x^n \left|\left(\frac{1}{x}\frac{d}{dx}\right)^k H_\nu \varphi(x)\right| \leq \int_0^\infty y^{2(\nu+k)+n+1} \left|\left(\frac{1}{y}\frac{d}{dy}\right)^n \varphi(y)\right| \frac{|J_{\nu+k+n}(xy)|}{(xy)^{\nu+k}} dy$$

and boundedness of $J_{\nu+k+n}(t)/t^{\nu+k}$ on $\mathbf{R}_+$ yields

$$||H_\nu \varphi||_{n,k} \leq C \sum_{j=0}^{m+1} ||\varphi||_{2j,n},$$

where $m \geq \nu + k + (n+1)/2$ is an integer (cf. the proof of Lemma 8 in [8]). This and Lemma 1 show that H_ν does map $\mathcal{S}_e(\mathbf{R})$ into itself and is continuous. The inversion formula guarantees that H_ν is also a bijection. This finishes the proof of Proposition 2. ∎

Given f, an integrable (with respect to the Lebesgue measure) function on $\mathbf{R}_+$, its Hankel transform $\mathcal{H}_\nu f$ is defined by

$$\mathcal{H}_\nu f(x) = \int_0^\infty (xy)^{1/2} J_\nu(xy) f(y) dy, \quad x > 0.$$

Both transforms are related to each other by

$$H_\nu f(x) = x^{-(\nu+1/2)} \mathcal{H}_\nu((\cdot)^{\nu+1/2} f(\cdot))(x) \tag{6}$$

whenever $x^{\nu+1/2} f$ is an integrable function on $\mathbf{R}_+$ and $\int_0^\infty |f(x)| x^{2\nu+1} dx < \infty$. In particular,

$$M_\nu H_\nu \varphi = \mathcal{H}_\nu M_\nu \varphi, \quad \varphi \in \mathcal{S}_e(\mathbf{R}) \tag{7}$$

where M_ν denotes the operator of multiplication by $x^{\nu+1/2}$. Consider the space $\mathcal{S}_\nu = x^{\nu+1/2} \cdot \mathcal{S}_e(\mathbf{R})$ with the topology translated from $\mathcal{S}_e(\mathbf{R})$ by means of the operator M_ν, that is the topology induced by the family of semi-norms

$$p_{n,k}(x^{\nu+1/2}\varphi) = ||\varphi||_{n,k}, \quad \varphi \in \mathcal{S}_e(\mathbf{R}).$$

We denote this topology by $\mathcal{T}_\nu(p)$.

COROLLARY 3. *The Hankel transform $\mathcal{H}_\nu$ is an automorphism of $(\mathcal{S}_\nu, \mathcal{T}_\nu(p))$.*

Proof. The corollary is a consequence of Proposition 2, (7) and the definition of the space and topology. ∎

PROPOSITION 4. *The spaces $\mathcal{Z}_\nu$ and $\mathcal{S}_\nu$ and the topologies $\mathcal{T}_\nu(\gamma)$ and $\mathcal{T}_\nu(p)$ coincide.*

Proof. Assuming we know that $\mathcal{Z}_\nu = \mathcal{S}_\nu$ the second part of the statement is obvious since $\gamma_{n,k}(\psi) = p_{n,k}(\psi)$, $n, k = 0, 1, \dots$, for every $\psi = x^{\nu+1/2}\varphi$, $\varphi \in \mathcal{S}_e(\mathbf{R})$. To prove the first part of the statement note that by the definition of $\mathcal{Z}_\nu$ it is sufficient to show that $\mathcal{Z}_{-1/2} = \mathcal{S}_e(\mathbf{R})$ and this was done in Lemma 1. ∎

Remarks. 1) It is perhaps interesting to note that $\mathcal{T}$ coincides with the Frechet topology $\mathcal{T}_1$ defined on $\mathcal{S}_e(\mathbf{R})$ by the family of semi-norms

$$|\varphi|_{n,k} = \sup_{x>0} \left| x^n \left(\frac{d}{dx}\right)^k \varphi(x) \right|, \quad n, k = 0, 1, 2, \dots.$$

To see this, note that by (2)

$$\left|x^n\left(\frac{d}{dx}\right)^{2k}\varphi(x)\right| \le \sum_{j=0}^{k}|A_{kj}|\cdot\left|x^{n+2(k-j)}\left(\frac{1}{x}\frac{d}{dx}\right)^{2k-j}\varphi(x)\right|.$$

Hence

$$|\varphi|_{n,2k} \le \sum_{j=0}^{k}|A_{kj}|\cdot||\varphi||_{n+2(k-j),2k-j},$$

and a similar inequality follows from (3) for $2k+1$ in place of $2k$. This means that the identity map

$$id : (\mathcal{S}_e(\mathbf{R}),\mathcal{T}) \to (\mathcal{S}_e(\mathbf{R}),\mathcal{T}_1)$$

is continuous. The Open Mapping Theorem for Frechet spaces now shows that this mapping is open, hence $\mathcal{T}=\mathcal{T}_1$.

2) Explicit representation of functions in $\mathcal{Z}_\nu$ given in Proposition 4 immediately shows that $\mathcal{Z}_{\nu+2k}\subset\mathcal{Z}_\nu$, $k=1,2,\dots$, [8, Lemma 1], and no other inclusion is possible on the scale of Zemanian spaces. Also, in another contexts the representation occurs to be useful. For instance, consider the problem of characterizing these functions m on $\mathbf{R}_+$ that possess the property that the operator of multiplication by m is a continuous endomorphism of a fixed Zemanian space $\mathcal{Z}_\nu$, cf. [3]. Keeping in mind the identity $\mathcal{Z}_\nu = x^{\nu+1/2}\mathcal{S}_e(\mathbf{R})$ the problem turns out to be equivalent to the following: characterize all functions $m:\mathbf{R}_+\to\mathbf{C}$ such that $m\mathcal{S}_e(\mathbf{R})\subset\mathcal{S}_e(\mathbf{R})$ continuously. In the classical case (when the symbol e is dropped the answer is: C^∞ functions of polynomial growth (which means polynomial growth of the function and all its derivatives). In the case we consider we have, clearly, to restrict the attention to the even C^∞ functions of polynomial growth (the condition (i) from Theorem 2.3 in [3] is equivalent to the statement that $\theta(x)$ is a restriction to $\mathbf{R}_+$ of a C^∞ function on $\mathbf{R}$ of polynomial growth).

3) Using the identity $(\frac{d}{dx})^2 = x^2(\frac{1}{x}\frac{d}{dx})^2 - \frac{1}{x}\frac{d}{dx}$ it follows by induction on k that

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k\left(\frac{d}{dx}\right)^2 = (1+x^2)\left(\frac{1}{x}\frac{d}{dx}\right)^{k+1} + 2(k-1)\left(\frac{1}{x}\frac{d}{dx}\right)^k, \quad k=1,2,\dots.$$

Hence, for any $n=0,1,2,\dots$, $||(\frac{d}{dx})^2\varphi||_{n,0} \le ||\varphi||_{n,1} + ||\varphi||_{n+2,2}$, $\varphi\in\mathcal{S}_e(\mathbf{R})$, and

$$\left\|\left(\frac{d}{dx}\right)^2\varphi\right\|_{n,k} \le ||\varphi||_{n,k+1} + ||\varphi||_{n+2,k+1} + 2(k-1)||\varphi||_{n,k}, \quad k=1,2,\dots.$$

This means that $(\frac{d}{dx})^2$, as well as the operator

$$D_\nu = \left(\frac{d}{dx}\right)^2 + \frac{2\nu+1}{x}\frac{d}{dx},$$

is continuous on $(\mathcal{S}_e(\mathbf{R}), \mathcal{T})$ (recall that $\frac{1}{x}\frac{d}{dx}$ does act on $\mathcal{S}_e(\mathbf{R})$). If we denote $\phi_y^\nu(x) = \frac{J_\nu(xy)}{(xy)^\nu}$, $y > 0$, then $D_\nu \phi_y^\nu = -y^2 \phi_y^\nu$ and

$$D_\nu H_\nu \varphi = -H_\nu(y^2\varphi), \quad H_\nu(D_\nu\varphi) = -x^2 H_\nu \varphi,$$

$\varphi \in \mathcal{S}_e(\mathbf{R})$. Analogously, the second order differential operator

$$\mathcal{D}_\nu = \left(\frac{d}{dx}\right)^2 + \frac{1/4-\nu^2}{x}$$

is continuous on $(\mathcal{Z}_\nu, \mathcal{T}(\gamma))$, [8, p. 565], and satisfies $\mathcal{D}_\nu \varphi_y^\nu = -y^2 \varphi_y^\nu$, where $\varphi_y^\nu(x) = (xy)^{1/2} J_\nu(xy)$, $y > 0$. Moreover

$$\mathcal{D}_\nu \mathcal{H}_\nu \psi = -\mathcal{H}_\nu(y^2\psi), \quad \mathcal{H}_\nu(\mathcal{D}_\nu\psi) = -x^2 \mathcal{H}_\nu \psi.$$

Since

$$M_\nu D_\nu \varphi = \mathcal{D}_\nu M_\nu \varphi, \quad \varphi \in \mathcal{S}_e(\mathbf{R}), \tag{8}$$

the last facts concerning the operator $\mathcal{D}_\nu$ follow from corresponding statements for D_ν by using (7) and (8).

4) There are different alternative descriptions of the Zemanian spaces. For this and another Hankel invariant test function spaces see [2, 4, 5] and papers cited there.

5) The fact that H_ν acts as an authomorphism on $(\mathcal{Z}_{-1/2}, \mathcal{F}_{-1/2}(\gamma))$ was proved in [1]. But due to (6) this is nothing else but merely an equivalent reformulation of the original Zemanian result that $\mathcal{H}_\nu$ acts as an authomorphism on $(\mathcal{Z}_\nu, \mathcal{F}_\nu(\gamma))$.

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