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EXACT SOLUTIONS FOR THE GENERALIZED SHALLOW WATER WAVE EQUATION BY THE GENERAL PROJECTIVE RICCATI EQUATIONS METHOD

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Abstract. We present the projective Riccati equations method to obtain exact solutions for the generalized shallow water wave equation (GSWW). We show the behavior of solutions with a graph.

Key words and phrases. Nonlinear differential equation, travelling wave solution, Mathematica, projective Riccati equation method.

Resumen. Presentamos el método proyectivo de ecuaciones de Riccati para obtener soluciones exactas de la ecuación generalizada de onda de agua poco profunda (GSWW). Mostramos el comportamiento de las soluciones con una gráfica.

Palabras claves. Ecuación diferencial no lineal, solución de onda viajera, Mathematica, método proyectivo de ecuaciones de Riccati.

Introduction

The search of exact solutions of partial differential equations is of great importance, because these equations appear in complex physics phenomena, mechanics, chemistry, biology and engineering. A variety of powerful and direct methods have been developed in this direction. Among these are the inverse scattering method, Hirota´s method, Bäcklund transformations, tanh-function method, extended tanh-function method. In the literature, Conte et al.[1]

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presented a general ansatz to seek more new solitary wave solutions of some NLPDE´s. More recently, Yan [4] developed Conte´s method and presented the general projective Riccati equations method . In this paper, we will use the *general projective Riccati equations method* to construct exact solutions for the generalized shallow water wave equation.

1. The general projective Riccati equations method

For given a nonlinear equation that does not explicitly involve independent variables

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{1.1}$$

when we look for its *travelling wave solutions*, the first step is to introduce the wave transformation, which have by definition the form

$$u(x,t) = v(\xi), \quad \xi = x + \lambda t, \tag{1.2}$$

where λ is a constant and change (1.1) to an ordinary differential equation (ODE) for the function $v(\xi)$

$$P(v, v', v'', \ldots) = 0. \tag{1.3}$$

The next crucial step is to introduce new variables $\sigma(\xi)$, $\tau(\xi)$ which are solutions of the system

$$\begin{cases} \sigma'(\xi) = e\sigma(\xi)\tau(\xi) \\ \tau'(\xi) = e\tau^2(\xi) - \mu\sigma(\xi) + r. \end{cases} \tag{1.4}$$

It is easy to see that the first integral of this system is given by

$$\tau^2 = -e[r - 2\mu\sigma(\xi) + \frac{\mu^2 + \rho}{r}\sigma^2(\xi)], \tag{1.5}$$

where $\rho = \pm 1$. From this integral we obtain the following particular solutions:

(1) Case I:
If $r = \mu = 0$ then

$$\tau_1(\xi) = -\frac{1}{e\xi}, \quad \sigma_1(\xi) = \frac{C}{\xi}. \tag{1.6}$$

(2) Case II:
If $e = 1$ and $\rho = -1$

$$\begin{cases} \tau_2 = \dfrac{\sqrt{r}\tan(\sqrt{r}\xi)}{\mu\sec(\sqrt{r}\xi) + 1} \quad (r > 0) \\ \sigma_2 = \dfrac{r\sec(\sqrt{r}\xi)}{\mu\sec(\sqrt{r}\xi) + 1} \quad (r > 0). \end{cases} \tag{1.7}$$

(3) Case III:
If $e = -1$ and $\rho = -1$

$$\begin{cases} \tau_3 = \dfrac{\sqrt{r}\tanh(\sqrt{r}\xi)}{\mu \mathrm{sech}(\sqrt{r}\xi) + 1} & (r > 0) \\ \sigma_3 = \dfrac{r \mathrm{sech}(\sqrt{r}\xi)}{\mu \mathrm{sech}(\sqrt{r}\xi) + 1} & (r > 0). \end{cases} \tag{1.8}$$

(4) Case IV:
If $e = -1$ and $\rho = 1$

$$\begin{cases} \tau_4 = \dfrac{\sqrt{r}\coth(\sqrt{r}\xi)}{\mu \mathrm{csch}(\sqrt{r}\xi) + 1} & (r > 0) \\ \sigma_4 = \dfrac{r\, \mathrm{csch}(\sqrt{r}\xi)}{\mu\, \mathrm{csch}(\sqrt{r}\xi) + 1} & (r > 0). \end{cases} \tag{1.9}$$

(5) Case V:
If $e = 1$ and $\rho = 1$

$$\begin{cases} \tau_5 = \dfrac{-\sqrt{-r}\coth(\sqrt{-r}\xi)}{\mu \mathrm{csch}(\sqrt{-r}\xi) + 1} & (r < 0) \\ \sigma_5 = \dfrac{r\, \mathrm{csch}(\sqrt{-r}\xi)}{\mu\, \mathrm{csch}(\sqrt{-r}\xi) + 1} & (r < 0). \end{cases} \tag{1.10}$$

We seek a solution of (1.1) in the form

$$u(x,t) = v(\xi) = a_0 + \sum_{i=1}^{M} \sigma^{i-1}(\xi)(a_i\sigma(\xi) + b_i\tau(\xi)), \tag{1.11}$$

where $\sigma(\xi)$, $\tau(\xi)$ satisfy the system (1.4). The integer M can be determined by balancing the highest derivative term with nonlinear terms in (1.3), before the a_i and b_i can be computed. Substituting (1.11), along with (1.4) and (1.5) into (1.3) and collecting all terms with the same power in $\sigma^i(\xi)\tau^j(\xi)$, we get a polynomial in the variables $\sigma(\xi)$ and $\tau(\xi)$. Equaling the coefficients of this polynomial to zero, we can obtain a system of algebraic equations, from which the constants μ, r, λ, a_i, b_i ($i = 1, 2, \ldots, M$) are obtained explicitly. Lastly, we substitute the solution from system in (1.11) along with (1.6), (1.7), (1.8), (1.9) and (1.10) and reversing, we obtain the explicit solutions for (1.1) in the original variables.

2. The shallow water wave equation

The shallow water wave equation is

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - \gamma u_{xx} = 0, \tag{2.12}$$

where subscripts indicate partial derivatives, u is a real scalar function of the two independent variables x and t, while α, β and γ are all model parameters and they are arbitrary, nonzero constants. Some particular cases of (2.12) have been discussed in the literature, $\alpha = \beta$, and $\alpha = 2\beta$, $\alpha = \beta = -3$ and $\gamma = 0$, $\alpha = -4$, $\beta = -2$ and $\gamma = 0$. In this paper, we obtain solution for many another values of α, β and γ.

Substituting (1.2) in (2.12), we obtain the ordinary differential equation

$$-(\frac{\gamma+\lambda}{\lambda})\, v''(\xi) + (\alpha+\beta)\, v'(\xi)\, v''(\xi) + v^{(4)}(\xi) = 0, \tag{2.13}$$

where

$$\alpha+\beta \neq 0, \quad \frac{\lambda+\gamma}{\lambda} \neq 0.$$

According to the method described above, we seek solutions of (2.15) in the form

$$v(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi), \tag{2.14}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy the system (1.4). Substituting (2.14), along with (1.4) and (1.5) into (2.13) and collecting all terms with the same power in $\sigma^i(\xi)\tau^j(\xi)$, $(i = 0, 1, \ldots, 5)$, $(j = 0, 1)$ we get a polynomial in the two variables $\sigma(\xi)$ and $\tau(\xi)$. Equaling the coefficients of these polynomial to zero and after simplifications with $e = \pm 1$ and $r \neq 0$ we get the following algebraic system:

- $\sigma(\xi)$:
  $era_1(\gamma + \lambda(1 + er)) = 0,$
- $\sigma(\xi)^2$:
  $e\mu a_1(-3\gamma + \lambda - e\lambda(4e + 15r) - 2r\lambda b_1(\alpha+\beta)) = 0,$
- $\sigma(\xi)^3$:
  $ea_1(2e(3r\lambda\mu^2 + e(\gamma+\lambda)(\mu^2+\rho) + 24r\lambda(3\mu^2+\rho) - 2r\lambda(25\mu^2+7\rho) + r\lambda b_1(\alpha+\beta)(8\mu^2+3\rho))) = 0,$
- $\sigma(\xi)^4$:
  $10e\lambda\mu a_1(\mu^2+\rho)(6e + (\alpha+\beta)b_1) = 0,$
- $\sigma(\xi)^5$:
  $4e\lambda a_1(\mu^2+\rho)^2(6e + (\alpha+\beta)b_1) = 0,$
- $\sigma(\xi)\tau(\xi)$:
  $e\mu b_1(\gamma + \lambda(1 + er)) = 0,$
- $\sigma(\xi)^2\tau(\xi)$:
  $-er^2\lambda a_1^2(\alpha+\beta) + b_1(2e(15r\lambda\mu^2 + e(\gamma+\lambda)(\mu^2+\rho) + 24r\lambda(3\mu^2+\rho) - 20r\lambda(4\mu^2+\rho)) + r(\alpha+\beta)\lambda\mu^2 b_1) = 0,$
- $\sigma(\xi)^3\tau(\xi)$:
  $\lambda\mu(-3ra_1^2(\alpha+\beta) + 3eb_1(\mu^2+\rho)(12e + b_1(\alpha+\beta))) = 0,$
- $\sigma(\xi)^4\tau(\xi)$:
  $2\lambda(\mu^2+\rho)(-ra_1^2(\alpha+\beta) + eb_1(\mu^2+\rho)(12e + b_1(\alpha+\beta))) = 0.$

Solving the previous system respect to unknowns variables r, a_1, b_1, μ we obtain the following solutions:

(1) $a_1 = 0$, $\mu = 0$, $r = \dfrac{e(\gamma+\lambda)}{4\lambda}$, and $b_1 = -\dfrac{12e}{\alpha+\beta}$.

(2) $r = -\dfrac{\gamma+\lambda}{e\lambda}$, $a_1 = \pm\dfrac{6\sqrt{\lambda(\mu^2+\rho)}}{(\alpha+\beta)\sqrt{\gamma+\lambda}}$, $b_1 = -\dfrac{6e}{\alpha+\beta}$.

In all cases $\rho = \pm 1$, $e = \pm 1$. Therefore, according (2.14) and using (1.6) to (1.10), after simplifications we obtain the following classification of some exact solutions for the equation (GSWW): (In all cases, $m = \alpha+\beta$, $n = \frac{\lambda+\gamma}{\lambda}$ and $\xi = x + \lambda t$)

N°	r	μ	a_1	b_1	u	
1	$-n$	$\|\mu\| \le 1$	$\frac{6\sqrt{1-\mu^2}}{m\sqrt{-n}}$	$-\frac{6}{m}$	$a_0 + \frac{6\sqrt{-n}(\sqrt{1-\mu^2} - \sin(\sqrt{-n}\,\xi))}{m(\mu+\cos(\sqrt{-n}\,\xi))}$	$(n<0)$
2	$-n$	$\|\mu\| \le 1$	$-\frac{6\sqrt{1-\mu^2}}{m\sqrt{-n}}$	$-\frac{6}{m}$	$a_0 - \frac{6\sqrt{-n}(\sqrt{1-\mu^2} + \sin(\sqrt{-n}\,\xi))}{m(\mu+\cos(\sqrt{-n}\,\xi))}$	$(n<0)$
3	n	$\|\mu\| > 1$	$\frac{6\sqrt{\mu^2-1}}{m\sqrt{n}}$	$\frac{6}{m}$	$a_0 + \frac{6\sqrt{n}(\sqrt{\mu^2-1} + \sinh(\sqrt{n}\xi))}{m(\mu+\cosh(\sqrt{n}\xi))}$	$(n>0)$
4	$-n$	0	$\frac{6}{m\sqrt{n}}$	$-\frac{6}{m}$	$a_0 - \frac{6\sqrt{-n}\tan(\frac{1}{2}\sqrt{-n}\xi)}{m}$	$(n<0)$
5	$-\frac{n}{4}$	0	0	$-\frac{12}{m}$	$a_0 + \frac{6\sqrt{-n}\cot(\frac{1}{2}\sqrt{-n}\xi)}{m}$	$(n<0)$
6	$-n$	$-i$	0	$-\frac{6}{m}$	$a_0 - \frac{6\sqrt{-n}\cot(\sqrt{-n}\xi)}{m(-1+\csc(\sqrt{-n}\xi))}$	$(n<0)$
7	$-n$	i	0	$-\frac{6}{m}$	$a_0 + \frac{6\sqrt{-n}\cot(\sqrt{-n}\xi)}{m(1+\csc(\sqrt{-n}\xi))}$	$(n<0)$
8	n	μ	$-\frac{6\sqrt{\mu^2+1}}{m\sqrt{n}}$	$\frac{6}{m}$	$a_0 - \frac{6\sqrt{n}(\sqrt{\mu^2+1} - \cosh(\sqrt{n}\xi))}{m(\mu+\sinh(\sqrt{n}\xi))}$	$(n>0)$
9	n	μ	$\frac{6\sqrt{\mu^2+1}}{m\sqrt{n}}$	$\frac{6}{m}$	$a_0 + \frac{6\sqrt{n}(\sqrt{\mu^2+1} + \cosh(\sqrt{n}\xi))}{m(\mu+\sinh(\sqrt{n}\xi))}$	$(n>0)$

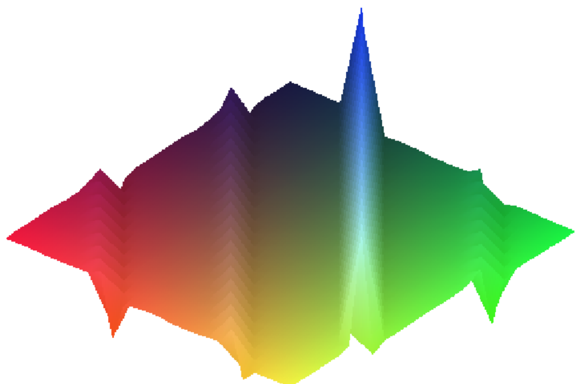

Figure 1

The surface correspond to solution (1) with $a_0 = 0$, $\xi = x + \lambda t$, $\gamma = 51$, $\mu = 0.9$, $\alpha + \beta = 10$ and $\lambda = -1$. For $x = -1$ to $x = 1$ and $t = -1$ to $t = 1$.

Figure 2

The surface correspond to solution (9) with $a_0 = 0$, $\xi = x + \lambda t$, $\mu = 2$, $\alpha + \beta = 10$, $\lambda = -1$ and $\gamma = 0.95$. For $x = -50$ to $x = 50$ and $t = -1$ to $t = 1$.

3. Conclusions

The projective Riccati equation method is a powerful method to search exact solutions for NLPDE´s . The projective method is more complicated than other method, in the sense that demands more computer resources since the algebraic system may require a lot of time to be solved. In some cases, this system is so complicated that no computer algorithm may solve it, specially if the value of M is greater than four. In this paper, this method has been applied to generalized shallow water wave equation with $M = 1$, to obtain doubly periodic wave and solitary wave solutions of the (GSWW). Some of the behaviors of the solutions can be inferred from Figures 1 and 2. The physical relevance of solitons solutions and periodic solutions is clear to us.

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