SOLVING PROBLEMS ON FUNCTIONS: ROLE OF THE GRAPHING CALCULATOR

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To study the roles that the graphing calculator plays in solving problems about functions, a small quasi-experimental study was conducted with four pairs of undergraduate students solving problems with and without the graphing calculator. The analysis of the protocols of the sessions did not reveal major differences that could be attributed to the presence or absence of the tool but indicated differences in strategies used with each problem that could be explained in terms of the nature of the knowledge at stake and to students' availability of that knowledge. The study suggests a model for conducting research that looks for explaining the effects of technology in learning and instruction.

Keywords: Functions; Graphing calculators; Problem solving

Resolución de Problemas sobre Funciones: Papel de la Calculadora Gráfica

Con el fin de analizar el papel que la calculadora gráfica juega en la resolución de problemas sobre funciones, se hizo un pequeño estudio cuasi-experimental con cuatro pares de estudiantes de pre-grado variando la condición de la disponibilidad de la calculadora. El análisis de los protocolos de las sesiones revela que no hay mayores diferencias que se puedan atribuir a la presencia o ausencia de la calculadora gráfica; sin embargo, las diferencias observadas en el uso de estrategias que se usaron en cada problema pueden explicarse en términos de la naturaleza del conocimiento en juego y de la disponibilidad de tal conocimiento para los estudiantes. El estudio sugiere además un modelo para realizar investigaciones que busquen explicar los efectos de la tecnología en el aprendizaje y en la instrucción.

Términos clave: Calculadoras gráficas; Funciones; Resolución de problemas

Graphing calculators have become part of high school mathematics classrooms. A survey of calculator usage in high schools commissioned by the College Board

Mesa, V. (2008). Solving problems on functions: role of the graphing calculator. *PNA, 2*(3), 109-135.

(Dion et al., 2001) indicated that graphing calculators are either required or allowed in at least 87% of the mathematics classes offered in high schools (p. 430). This imposes an interesting challenge to both college mathematics teachers and to mathematics educators who are responsible of preparing future mathematics teachers, as many of their students may come with experience with graphing calculators from their high school. A review of the research involving graphing calculators at the undergraduate level shows at least two types of studies. On the one hand, there are studies that investigate the *impact* of introducing graphing calculators in the classrooms on students' motivation, attitude, achievement, and retention (Hennessy, 1997; Hollar & Norwood, 1999; Quesada & Maxwell, 1994; Smith & Schotsberger, 1997). On the other hand, there are studies that investigate students' *understanding* of the content or their discursive practices in the classroom in relation to the representations offered by graphing calculators (Dick, 2000; Kaput, 1992; Roschelle, Pea, Hoadley, Gordin, & Means, 2000; Ruthven, 1990; Shumway, 1990; Slavit, 1994). Both types of studies work under the assumption that the immediate availability of multiple representations of mathematical objects facilitate the process of making connections among those representations which in turn produces more robust or *connected* learning (Hiebert & Carpenter, 1992; Schoenfeld, 1987). However, using the graphing calculator efficiently in the classroom or documenting what actually is done with the tool has proven to be more difficult to accomplish. Teachers' beliefs and how students organize themselves to work on problems have been cited as reasons why implementations with graphing calculators do not work as expected (Demana, Schoen, & Waits, 1993; Simmt, 1997).

In this article I want to suggest that the nature of the tasks, students' previous mathematical knowledge, and their experiences with graphing technology—independently of the availability of the graphing calculator—shape the collaborative construction of solutions among pairs of students. The present study was carried out to investigate the roles that the graphing calculator played when students had controlled access to it in a problem solving session. Studies that look at large effects of introducing the graphing calculator in classrooms (e.g., contrast overall achievement of a group of students when technology is present vs. not present) overlook the fact that the curriculum that is offered to each group is not comparable, and therefore it is not possible to conclude that differences in achievement, attitudes, or retention could be attributed only to the presence of the graphing calculators. And studies that look very closely at what happens when graphing calculators are used in the classroom, can not attribute results to the presence of the graphing calculator because there is not much knowledge about the particular aspects related to how the graphing calculator is used in specially crafted situations or about how problems are solved without the graphing calculator. I contend that analyses of such situations may allow us to better understand which outcomes can be attributed to the tool itself and which to other factors. Moreover, a closer look at what students can do with and without the tool might better in-form the process of curriculum design and organization, as well as the quality of the interplay between the graphing calculator, the content, and the students.

In this study, I wanted to know how pairs of students solved problems that were produced under the assumption that multiple connections among representations of a mathematical concept strengthen understanding (Kaput, 1992) and to determine how problems were solved under two different conditions, with the graphing calculator and without the graphing calculator. I wanted to know if the availability of the tool triggered questions that either guided their solutions or changed their solution strategies, how much time students spent solving the problems under each condition, and the level of success solving the problems under each condition. For the purposes of this article, I will report on answers to the following three questions: (a) What strategies do the students choose as they attempt to solve a problem on functions? (b) How are the strategies different when the graphing calculator is present and when the graphing calculator is not present? And, (c) what roles does the graphing calculator play in the solution process?

In the next section, I present a brief review of studies that may be seen as paradigmatic of research with graphing calculators in general and of graphing calculators in problem solving in particular.

Problem Solving and Graphing Calculators

Reform documents such as the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) and *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) have put a strong emphasis on the use of technology within the classroom. According to Smith (1998), teachers facing the challenge of introducing graphing calculators into their classrooms have used them in four distinguishable types of enrichments: tools for expediency (they save time used for tedious or difficult procedures), amplifiers for conceptual understanding (they offer multiple linked representations), catalysts for critical thinking (they allow exploration of "what if" questions), and vehicles for integration with other disciplines (they facilitate the work in other disciplines, like physics or programming). These are *a posteriori* attributions that describe instruction with the tool; it is an open question whether these attributed roles depend on the particular tool (the same attributions can be made about a computer with a graphing program). A tool has a limited set of functions and operations; it is the mathematical activities that impose conditions on how the tool is used in a given situation (see Balacheff, 1993). A main advantage of graphing calculators and computers is that they ease teachers' burden of creating materials, such as overhead transparencies for example. And between graphing calculators and computers, the "power of the small and easy-to-use computer" (p. 1) does make an important difference in terms of portability[1].

A typical experimental study contrasting the outcomes of two classes, one intact and one that had graphing calculators, is Graham and Thomas's (2000). Their study aimed at evaluating the potential of a unit designed to help students understand algebraic variables[2]. The graphing calculator in the Graham and Thomas's study was taken as an amplifier for conceptual understanding, a catalyst for critical thinking, and as a vehicle for integrating other disciplines. The study involved 147 students using a graphing calculator with the experimental unit "Tapping into Algebra" and 42 students who followed their "normal algebra teaching whole class, skills-based instruction and assessment with their usual teacher presentation style" from six schools in New Zealand (p. 271). The activities used the graphing calculator storage variables: students would assign values to these variables and afterwards conjecture what would happen after modifications (numerical operations) were performed on those variables. Another activity consisted of students guessing the values of two variables when students knew results of some numerical operations (e.g., $A + B = 0$ and $A / B = -1$, p. 270). According to Graham and Thomas, in these activities the students were involved

> *in a* cybernetic *process where the technology reacts to the individual's actions according to pre-programmed and predictable rules. The environment provides consistent feedback in which students can predict and test, enabling them to construct an understanding of letters in algebra* (emphasis in original, p. 270).

There was a statistically significant difference in performance between the two groups of students—in favor of the experimental group—in a post-test that measured "understanding of the use of letters as specific unknown, generalized number and variable" (p. 272). The higher performance of the experimental group was independent of prior student ability. Students in the control group statistically outperformed students in the experimental group in only one of the procedural skill items in the posttest ("simplify $(a + b) + a$"). There were no significant differences between the two groups in other procedural items. From these results, authors concluded that students "can obtain an improved understanding of the use of letters as specific unknown or generalized number from a module of work based on the graphic calculator" (p. 278). When no significant differences are found between the two groups, as is in the case of the Smith and Schotsberger (1997) and Alkhateeb and Wampler (2002) studies, the implication is that the graphing calculator is not detrimental in terms of what students learn.

[1] The argument of portability has become increasingly more popular among advocates of handheld computers (Vahey, Tatar, & Roschelle, 2004).

[2] See also the following experimental studies at the college level: Alkhateeb and Wampler (2002) on derivatives; Hollar and Norwood (1999) and Quesada and Maxwell (1994) on functions; and Smith and Schotsberger (1997) on college algebra.

These studies illustrate one of the main difficulties with experimental research on innovations that incorporate the graphing calculator: the control group is never exposed to comparable mathematics as the experimental group, the assumption being that the control class should be intact. If one of the points at stake is to know whether the graphing calculator makes a difference, a more appropriate approach (and for some, more ethical) to this kind of research would be one in which the control group experiences a module that is *not* based on the graphing calculator but that preserves its spirit in every other aspect. If in this case, the cybernetic process attributable to the presence of the graphing calculator does not occur, statistically significant different results in performance in favor of the experimental group could be attributed to the absence of such cybernetic process. One of the reasons that no studies have been conducted in such a way may be that it is difficult to imagine what the dual environment might be. However, it might be possible to deal with this fundamental problem.

The issue raised about the difficulty of conducting these studies is connected to the types of activities teachers need to use when the graphing calculator is available, because the activities assume a re-conceptualization of what it means "to do mathematics". But when doing mathematics is equated to problem solving[3] (as it helps to build new knowledge, both within and outside mathematics, National Council of Teachers of Mathematics, 2000, pp. 52-53) a natural question that arises is: How do graphing calculators support problem solving activities? A number of studies exist that describe ways in which graphing calculators offer learning opportunities to students. Hennessy, Fung, and Scanlon (2001) reported several features of the technology that can structure and support collaborative problem solving. Their study dealt with undergraduate students using graphing calculators in an innovative course at the Open University that looked for fostering understanding of graphing. Drawing from Vygotsky's ideas of tool mediation, from collaborative learning theory, and from work on gender and computing, they devised three design principles for the activities used (p. 269):

- *Open-ended investigations in which graph recognition and interpretation are developed exploring different kinds of graphs.*
- *Personal ownership of the technology (one machine per student) and of the activity (student choice of problem solving approach).*
- *Collaboration in planning and problem solving and ample opportunities for discussion with peers; use of individual machines to work on a shared task; achievement of written consensus.*

The course was aimed at building students' confidence in using mathematics (p. 270). By the end of the course students reported positive feelings and attitudes with respect to doing and learning mathematics, and an appreciation of the capa-

[3] In this article a *problem* is defined as a task for which no clear pattern of solution exist.

bilities of the graphing calculator to visualize, to facilitate and accelerate computations, and to give immediate feedback (i.e., the cybernetic process described by Graham & Thomas, 2000). In a follow-up study after four months of taking the innovative course, students were observed "planning, executing, and reporting calculator actions and strategies" (p. 275). The students used some trial and error, but for most of the session, were confident and were not surprised by the results obtained in the graphing calculator; they also engaged in productive explorations. Students were observed performing actions that would not have been worthwhile to conduct manually (e.g., checking if the graphs of two expressions superimpose). Hennessy et al. (2001) conclude that (a) the main advantages of the graphing calculator in facilitating students' learning of graphing fall in three broad categories: visualization of functions, automatic translation between representations and immediate feedback, and rapid and easy graph plotting (p. 278-279); and (b) students working collaboratively was an important factor explaining the richness of the problem solving session, because students were working synergistically (Noss & Hoyles, 1996) between interdependence and autonomy. The "use of technology was firmly embedded within and inseparable from the mathematical activity being undertaken" (Hennessy et al., 2001, p. 282)[4].

These studies highlight another difficulty for conducting investigations that analyze the impact of the graphing calculator in a particular setting. Authors make explicit the principles that guide the design of the tasks used in the courses—which also guides the design of the tasks for the case studies. Such principles, backed up by strong theoretical frameworks, assist authors in producing activities that help students develop certain mathematical notions with technology, with technology and collaborative work playing a fundamental role. Researchers collect information about students on these aspects, and find that, not only do students appreciate this way of teaching mathematics, but that in problem solving sessions, they exhibit work in which the "use of technology [is] firmly embedded within and inseparable from the mathematical activity being undertaken." I wonder, could this be otherwise? Is it possible, after creating such a course, that the use of technology was neither firmly embedded within nor inseparable from the mathematical activity being undertaken? Or, that the students did not work collaboratively to solve the problems? The difficulty arises because more than the presence of the tools—such as graphing calculator or collaborative work—it is the quality and nature of the mathematics that is being learned that has dramatically changed. Our recognition of this different mathematics learning confounds our appreciation of the actual role of technology and of group work.

The present study is a contribution to begin to understand these problems of investigating the role that technology in general, and the graphing calculator in

[4] See Figg and Burson (2005) and Roschelle et al. (2000) for similar studies that illustrate how graphing calculators, computers, and handheld technologies can be used to improve the way in which students learn.

particular, plays in problem solving sessions. The main questions—What strategies do students choose as they attempt to solve a problem on functions? How are the strategies different when the graphing calculator is present from when the graphing calculator is not present? And what roles does the graphing calculator play in the solution process?—will be explored in a more "controlled" setting, as is explained in the next section.

METHOD

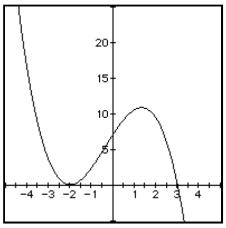
Instruments

Two problems were adapted from Gómez and Mesa (1995), a book that contains about 100 problems on pre-calculus. The problems are the result of a collective work that studied the effects of introducing the graphing calculator in a pre-calculus course (Carulla & Gómez, 1996; Gómez & Fernández, 1997; Mesa & Gómez, 1996; Valero & Gómez, 1996) and were created with the purpose of developing students' higher-order mathematical thinking (Gómez & Mesa, 1995, p. 5), according to Resnick's (1987) characterization (i.e., non-algorithmic problems having multiple solutions, a high level of uncertainty, and requiring self-regulation and a great deal of effort). Both problems required students to propose expressions for functions that would satisfy a given set of conditions. Problem 1 provided symbolic forms for the functions with parameters for the coefficients of the independent variable, and Problem 2 provided nine graphs of polynomial functions, some of them related, with very few precise numerical referents (see Figure 1).

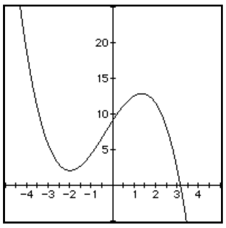
A full description of the content addressed, the reasons for choosing the wording, predicted strategies, and difficulties, and the possible hints that could be offered for overcoming the difficulties are described at length in Mesa (1996). The problems do not preclude the use of solutions based exclusively on symbolic approaches but symbolic approaches are not practical to find suitable solutions to the problems.

1. Find two functions, $f(x)=(x-k)^2+k$ *and* $g(x)=a|x-b|+c$ *such that the solution to the inequality* $f(x)\le g(x)$ *is the interval* $[2,5]$.

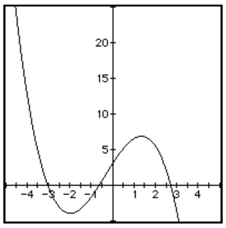
2. Give one expression for each of the functions shown:



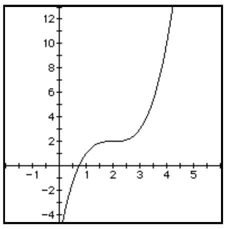
Graph 1



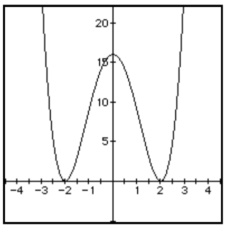
Graph 2



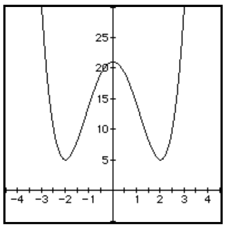
Graph 3



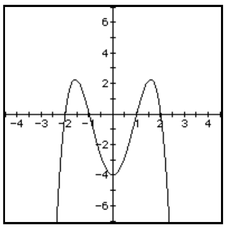
Graph 4



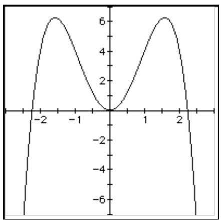
Graph 5



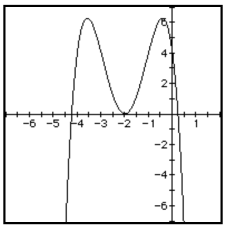
Graph 6



Graph 7



Graph 8



Graph 9

Figure 1. *Problems selected for the study (adapted from Gómez and Mesa, 1995, pp. 93, 102-103)*

The list of strategies, difficulties, and solutions that could emerge in the solution process were compiled from previous experiences with the problems, and were instrumental during the problem solving session to assist students during the session and for the analyses of the data that emerged.

Two versions of a problem-solving instrument were prepared. Each contained the two problems but with different instructions on the use of the graphing calculator. In the first version, the students were not allowed to use the graphing calculator in working Problem 1 but could use it for solving Problem 2. In the second version, these instructions were reversed. The purpose of using two versions was to contrast the processes used when the graphing calculator was available with those when the graphing calculator was not available. The graphing calculator used was the Texas Instruments TI-82.

A set of three questions—(a) How would things change if you were or were not allowed to use the graphing calculator for solving each problem?, (b) What knowledge do you think this kind of problems require?, and (c) What is your opinion about the problems?—were prepared for having an informal discussion once the session was over. With these questions I wanted to get the participants' perspective on the problems and on the appropriateness of using the graphing calculator in teaching mathematics.

Participants

Because of the nature of the problems I chose, I needed to carefully select the participants in order to guarantee that the problems would not look too alien to them. I asked permission to conduct in-class problem-solving sessions in two courses for students majoring in secondary mathematics education at a large southern university. I visited the secondary school mathematics curriculum class and the research seminar in mathematics education class, classes taken by the students in their third and fourth year in the program. During these in-class problem-solving sessions, I gave all the students a list of 9 problems that covered topics in linear, quadratic, cubic, quartic, and rational functions. I used these sessions for identifying participants and for fine-tuning the characteristics of the activities that would be undertaken during the research problem-solving session. The participants were chosen for their engagement with the class activities, the different ways in which they addressed questions, their interest in using the graphing calculator, and their willingness to participate in the study. I asked them to participate as pairs to facilitate their interactions during the problem solving session. Four pairs of students, A: Alice and Amy, B: Ben and Bill, C: Cindy and Connie, and D: Dina and Donna (all pseudonyms), were selected; all students had worked together frequently in class, except for Dina and Donna, who had worked together only occasionally. All students had completed the mathematics requirements of their programs (precalculus, calculus, linear algebra, modern algebra, geometry, discrete structures, history of mathematics, computers and algorithms, higher mathematics, and problem solving); one participant, Dina, had also taken differential equations and complex variables. Other characteristics (mathematical confidence and performance and interest in inquiring and extending problems and in using technology) were collected after students had agreed to participate in the problem solving session. Most of the pairs of students were similar in these characteristics, except for pair D, which had the most differences among them, except that both students exhibited low interest in using technology.

Procedure

The data were collected during the spring quarter of 1996. The design was quasi-experimental, with pairs of students solving both problems under different conditions. Each version of the problem-solving instrument was given twice, each time to a different pair of students. Pairs A and C solved the first version; pairs B and D solved the second version (see Table 1). Each session was recorded and videotaped with two video cameras; one captured the students' use of the graphing calculator—which allowed me to record their keystrokes—and the other captured the students' interactions and gestures.

Table 1
Availability of graphing calculator on problems for each group

	Problem	
Group	1	2
A: Amy and Alice	No	Yes
B: Ben and Bill	Yes	No
C: Cindy and Connie	No	Yes
D: Donna and Dina	Yes	No

Each pair of students was told that they had 50 minutes to work on both problems[5], that I would ask them a few questions about the process at the end, that they needed to talk "as much as possible," and that I would interact with them only if they asked for help. The students were told to work together, speaking their thinking out loud, and were reminded of the time limitation, which was thought of as a pressure for them to negotiate a plan for arriving at some solution. I provided a TI-82 graphing calculator to each student when the calculator could be used. Once the pair had finished the first problem, I collected all the written material and gave them the second problem. Field notes were taken during the session and I wrote a full report later. When students indicated that they had finished or when time was over for the session, I conducted the exit interview.

Data Analysis

Each problem-solving session was transcribed and eight protocols obtained, one from each pair solving a problem. The protocols were supplemented with information from the videotapes and my field notes. I produced detailed descriptions of the students' solution to each problem. In order to better identify differences in

[5] The estimation of time was based on previous experiences with these problems. Pairs of students who took part of the reformed pre-calculus course described in Gómez and Mesa (1995) required an average of 15 minutes to find suitable solutions.

the solution processes under the two different conditions and the roles that the graphing calculator played when it was available, I combined two frameworks, Schoenfeld's (1983) and Artzt and Armour-Thomas's (1990) to parse the protocols. Schoenfeld's framework was used to determine main episodes in each protocol by identifying all the points in which managerial decisions were made (e.g., "Let's graph this equation on the graphing calculator"), and Artzt and Armour-Thomas's operationalization of cognitive and metacognitive processes was used to identify points of managerial decision that were not accounted for with Schoenfeld's framework (e.g., students' asking for clarification). Once the episodes were defined, they were classified as belonging to one of the following categories: read, analyze, explore, plan, implement, plan and implement, verify, and new information and local assessment, using both Shoenfeld's and Artzt and Armour-Thomas's description. By parsing the protocols, I was able to identify different stages of the problem solving process followed by each pair with each problem. The parsing and their pictorial representation were used to make claims about similarities or differences between situations and to establish in which episodes the graphing calculator was most commonly used. Table 2 presents the definition of the categories used to classify episodes in each protocol.

Table 2
Categories of the analytical framework used for parsing the problem solving protocols

Category	Description
Read	The student reads the problem; includes consideration of the problem conditions.
Analyze	The student decomposes the problem into its basic elements and examines the implicit or explicit relations between the givens and goals of the problem. The student may simplify or reformulate the problem.
Explore	The student searches for relevant information that can be incorporated into the analysis-plan-implement sequence. He or she uses different problem-solving heuristics, examines related problems, or uses analogies. Trial-and-error strategies are common.
Plan	The student selects steps for solving the problem and a strategy for combining them that might potentially lead to a problem solution if implemented.
Implement	The student executes each of the steps defined in the plan. The student's actions are systematic and deliberate in transforming the givens into the goals of the problem.

Table 2
Categories of the analytical framework used for parsing the problem solving protocols

Category	Description
Plan and Implement	This category comprises those episodes in which the student does not make the plan explicit, but one can be inferred from the student's deliberate actions.
Verify	The student evaluates the outcome of the work so far, for example, by a recalculation of the computations.
New Information and Local Assessment	Points that can trigger a change in types of episodes. New information points are items in which a previously unnoticed piece of information–data or heuristics–is obtained or recognized. Local assessment is an evaluation of the current state of the solution at a microscopic level.
Transition	When either the new information or the local assessment produced a change in the character of the episode, the triggering elements were categorized as transition.

Consistency in the coding of the episodes was established by inter-rater agreement; a graduate student in mathematics education, not attached to the investigation, parsed two protocols using the descriptions given in Table 2. The agreement (measured as number of utterances classified as belonging to the same episode divided by the total number of utterances of the protocol) was 90% in one protocol and 100% in the other.

RESULTS

The protocol parsing was used to determine whether there were patterns common for the cases in which the graphing calculator was available as contrasted to the cases in which the graphing calculator was not available. In general, the parsing reflects a problem solving behavior that is consistent with that of novice problem solvers (Schoenfeld, 1992): students read the problem, sometimes analyzed the situation (about 35 minutes in total for all groups) but usually began immediately to implement a solution (approximately 70 minutes in total for all groups) or to explore an alternative (about 75 minutes in total), without making a plan explicit, and rarely conducting assessment of overall progress (about 10 minutes total for all groups) or verification of solutions (about 10 minutes for all groups). The main differences observed were across problems, with Problem 1 having twice as much time spent analyzing the situation (23 minutes in Problem 1 vs. 12 minutes in Problem 2) and Problem 2 exhibiting considerable portions of exploration (70 minutes total for all groups) that did not happen in Problem 1 (5 minutes for only one group). For more details about the parsing and the results, see Mesa (1996). Given that the major differences seem to be associated with the problems, I turn now to answer Question (a)—What strategies do students choose as they attempt to solve a problem on functions?—by describing the solutions the groups proposed for each problem and then presenting the uses of the graphing calculator by the groups that had it available. In the discussion section, I will present answers to the remaining two questions, How are strategies different when the graphing calculator is present from when the graphing calculator is not present?, and What role does the graphing calculator play in the solution process?

The Students' Solutions

In solving Problem 1, two different solutions were observed: *Fix parameters and solve the equations*, followed by pairs A, B, and C, and *Solve inequalities symbolically* followed by pair D. In the first strategy students created sketches of the functions given and then selected arbitrary numbers for three of the parameters in the expressions. Next, they set up a system of two equations and two unknowns and solved for the two remaining parameters. A summary of the work by students in pair C, who did not have the graphing calculator available for this problem, is given in Table 3. Only one group attempted a verification of the solution they found.

Table 3
Solution to Problem 1 by Group C

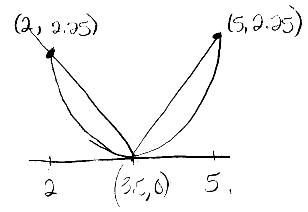
Field Notes	Students' Work
They assigned values to *h* and *k*.	$h = 3.5$ and $k = 0$
They found $f(2)$ by direct substitution in $f(x)$, and found $f(5)$ by symmetry in the graph.	$f(2) = f(5) = 2.25$ 

Table 3
Solution to Problem 1 by Group C

Field Notes	Students' Work
They assigned values to *h* and *k*.	$h = 3.5$ and $k = 0$
Next, they fixed *b*.	$b = 3.5$; therefore, $y = a\|x - 3.5\|$
They substituted $y = 2.25$ in the expression for $g(x)$ and solved for *a*, using both 2 and 5 in place of *x*.	$g(x) = 1.5\|x - 3.5\|$

In the second strategy, followed by pair D, who had the graphing calculator for this problem, the students wrote explicitly the inequality, replaced $x = 2$ and $x = 5$ into the inequality, and obtained a system of two inequalities:

$$a|2 - b| + c - (2 - h)^2 - k \geq 0$$
$$a|5 - b| + c - (5 - h)^2 - k \geq 0$$

Taking cases for the absolute value, they generated four inequalities, and after some manipulations—some of them incorrect—reduced the problem to one inequality that related two of the four parameters:

$$a|2 - b| - a|5 - b| \leq \left(4 + 4h + h^2\right) + \left(25 - 10h + h^2\right)$$

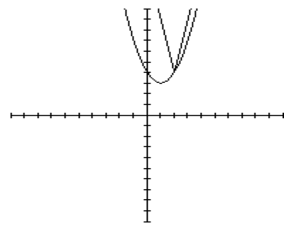
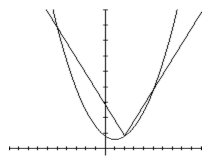
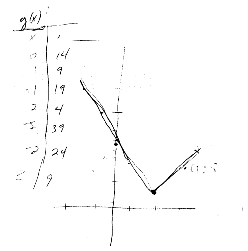
The students did not incorporate graphs or sketches of the functions into their solution and were unable to provide a pair of functions that satisfied the conditions given.

The solution by Group B, which had the graphing calculator, is worth discussing further, because this is the only group that attempted a verification of their solution with the tool, a move that helped them to correct their original "solution" (see Table 4).

Table 4
Excerpts from the solution to Problem 1 by Group B with graphing calculator

Field notes	Students' Work
They wrote an inequality using the functions with parameters and wrote two equations, one obtained by substituting $x = 2$ the other by substituting $x = 5$.	$(2 - h)^2 + k = a\|2 - b\| + c$ $(5 - h)^2 + k = a\|5 - b\| + c$
They substituted values for *h* and *k* (i.e., fixed the parabola by setting $h = 1$ and $k = 3$).	$(2 - 1)^2 + 3 =$
They set *b* equal to 2 and solved the resulting system	$4 = a\|2 - 2\| + c$

Table 4
Excerpts from the solution to Problem 1 by Group B with graphing calculator

Field notes	Students' Work
of two equations and two unknowns for a and c using the values of $f(2)$ and $f(5)$ getting $a=5$, $c=4$.	$19 = a\lvert 5-2\rvert + c$
They wrote down the expressions and entered them into the graphing calculator: $f(x)=(x-1)^2+3$; $g(x)=5\lvert x-2\rvert+4$.	
They were about to say that the problem was solved, but the wider range of points in the graphing calculator and the fact that the table showed three points at which the functions were equal helped them state that there was an error. They went over the process but could not say what might be done to fix the problem.	
They constructed a table of ordered pairs for the function of the absolute value and made its graph on paper. With these data they verified that there was a third intersection point.	
After exploring with the graphing calculator they found that moving the vertex of the function with the absolute value away from $x=2$ would eliminate the third intersection.	
They changed b to a number greater than 2 and 5, namely 8, and solved the two equations.	

Ben and Bill chose b, the x-coordinate of the vertex of the function with absolute value, as one of the extremes of the solution interval and a parabola that was "opening up." These selections forced the vertex of the absolute value function to be on the parabola, and "inside" it, but because quadratic functions grow faster than linear functions, a third intersection could be expected. Both of the students in this pair, however, were reluctant to accept the evidence shown in the graphing calculator and created a table, finding values with paper and pencil, to confirm the results. After some time spent in assimilating the evidence, students modified the solution, by moving the vertex of the parabola to the right of the interval, and repeating the process again. No attempt to verify this new solution was made.

In solving Problem 2, all the students began by recognizing the degrees of the polynomials depicted; they also recognized the transformations that were applied to some of them to obtain others. Almost all the students' solutions incorporated elements of three different strategies to find a basic function that could be transformed to obtain other graphs that were evidently derived from such functions[6]. The three strategies were (a) transform a general expression, (b) use roots observed to create factors, and (c) set up a system of linear equations. With the first strategy students began with a polynomial expression (e.g., $ax^3 + bx^2 + cx + d$ for a cubic polynomial) and using trial and error modified the parameters (*a*, *b*, and *c*) to obtain a graph that resembled the one sought. With the second strategy, students used the roots seen in the graphs to create a symbolic expression in the form $a(x - r_1)(x - r_2)\ldots(x - r_n)$, for a polynomial of degree *n*. With the third strategy, students selected points on the given graphs and substituted them into a polynomial expression to set up a system of linear equations that is solved by standard methods.

Groups A and C, who had the graphing calculator, used the first strategy; that is, they attempted to transform the general expression of a cubic function to fit Graph 1. They started by assigning parameters to the expression $ax^3 + bx^2 + cx + 10$ and checking the resulting graph against Graph 1. After unfruitful experimentation they asked for help. I suggested looking at the similarities between their graph and the graph of the expression $y = x^3 - x$. Finding the two graphs similar, the students began to use trial and error in modifying this expression to obtain the expected graph. After more intervention aimed at making explicit the relation between the roots of a polynomial and the factored expression (e.g., "factor the expression and consider when the function is zero"), the students in these two groups produced reasonable expressions for Graphs 1 and 2. Students who had access to the graphing calculator relied on the general shape of the graph to appraise the good fit of their proposed functions, rather than on the analytical tools that can be used to model the functions.

Of the two groups who solved this problem without the graphing calculator, Group D, used the second strategy, using the roots and provided expressions for graphs 4, 5, 6, 7, and 1 (in that order). Group B used the third strategy. Ben and Bill started with a factored expression to set up a system of linear equations to find Graph 1: They wrote three linear factors, $f(x) = -a(x + 2)(x - 3)(x - z)$, multiplied them together to obtain the general cubic equation, and solved the system of two equations in two unknowns [choosing $f(1) = 7$ and $f(0) = 8$], which led them to the final expression:

[6] The activity assumed that students would be generating the polynomial function of smallest degree whose behavior in the given interval was the same as the behavior illustrated in the figures. Without this condition, we have infinitely many expressions that could fit each graph in the interval given.

$$f(x) = -\frac{7}{45}(x+2)(x-3)(x-\frac{60}{7}).$$

They made some computational mistakes in the process. The students did not attempt to verify the correctness of their proposal and did not find other expressions.

Students' Use of the Graphing Calculator

It has been suggested that the graphing calculator is an important tool during exploratory work in problem solving activities. In this study, students seemed to prefer to use the graphing calculator for verification in Problem 1 and for exploration in Problem 2. In solving Problem 1 only Group D used the graphing calculator in an exploratory way, to find out the shape of the absolute value function, but did not use the result obtained nor did they attempt to use the graphing calculator again:

Dina: Okay, well let's see what this [absolute value function] graph looks like. Can you find what this graph looks like?

Donna: With this? [Pointing to the graphing calculator.]

Dina: Uh-huh.

[After some difficulties with the range of the display screen, Donna produces the graph of $g(x) = |x-1| + 3$]

Donna: Okay.

Dina: That is what it looks like? Neat.

Donna: Well, it depends on what your constants are.

Dina: Right. Okay. [Returns to her symbolic manipulations]

Later, the possibility of using the graphing calculator is ruled out:

Donna: I can't do the graph of them [in the graphing calculator] until we know...

Dina: Yes, what these numbers are!

Because the problem asked for five parameters and Group D did not know them, they decided that using the graphing calculator was inappropriate. They felt that they could not use what they were supposed to find as part of the solution. The situation did not make it possible for them to use the graphing calculator as they expected, in checking the correctness of the graphs of the functions that they were looking for.

Group B used the graphing calculator in Problem 1 during a verification episode that in the long run allowed them to carry out some exploratory work. Recall that when conducting the verification they found three intersection points instead of two and that grappling with the mistake was not straightforward. Their first reaction was to deny the evidence and demonstrate that there was a mistake in the information provided by the calculator. One argument used was that the parabola was wide enough which would make a third intersection point unlikely to occur. Ben checked, both *by hand* and by using the table option in the calculator, that the functions intersected at 2 and 5. When a wider window showed them the three points of intersection, they *drew by hand* the graph of the absolute value using their hand-generated table of ordered pairs. Once they were convinced that the two graphs also intersected at –5, they proceeded to see how to resolve the conflict in a cycle of explorations. But in spite of this cycle of explorations, Ben and Bill, appeared to use the graphing calculator as a verification tool during the problem solving session. The fact that the graphing calculator did not produce the expected results and their reluctance to accept the evidence indicates that they expected to check their answers with the calculator.

During the post-interview, the groups that did not use the graphing calculator indicated that they would have used it in Problem 1 for checking their answers. Group A said:

Alice: But the main point was the full process. As for where do you start and where do you put that vertex and, you know, those things, you can't really use the graphing calculator with them yet. ... We kind of experiment if there were different parabolas than what we thought and then we graph it quickly for us instead of doing it by hand. But I think that the by-hand part helped us more than anything.

Amy: Or if we do not know what the absolute value graph was, we just can graph a quick and easy absolute value and see what it was. But–.

Alice: Right.

Amy: But it is as easy to do it by hand as it is to do with the calculator.

Group C gave a similar response:

Connie: I think that having the graphing calculator, you can do it [solve the problem], but playing around more.

Cindy: You mean more trial and error?

Connie: Yeah, I don't, you know, I usually do not think about what I am doing when I am using it [the graphing calculator] a lot of the time.... I mean, I do [think], but I draw less.…

Cindy: This is more of a process, you know. We have numbers. We would be more totally guessing [by using the graphing calculator].

Note, in this last excerpt, how the role of the graphing calculator in exploring is downplayed ("totally guessing", "more trial and error"). In contrast, in Problem 2 the graphing calculator was used principally in exploration episodes to test the effects of different coefficient values on the graph of the function. Students who had the graphing calculator tested many parameters and suggested several hypothesis about the connections between those parameters and what they were getting, very similar in what Graham and Thomas (2000) describe as a cybernetic process. However, in the absence of the knowledge about the relation between the roots of a polynomial as seen in its graphs and the polynomial's factored expression most of these explorations were not productive. Although all four groups made conjectures as to the value of the parameters, the graphing calculator made a difference in the time used to answer the same questions. On average, the groups without the graphing calculator tested two functions in 3.5 minutes, whereas the groups with the graphing calculator tested an average of five functions in 3 minutes. That the graphing calculator was expected to be useful in their exploration was also evident by Group D's "Oh! We can not use the graphing calculator for this!" uttered with disappointment at the beginning of their work on this problem. Pairs A and C who had access to the graphing calculator spent about 46 minutes in explore episodes, 40 of which were done with the graphing calculator.

An important difference to note between the groups with graphing calculators and the groups that did not have them, was the time spent in solving the problems. Table 5 shows the time in minutes each group spent working on each problem. The underlined number corresponds to the session when the graphing calculator was available: When the students had the graphing calculator, they took more time—in one case twice as much as—than the time used when the graphing calculator was not available.

Table 5
Time in minutes to solve each problem by each group

Group	Problem 1	Problem 2	Total Time
A	14	<u>39</u>	53
B	<u>34</u>	21	55
C	16	<u>26</u>	42
D	<u>27</u>	22	49

Note: The underlined number corresponds to the minutes spent in the session when the graphing calculator was available.

Discussion

Two key observations emerge from these results. There is not much reason to believe that there were substantial differences in how the students solved the problems when the graphing calculator was available as compared with when it was not available. Second, the students opted for strategies in which the graphing calculator would back up their answers to solve Problem 1, or would test their hypothesis about the coefficients for Problem 2. The only advantage the students with the graphing calculator had in solving Problem 2 was that they could test more functions in the same amount of time.

Strategies Are not That Different

Although Problem 1 is not a standard inequality problem (such as find the solution to $(4x-2)^2-1 \leq 3|x-2|+2$), it contained elements that might have induced the recall of certain approaches (decompose the absolute value, find the points when an equality is obtained). However, the task is designed in such a way that a link between the symbolic representation of the functions and its graphical representation is useful for progressing towards a solution of the problem. Failure to use this link kept Group D from solving the problem. The graphical representation can come into play in two ways: in depicting how the functions look (something that all groups did) and in confirming that the inequality had the given solution set. For none of the groups was the graphing calculator crucial for depicting the functions: Groups A, B, and C knew the form of the graphs they were dealing with whereas Group D thought they did not have the right to use it. But for all the groups, the graphing calculator was fundamental for verifying that the functions satisfied the given conditions. Students who did not have access to the graphing calculator were able to find a solution to the problem whenever they knew general shapes of the functions involved, and for at least one group the graphing calculator was important in verifying that the functions proposed satisfied the conditions given. A symbolic approach to verification would require selecting a representative value from three different sets (a number less than 2, another between 2 and 5, and another greater than 5) and to establish in each case whether the left side of the inequality is greater than, equal to, or less than the right side. A graphical representation simplifies this task by showing the intervals in which one graph is above, coincides with, or is below the other. Thus it seems that because students in Groups A, B, and C could connect the symbolic and graphical representations, they were able to engage in a process that led them to produce a solution, whereas the lack of this link made it impossible for Group D to find a solution independently of the availability of the graphing calculator.

The students in this study did not attempt to find other possible solutions to Problem 1. From previous experiences, I have found that a visual representation of the functions facilitates "dropping" conditions implicitly imposed on the solution (e.g., students tend to assume that the vertices of the parabola and the absolute value should coincide with the half point of the interval [2, 5], or that a in function $g(x)$ must be different from 0). The visual representation provided by the graphs allows students to identify the two points of intersection as key for the solution: these points, and the relation between the functions, should be kept. Even in environments when the graphing calculator is not available, students tend to imagine how the vertex of the absolute value could be "moved up and down," satisfying the conditions and generating hypothesis about the values of the parameters. One such attempt could be perceived by Group B's resolution of their conflict when they "dragged" the vertex of the absolute value away from the solution interval. It might be possible that further probing could have helped these students realize that there were many more pairs of functions that satisfied the given conditions.

Problem 2 illustrates a different issue. The most difficult task for the students was to produce a basic expression that could be transformed to generate a new expression that would match the graph given. Students in groups A, B, and C were confident with the transformations but lacked familiarity with the mathematical content that would facilitate the production of a basic expression for these types of polynomials (recall that none of these students had trouble with linear and quadratic expressions). Students were resourceful in using their knowledge (of general aspects such as shape and symbolic representation most likely to produce such outcomes) either as a basis to begin an exploration of parameters or for finding alternate routes (e.g., setting up a system of equations) to solve the problem. However, once the connection between the factored expression of a polynomial and the roots as illustrated in a graph was made available for the students—that was Group D's initial knowledge—the solution process became more straightforward. Again, the availability of the knowledge was a better "predictor" of the outcomes than the presence or absence of the graphing calculator.

Roles of the Graphing Calculator

In both problems, the main use of the graphing calculator was to find the graph of a function. Graphing a function, however, served different purposes in solving the two problems: to verify solutions in Problem 1 and to test proposed values for parameters in Problem 2. Students' previous encounters with graphing calculators, the nature of the tasks, and the knowledge the tasks put in play may help explain these results.

The graphing calculator allows students to work on problems in which a family of functions is described based on results of multiple graphs of a parameterized expression (e.g., $ax^2 + bx + c$). Another common use consists in looking at the different representations available (tables, graphs, symbolic expressions), altering them, and contrasting the results. Yet other uses involve the interpolation of data to find expressions that fit a given data set. All these tasks, in essence, take advantage of what the graphing calculator can do, overlooking to some extent the questions of what mathematics is worth teaching given these capabilities of the tool (Williams, 1993). Thus in Problem 1, in which the students had the expressions and there was not an explicit requirement of describing the family of functions depicted, students chose to use the graphing calculator to verify the solutions obtained. Problem 2, on the other hand, required students to re-create the expression that would fit the graphs given. Even in this case, the students opted for a parameter testing procedure applied to a given expression, taking advantage of what the calculator can do. Students' appraisal of the goodness of fit was based on the graphical resemblance rather than on structural properties of the parameters involved.

The knowledge at stake, that is, the knowledge that is needed in order to be able to engage productively in the activities, may also explain the different purposes for graphing a function. In Problem 1, Groups A, B, and C were familiar with the general characteristics of the functions and with the relationship between the solution to an inequality and the graphical representation of the expressions involved. Group D lacked this knowledge; therefore, they could not advance productively in solving the problem in spite of the availability of the graphing calculator, and in spite of their using it in a brief exploratory episode. However, for all groups, familiarity and confidence in their solution process made them use the graphing calculator in verifying their solutions. Only when the graphing calculator gave results that were unexpected, did the act of graphing have a different purpose. In Problem 2, the knowledge at stake, that is the fundamental theorem of algebra, was not in students' repertoire. In this case, the function-graphing capability of the graphing calculator became the only resource students could use more or less successfully, in order to deal with the problem. Interestingly, once a basic function was found, the process of transforming the function to obtain related functions became a paper and pencil activity, with the graphing calculator playing a very limited role in verifying that the functions proposed met the expected conditions.

Finally, how students have experienced solving mathematical problems with the graphing calculator may account for students' use of the tool. The students in this study had a varied set of experiences with and interest in using the graphing calculators, but such diversity does not seem to have had an influence on the uses they gave to the tool—the uses seem independent of such experiences. A larger sample of students may be more suitable for establishing the extent to which previous experiences with the graphing calculator can affect how the graphing calculator is used in a problem solving session.

Conclusions

The results of this study suggest that the role of the graphing calculator in guiding students' active construction of solutions to problem solving activities—at least when they relate to functions—merits more careful analysis. There is some reason to believe that what guides the problem solving efforts beyond the availability of the tools is (a) the way in which knowledge is to be used and (b) students' familiarity with that knowledge.

The students in this study were not trained to use the graphing calculator in particular ways; nor did they declare a heavy use of graphing calculator as undergraduates. This fact adds to these findings, because it shows that students were actually responding to the constraints in the tasks and not to the cues that may be established when there is an instructional agenda that supports the use of graphing calculators as in the studies reported in the literature review section. Indeed, it is difficult to control for instruction in such studies; but instruction should be taken into account when making claims about the effects of innovations in the classroom. On a recent observation, a group of about 15 second-year undergraduate students were given Problem 1 after receiving a 5-minute refresher on linear and quadratic functions. The refresher was delivered using the blackboard and overhead projector and transparencies that were moved up and down to illustrate effect of parameters on linear functions. In spite of the availability of the graphing calculator, none of these students opted for using it; they arrived at solutions in less time than what was observed in this study; and were able to suggest families of solutions, to generalize, and drop some of the conditions. My short "instruction" in this case pointed at the *relevant knowledge* students needed to be able to engage with the task. So, in this case, with students who were not familiar with the problem the overhead projector and the transparencies fulfilled the purpose of illustrating the connections between representations, which was a crucial content for attempting the problem. Thus the claim that graphing calculators can be used as amplifiers for conceptual understanding, as catalysts for critical thinking, or as vehicles for integration, as suggested by Smith (1998), is a claim that may apply to other tools, given that *instruction*, with those tools, includes such purposes.

Smith also notes that graphing calculators are seen as tools for expediency, because they save time that otherwise would be used in tedious or difficult procedures (e.g., estimating a best fit regression line). However, the results of this study suggest that the time saved in computations will be used in other kinds of activities that may be more time consuming. The illustration for this phenomenon is the time spent in exploration by the groups who had the graphing calculator in Problem 2: of the 65 minutes that the groups spent solving Problem 2, 46 minutes (71%) were devoted to exploration; and of these 46 minutes, 87% was devoted to exploration with the graphing calculator. This is a sizable amount of time, especially considering that for the most part the explorations did not help students find suitable solutions, until an intervention helped students reorganize their exploration ("factor the expression $y = x^3 - x$ and consider when the function is zero"). This phenomenon raises an important issue for practice because it shows that even though the cybernetic process is a key feature that the graphing calculator affords, if the process is not informed by mathematical principles, knowledge reorganization may not occur after all.

That students used the graphing calculator to verify the solutions when they were more comfortable with the knowledge and to explore when they were not, may give some root to the perception that real mathematics is not really carried out with calculators, that the mathematics that counts is the one that is done with paper and pencil. Thus in spite of the importance of the tool for assisting in finding solutions, the tool is after all, a crutch, something you may dispose of when you become proficient with the knowledge. Further research in this area could consider students' beliefs about the role of technology in doing mathematics.

Finally, this study was sensitive to offering the same mathematical opportunities to both groups of students; that is it looked for maintaining the mathematics at stake a "constant." Ethical issues may arise when we conduct studies that benefit some and not other groups of students, as it may happen in controlled experiments in education. However, the issue of crafting activities that put interesting mathematical knowledge at stake is a difficult one, more so when we also want to study the actual role that tools play in those situations. The alternative proposed in this study was to not have the tool, and to study that situation *vis a vis* the situation in which the tool was present. Could we claim that there was not a cybernetic process? Or that there were not opportunities for conceptual understanding? Or that critical thinking did not happen? As illustrated by the results, these activities happened in both situations. Thus the tool itself, although it may have been seen as the reason for some of these opportunities to happen, may only be playing a secondary role to that of the transformation of instruction and curriculum. The study of these transformations may be a more fruitful area for further investigation of students' uses of technology in the classroom.

Acknowledgements

This paper is based on the author's master's thesis completed at the University of Georgia under the direction of Jeremy Kilpatrick. Many thanks to Jeremy Kilpatrick, Dan Chazan, Patricio Herbst, Edward Silver, Jack Smith, and two anonymous reviewers for comments to previous versions of the paper.

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This document was originally published as Mesa, V. (2007). Solving problems on functions: Role of the graphing calculator. *Focus on Learning Problems in Mathematics*, *29*(3), 30-54.

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