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AN EXTENSION OF THE STONE DUALITY: THE EXPANDED VERSION

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Abstract. This paper deals with a duality between two categories extending the classical Stone Duality between totally disconnected compact Hausdorff spaces (Stone spaces) and Boolean rings with unit. This duality was announced and very briefly sketched in [7]. The first category denoted by **RHQS** has as objects the representations of Hausdorff quotients of Stone spaces and as morphisms all compatible continuous functions. The second category denoted by **BRLR** has as objects all Boolean rings with unit endowed with a link relation and as morphisms all compatible Boolean rings with unit morphisms. Furthermore, we study connectedness from an algebraic point of view, in the context of the proposed generalized Stone duality.

Key words and phrases. Stone duality, Boolean rings, quotients of Stone spaces, continua, Cantor space

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Resumen. Este artículo se trata de una dualidad entre dos categorías, que extiende la dualidad clásica de Stone entre espacios de Hausdorff, compactos, totalmente disconexos (espacios de Stone) y anillos Booleanos con unidad. Esta dualidad fue enunciada y muy brevemente bosquejada en [7]. La primera categoría denotada por **RHQS** tiene como objetos las representaciones de cocientes de Hausdorff de espacios de Stone y como morfismos todas las funciones continuas compatibles. La segunda categoría denotada por **BRLR** tiene como objetos todos los anillos Booleanos con unidad dotados de una relación de ligazón, y como morfismos todos los morfismos de anillos Booleanos con unidad, compatibles. Además estudiamos la conexidad desde un punto de vista algebraico, en el contexto de la propuesta dualidad de Stone generalizada.

Palabras claves. Dualidad de Stone, anillos Booleanos, cocientes de espacios de Stone, continuos, espacio de Cantor.

1. Introduction

The well-known Stone's Representation Theorem [9], [10] for Boolean rings with unit, establishes the equivalence of the categories: *Boolean rings with unit-Boolean ring morphisms which preserve the unit* and *Stone spaces-continuous functions.*

In 1937, **M. H. Stone** [11] considered some of his own ideas applied to distributive lattices and generalized his representation theorem to non-Boolean distributive lattices. So, algebraic properties of distributive lattices were related to properties of a certain topological space of prime ideals of the lattice. **G. Gratzer** [3], in 1963, proved a representation theorem for Stone lattices, that is, a lattice which is distributive and pseudo-complemented, and in which the formula $a^* \cup a^{**} = 1$ holds identically; Stone lattices are a generalization of Boolean algebras. Gratzer proved that every Stone lattice is isomorphic to a sublattice of the lattice of all ideals of a complete and atomic Boolean algebra. **T. P. Speed** [8], in 1969, studied connectedness, irreducibility , combinatorial dimension and the noetherian property of the spaces of prime ideals of a distributive lattice. **H. A. Priestley** [5], [6], in 1970, showed that by defining a topology and an order relation on the set of 2-valued homomorphisms of a distributive lattice, one could obtain a space dual to the given lattice: a distributive lattice is isomorphic to the lattice of clopen increasing subsets of its dual space, and every compact totally order disconnected space arises as the dual of a distributive lattice. With these results, Priestley related properties of a lattice to properties of its dual space.

In the present work, a generalization of the Stone duality for Boolean rings with unit is obtained in a different way, rather than omitting conditions in the definition of Boolean ring with unit, they are considered enriched with a certain relation: given a pair (A, α), where A is a Boolean ring with unit and α is a link relation (Definition 2.9), by defining a topology and a closed equivalence relation on the set of ultrafilters of A, a dual space to the given (A, α) is obtained. Every pair (A, α) is isomorphic to the Boolean ring of clopen subsets of the spectrum of A endowed with a certain relation, and every Stone space endowed with a closed equivalence relation (that is, every representation of a Hausdorff quotient of a Stone space) is viewed as the dual of a Boolean ring with unit endowed with a link relation. The pairs (A, α) are a generalization of Boolean algebras and the representations of Hausdorff quotients of Stone spaces, are a generalization of Stone spaces.

This paper falls into two main parts: Section 2 establishes (Corollary 2.16) the duality previously described as an extension of the Stone duality for Boolean rings with unit. some other analogous dual pairs of categories are also considered in this section. Section 3, relates the topological property of connectedness of Hausdorff quotients of Stone spaces, to properties of its dual space. In this section, an algebraic characterization of continua, is presented (Corollary 3.1.5).

Throughout this paper, if A is a Boolean ring, then $Spec(A)$ denotes the set of ultrafilters in A, endowed with the topology whose basic open sets are

$$\mathbb{D}(a) := \{ U \in Spec(A) \mid a \in U \}, \quad \forall a \in A.$$

On the other hand, if X is a Stone space then $\mathbb{A}(X)$ denotes the Boolean ring of the clopen subsets of X.

2. The categories $\mathbf{BR\mathcal{L}_1R}$ versus **RHPS** and **BRLR** versus **RHQS**

In this section, two extensions of the Stone duality are established. The functors establishing these extensions are defined in terms of the following relations.

Definition 2.1. Let X be a set and α be a relation on X. The R_α and R^α relations in any subfamily of $\mathcal{P}(X)$ are defined as follows. Let C and D be subsets of X,

$$CR_\alpha D \iff (\exists x)(\exists y)(x \in C,\ y \in D,\ \text{and } x\alpha y),$$

$$CR^\alpha D \iff (\forall x)(\forall y)(x \in C, y \in D \implies x\alpha y).$$

Definition 2.2. The category **BR$\mathcal{L}$R (Boolean rings with unit and with a $\mathcal{L}$-relation)** is defined as follows.

- Objects: Pairs (A, α) where A is a Boolean ring with unit and α is a relation on $A \setminus \{0\}$, that satisfies the following properties:
  (L1) α is reflexive;
  (L2) α is symmetric;
  (L3) $(\forall c, d \in A)\ (c\alpha d,\ c \leq a,\ d \leq b \Longrightarrow a\alpha b)$.

  We will call **$\mathcal{L}$-relation** a relation satisfying (L1), (L2) and (L3).
- Morphisms: $f : (A, \alpha) \longrightarrow (A', \alpha')$, morphism of Boolean ring with unit, such that $f(c)\alpha' f(d)$ implies $c\alpha d$, $\forall c, d \in A$.

Definition 2.3. The category **RPS (Representations of "pre-quotients" of Stone spaces)** is defined as follows.

- Objects: Pairs (X, γ), where X is a Stone space and γ is a reflexive and symmetric relation on X.
- Morphisms: $f : (X, \gamma) \longrightarrow (X', \gamma')$, continuous functions such that $x\gamma y$ implies $f(x)\gamma' f(y)$, $\forall x, y \in X$.

The proof that **BR$\mathcal{L}$R** and **RPS** are in fact categories is straightforward and so omitted.

Definition 2.4. The functors $\mathbb{S}$ and $\mathbb{A}$ are defined by the diagrams of Figures 1 and 2 respectively, where, if $f : A \longrightarrow A'$ then $f^! : Spec(A') \longrightarrow Spec(A)$ is defined by:

$$f^!(U) = f^{-1}(U), \quad \forall U \in Spec(A')$$

and similarly, if $f : X \longrightarrow X'$ then $f^! : \mathbb{A}(X') \longrightarrow \mathbb{A}(X)$ is defined by:

$$f^!(C) = f^{-1}(C), \quad \forall C \in \mathbb{A}(X')$$

$$\begin{array}{ccc}
\mathbf{BR\mathcal{L}R} & \xrightarrow{\ \mathbb{S}\ } & \mathbf{RPS} \\
(A, \alpha) & \longmapsto & (Spec(A), R^{\alpha}) \\
\Big\downarrow f & & \Big\uparrow \mathbb{S}(f) = f^! \\
(A', \alpha') & \longmapsto & (Spec(A'), R^{\alpha'})
\end{array}$$

Figure 1

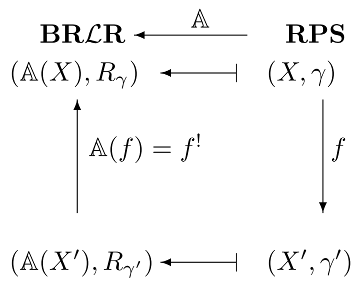

Figure 2

Properties (L1) and (L3) guarantee that R^α is reflexive in Spec(A), whereas (L2) guarantees that R^α is symmetric. On the other hand, the reflexivity and symmetry of γ imply for R_γ the properties (L1) and (L2) respectively, while the property (L3) for R_γ is deduced easily.

Furthermore, if $f : (A, \alpha) \longrightarrow (A^{'}, \alpha^{'})$ is a morphism of $\mathbf{BR\mathcal{L}R}$, U, $G \in Spec(A^{'})$ and $UR^{\alpha^{'}}G$, then $f^!(U)R^\alpha f^!(G)$, that is $f^{-1}(U)R^\alpha f^{-1}(G)$. In fact: let $c \in f^{-1}(U)$ and $d \in f^{-1}(G)$; then $f(c) \in U$ and $f(d) \in f(G)$, which implies $f(c)\alpha^{'}f(d)$ and since f is a morphism of $\mathbf{BR\mathcal{L}R}$, then $c\alpha d$ and hence $f^{-1}(U)R^\alpha f^{-1}(G)$.

Now, if $f : (X, \gamma) \longrightarrow (X', \gamma')$ is a morphism of **RPS**, $C, D \in \mathbb{A}(X')$ and $f^{-1}(C)R_\gamma f^{-1}(D)$, there exist $x \in f^{-1}(C)$ and $y \in f^{-1}(D)$ such that $x\gamma y$, since f is a morphism of **RPS**, we have $f(x)\gamma' f(y)$, but clearly $f(x) \in C$ and $f(y) \in D$, therefore $CR_\gamma D$. On the other hand, it is clear that $\mathbb{S}$ and $\mathbb{A}$ respect the composition and the identities. Using the standard Stone's theory we have that $\mathbb{S}$ and $\mathbb{A}$ are in fact functors between the categories $\mathbf{BR\mathcal{L}R}$ and **RPS**, thus we can state:

Proposition 2.5. The functors $\mathbb{S}$ and $\mathbb{A}$ are adjoint.

Proof. Let (A, α) be an object of $\mathbf{BR\mathcal{L}R}$ and (X, γ) be an object of **RPS**. Let

$$[(X, \gamma), (Spec(A), R^\alpha)]_{RPS} \xrightarrow{\Theta} [(A, \alpha), (\mathbb{A}(X), R_\gamma)]_{BR\mathcal{L}R}$$

defined in the following manner: if $f : (X, \gamma) \longrightarrow (Spec(A), R^\alpha)$ is a morphism of **RPS**, then

$$\begin{aligned}\Theta f:\ (A,\alpha) &\longrightarrow (\mathbb{A}(X), R_\gamma)\\ c &\longmapsto \Theta f(c) := f^{-1}(\mathbb{D}(c)).\end{aligned}$$

Therefore Θ is a natural bijection between (A,α) and (X,γ). □

The adjunction established in the previous proposition is not an equivalence as the following example shows:

Example 2.6. Let $(\mathcal{P}(\mathbb{N}),\alpha)$, with

$$\alpha = \{(C,D) \mid C\cap D \neq \emptyset\} \cup \{(E,O),(O,E)\},$$

where E and O are the even and odd natural number sets, respectively. Then it is easy to see that α satisfies $(L1)$, $(L2)$ and $(L3)$. In this case $Spec(\mathcal{P}(\mathbb{N})) = \beta\mathbb{N}$ is the Stone-$\check{C}$ech compactification of $\mathbb{N}$ and one can prove that $(\mathcal{P}(\mathbb{N}),\alpha)$ and $(\mathbb{A}(\beta\mathbb{N}), R_{R^\alpha})$ are **not** isomorphic objects in the category **BR$\mathcal{L}$R**.

Definition 2.7. We denote by **BR$\mathcal{L}_1$R (Boolean rings with an $\mathcal{L}_1-$ relation)** the category of pairs (A,α) where A is a Boolean ring with unit and α is a $\mathcal{L}$-relation in A which moreover satisfies:

$(L4)$ $(\forall a,b,c\in A)$ $(c\ \alpha\ a\vee b \implies c\alpha a\ \vee\ c\alpha b)$.

The morphisms are taken to be the same as in **BR$\mathcal{L}$ R**.
We will call **$\mathcal{L}_1$-relation** a relation that satisfies $(L1)$ to $(L4)$.

Definition 2.8. We denote by **RHPS (Representations of Hausdorff "pre-quotients" of Stone spaces)** the category of pairs (X,γ), where X is a Stone space and γ is a relation on X, reflexive, symmetric and closed (that is, γ is a closed subset of $X\times X$), and the morphisms are taken to be the same as in **RPS**.

Definition 2.9. We denote by **BRLR (Boolean rings with a link relation)** the category of pairs (A,α) where A is a Boolean ring with unit and α is an $\mathcal{L}_1$-relation in A which moreover satisfies:

(L5) R^α is transitive in $Spec(A)$.

The morphisms are the same as in **BR$\mathcal{L}$ R**.
We will call **link relation** a relation satisfying $(L1)$ to $(L5)$.

Definition 2.10. We denote by **RHQS (Representations of Hausdorff quotients of Stone spaces)** the category of pairs $(X, \sim)$, where X is a Stone space and $\sim$ is a closed equivalence relation on X and morphisms are taken to be the same as in **RPS**.

It is clear that the categories $\mathbf{BR}\mathcal{L}_1\mathbf{R}$ and **RHPS** are full subcategories of the categories $\mathbf{BR}\mathcal{L}\mathbf{R}$ and **RPS** respectively, then the categories **BRLR** and **RHQS** are full subcategories of the categories $\mathbf{BR}\mathcal{L}_1\mathbf{R}$ and **RHPS** respectively.

Proposition 2.11. The categories $\mathbf{BR}\mathcal{L}_1\mathbf{R}$ and **RHPS** are equivalent.

Before proving this proposition, four preliminary results will be established:

Lemma 2.12. Let A be a Boolean ring with unit and α be a relation on $A \smallsetminus \{0\}$, satisfying the properties $(L2)$, $(L3)$ and $(L4)$. Let F and G be filters on A such that $FR^{\alpha}G$ and let $x \in A$. Then at least one of the following four statements holds:

i) $\forall y \in F$, $\forall z \in G$, $xy \;\alpha\; xz$,
ii) $\forall y \in F$, $\forall z \in G$, $xy \;\alpha\; x'z$,
iii) $\forall y \in F$, $\forall z \in G$, $x'y \;\alpha\; xz$,
iv) $\forall y \in F$, $\forall z \in G$, $x'y \;\alpha\; x'z$.

Proof. By contradiction, suppose that there exists $x \in A$ such that:

$$(\exists y_1 \in F,\ \exists z_1 \in G:\ xy_1 \neg\alpha\ xz_1) \text{ and } (\exists y_2 \in F,\ \exists z_2 \in G:\ xy_2 \neg\alpha\ x'z_2) \text{ and}$$
$$(\exists y_3 \in F,\ \exists z_3 \in G:\ x'y_3 \neg\alpha\ xz_3) \text{ and } (\exists y_4 \in F,\ \exists z_4 \in G:\ x'y_4 \neg\alpha\ x'z_4).$$

Since $FR^{\alpha}G$, then $y_1y_2y_3y_4 \;\alpha\; z_1z_2z_3z_4$ from which $(x \vee x')y_1y_2y_3y_4 \;\alpha\; (x \vee x')z_1z_2z_3z_4$ then $(xy_1y_2y_3y_4 \vee x'y_1y_2y_3y_4) \;\alpha\; (xz_1z_2z_3z_4 \vee x'z_1z_2z_3z_4)$. Using $(L2)$ and $(L4)$ we have $(xy_1y_2y_3y_4 \;\alpha\; xz_1z_2z_3z_4)$ or $(xy_1y_2y_3y_4 \;\alpha\; x'z_1z_2z_3z_4)$ or $(x'y_1y_2y_3y_4 \;\alpha\; xz_1z_2z_3z_4)$ or $(x'y_1y_2y_3y_4 \;\alpha\; x'z_1z_2z_3z_4)$, and applying $(L3)$, we have $xy_1 \;\alpha\; xz_1$ or $xy_2 \;\alpha\; x'z_2$ or $x'y_3 \;\alpha\; xz_3$ or $x'y_4 \;\alpha\; x'z_4$ □

In the following lemma, take $A = \{\ x_\lambda \mid \lambda \in \Omega\ \}$, where Ω is an ordinal number.

Lemma 2.13. Let A be a Boolean ring with unit and α be a relation in $A \smallsetminus \{0\}$ satisfying the properties $(L2)$, $(L3)$ and $(L4)$. If $c\alpha d$ $(c, d \in A)$ then for every $\lambda \in \Omega$ there exist filters F_λ, G_λ such that:

(i) $F_\beta \subseteq F_\lambda$ and $G_\beta \subseteq G_\lambda \;\; \forall \beta \leq \lambda$;
(ii) $F_\lambda R^\alpha G_\lambda$;
(iii) $c \in F_\lambda$ and ($x_\lambda \in F_\lambda$ or $x'_\lambda \in F_\lambda$);
(iv) $d \in G_\lambda$ and ($x_\lambda \in G_\lambda$ or $x'_\lambda \in G_\lambda$).

Proof. By transfinite induction: let 0 be the first element of Ω. Since $c \alpha d$ then $x_0 c \; \alpha \; x_0 d$ or $x_0 c \; \alpha \; x'_0 d$ or $x'_0 c \; \alpha \; x_0 d$ or $x'_0 c \; \alpha \; x'_0 d$ (applying $(L2)$ and $(L4)$). It suffices to take in each case: $F_0 :=< x_0 c >$ and $G_0 :=< x_0 d >$ or $F_0 :=< x_0 c >$ and $G_0 :=< x'_0 d >$ or $F_0 :=< x'_0 c >$ and $G_0 :=< x_0 d >$ or $F_0 :=< x'_0 c >$ and $G_0 :=< x'_0 d >$ respectively (observe that, since that α is defined on $A \setminus \{0\}$, then F_0 and G_0 are in fact filters). In any of the four cases it is easy to prove conditions $(i)-(iv)$. Now, suppose that the statement is valid for every $\beta < \lambda$.

Let $F := \bigcup_{\beta<\lambda} F_\lambda$ and $G := \bigcup_{\beta<\lambda} G_\lambda$. Using the inductive hypothesis it is easy to see that F and G are filters such that $F R^\alpha G$. By Lemma 2.12, one of the following four statements occurs:

(a) $\forall y \in F, \forall z \in G, \; x_\lambda y \; \alpha \; x_\lambda z$;
(b) $\forall y \in F, \forall z \in G, \; x_\lambda y \; \alpha \; x'_\lambda z$;
(c) $\forall y \in F, \forall z \in G, \; x'_\lambda y \; \alpha \; x_\lambda z$;
(d) $\forall y \in F, \forall z \in G, \; x'_\lambda y \; \alpha \; x'_\lambda z$;

and it suffices to take respectively the following pairs of filters:

(a) $F_\lambda :=< \{x_\lambda y \mid y \in F\} >$ and $G_\lambda :=< \{x_\lambda z \mid z \in G\} >$;
(b) $F_\lambda :=< \{x_\lambda y \mid y \in F\} >$ and $G_\lambda :=< \{x'_\lambda z \mid z \in G\} >$;
(c) $F_\lambda :=< \{x'_\lambda y \mid y \in F\} >$ and $G_\lambda :=< \{x_\lambda z \mid z \in G\} >$;
(d) $F_\lambda :=< \{x'_\lambda y \mid y \in F\} >$ and $G_\lambda :=< \{x'_\lambda z \mid z \in G\} >$.

In each case one can prove easily that F_λ and G_λ satisfy $(i)-(iv)$. $\square$

Proposition 2.14. Let A be a Boolean ring with unit and α be an $\mathcal{L}_1$-relation on A. Then the map

$$\begin{aligned} \mathbb{D} : A &\longrightarrow \mathbb{A}(Spec(A)) \\ c &\longmapsto \mathbb{D}(c) := \{\; U \in Spec(A) \mid c \in U \;\} \end{aligned} \qquad (*)$$

is an isomorphism of Boolean rings with unit such that for every pair $c, d \in A$,

$$\mathbb{D}(c) R_{R^\alpha} \mathbb{D}(d) \Longleftrightarrow c \alpha d.$$

Proof. It is known that $\mathbb{D}$ is an isomorphism of Boolean rings with 1. Suppose now $c,\ d \in A$ such that $c\alpha d$. Let $U_c := \bigcup_{\lambda\in\Omega} F_\lambda$ and $U_d := \bigcup_{\lambda\in\Omega} G_\lambda$, where F_λ and G_λ are the filters whose existence is secured by Lemma 2.13. Then U_c and U_d are ultrafilters which contain c and d respectively and $U_c R^\alpha U_d$, that is, $\mathbb{D}(c) R_{R^\alpha} \mathbb{D}(d)$. Conversely, if $\mathbb{D}(c) R_{R^\alpha} \mathbb{D}(d)$ then there exist $U \in \mathbb{D}(c)$ and $G \in \mathbb{D}(d)$ such that $U R^\alpha G$. It is clear that $c \in U$, $d \in G$ and therefore $c\alpha d$ □

Proposition 2.15. Let X be a Stone space and let γ be a closed relation on X. Then,

$$\begin{aligned} \mathfrak{U} : X &\longrightarrow Spec(\mathbb{A}(X)) \\ x &\longmapsto \mathfrak{U}_x := \{\ C \in \mathbb{A}(X) \mid x \in C\ \} \end{aligned}$$

is a homeomorphism that satisfies: $x\gamma y \Longleftrightarrow \mathfrak{U}_x R^{R_\gamma} \mathfrak{U}_y$, for every $x, y \in X$.

Proof. It is known that $\mathfrak{U}$ is a homeomorphism. Let $C \in \mathfrak{U}_x$ and $D \in \mathfrak{U}_y$. If $x\gamma y$ then clearly $C R_\gamma D$ and therefore $\mathfrak{U}_x R^{R_\gamma} \mathfrak{U}_y$. Conversely, suppose $\mathfrak{U}_x R^{R_\gamma} \mathfrak{U}_y$. It suffices to prove that (x, y) is a cluster point of $\gamma\ = \{(a, b) \in X \times X \mid a\gamma b\}$. Let $O_x \times O_y$ be an (basic) open set of $X \times X$, with $x \in O_x$ and $y \in O_y$. Since X is totally disconnected, we may assume that O_x and O_y are clopen sets. So, $O_x \in \mathfrak{U}_x$, $O_y \in \mathfrak{U}_y$, $O_x R_\gamma O_y$, from which there exist $a \in O_x$ and $b \in O_y$ such that $a\gamma b$. Therefore $(a, b) \in O_x \times O_y \cap \gamma$. □

Remarks

1. It is not difficult to prove that the isomorphisms in the category $\mathbf{BR}\mathcal{L}_1\mathbf{R}$ are the isomorphisms of Boolean rings with unit that respect the relations in both senses. Similarly the isomorphisms in the category **RHPS** are the homeomorphisms that respect the relations in both senses.
2. Since R^α is always closed for all relation α and since R_γ always satisfies $C R_\gamma A \cup D \Longrightarrow C R_\gamma A$ *or* $C R_\gamma D$, then the restrictions of the functors $\mathbb{S}$ and $\mathbb{A}$ to the subcategories $\mathbf{BR}\mathcal{L}_1\mathbf{R}$ and **RHPS** respectively determine functors between those two subcategories.
3. Similarly, since the transitivity of R^α in Spec(A) is exactly ($L5$) and on the other hand γ is transitive iff R^{R_γ} is transitive (it's a consequence of Proposition 2.15), then the restrictions of the functors $\mathbb{S}$ and $\mathbb{A}$ to

the subcategories **BRLR** and **RHQS** respectively, determine functors between them.

Proof of Proposition 2.11. It suffices to prove that the functor $\mathbb{S}$ (restricted to $\mathbf{BR}\mathcal{L}_1\mathbf{R}$) is faithful, full and representative.

• $\mathbb{S}$ is faithful: let (A,α) and (A',α') be objects of $\mathbf{BR}\mathcal{L}_1\mathbf{R}$. We must prove that

$$\mathbb{S}' : [(A,\alpha),\ (A',\alpha')] \longrightarrow [\mathbb{S}(A',\alpha'),\ \mathbb{S}(A,\alpha)]$$
$$f \longmapsto \mathbb{S}(f) = f^!$$

is one to one. Let $h, g \in [(A,\alpha),\ (A',\alpha')]$, and suppose $h \neq g$. There exists $k \in A$ such that $h(k) \neq g(k)$. If for example $h(k) \nleq g(k)$, let U be an ultrafilter such that

$$< h(k) \wedge (1 + g(k)) >= \{c \in A \mid h(k) \wedge (1 + g(k)) \leq c\} \subseteq U,$$

then $h(k) \in U$ and $g(k) \notin U$, therefore $h^{-1}(U) \neq g^{-1}(U)$ and so $\mathbb{S}(h) \neq \mathbb{S}(g)$. Similarly if $g(k) \nleq h(k)$.

• $\mathbb{S}$ is full: now, we must prove that $\mathbb{S}$ is onto. Let $t : (Spec(A'), R^{\alpha'}) \longrightarrow (Spec(A), R^{\alpha})$ a morphism of **RHPS**. Observe the diagram of Figure 3, where $\mathbb{D}$ and $\mathbb{D}'$ are the isomorphisms defined by $(*)$ in Proposition 2.14.

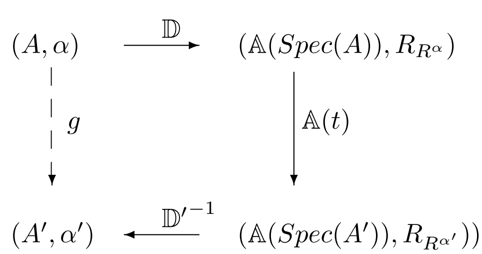

Figure 3

Let $g := \mathbb{D}'^{-1} \circ \mathbb{A}(t) \circ \mathbb{D}$. Then $g \in [(A,\alpha),(A',\alpha')]$ and $\mathbb{S}(g) = t$. We will prove that $\mathbb{S}(g) = t$. It is obvious that

$$\mathbb{S}(g) : (Spec(A'), R^{\alpha'}) \longrightarrow (Spec(A), R^{\alpha}).$$

Let $U \in Spec(A')$, then

$$\begin{aligned} b \in \mathbb{S}(g)(U) &\Longleftrightarrow b \in g^{-1}(U) \\ &\Longleftrightarrow b \in \mathbb{D}^{-1}(\mathbb{A}(t)^{-1}\mathbb{D}'(U)) \\ &\Longleftrightarrow \mathbb{D}(b) \in \mathbb{A}(t)^{-1}\mathbb{D}'(U) \\ &\Longleftrightarrow \mathbb{A}(t)(\mathbb{D}(b)) \in \mathbb{D}'(U) \\ &\Longleftrightarrow t^{-1}(\mathbb{D}(b)) \in \{\mathbb{D}'(u) \mid u \in U\} \\ &\Longleftrightarrow t^{-1}(\mathbb{D}(b)) = \mathbb{D}'(u), \text{ for some } u \in U \\ &\Longleftrightarrow \{G \in Spec(A') \mid b \in t(G)\} = \mathbb{D}'(u), \text{ for some } u \in U, \end{aligned}$$

but clearly $U \in \mathbb{D}'(u)$, therefore $b \in t(U)$ and so $\mathbb{S}(g)(U) \subseteq t(U)$. Since $\mathbb{S}(g)(U)$ and $t(U)$ are ultrafilters then they must be equal.

• $\mathbb{S}$ is representative: let (X, γ) be an object of **RHPS**. From Proposition 2.15 we have that $\mathbb{S}\mathbb{A}(X, \gamma) = (Spec\mathbb{A}(X), R^{R_\gamma}) \simeq (X, \gamma)$ (isomorphism in **RHPS**). This completes the proof. □

From Remark 3 and Proposition 2.11, we have:

Corollary 2.16. The categories **BRLR** and **RHQS** are equivalent too.

Example 2.17. Let $A = \mathcal{P}(X)$, where $X = \{a, b, c\}$ and

$\alpha = \alpha_A \cup \{(\{a\}, \{b\}), (\{b\}, \{a\}), (\{a, c\}, \{b\}), (\{b\}, \{a, c\}), (\{b, c\}, \{a\}), (\{a\}, \{b, c\})\}$,

where $\alpha_A = \{(C, D) \mid C \cap D \neq \emptyset\}$. Then α satisfies the properties $(L1)$ to $(L5)$. The topological representation of (A, α) is $(Spec(A), R^\alpha)$ with:

$$Spec(A) = \{< a >, \ < b >, \ < c >\},$$

where $< a >= \{\{a\}, \{a, b\}, \{a, c\}, X\}$, $< b >= \{\{b\}, \{b, c\}, \{a, b\}, X\}$, and $< c >= \{\{c\}, \{b, c\}, \{a, c\}, X\}$. Thus, $Spec(A)$ is a discrete space,

$$\begin{aligned} R^\alpha = \{&(< a >, < a >), (< b >, < b >), (< c >, < c >), \\ &(< a >, < b >), (< b >, < a >)\} \end{aligned}$$

and the corresponding quotient is (homeomorphic to) a discrete space with two points.

Example 2.18. Let $A = \mathcal{P}(\mathbb{N})$,

$$\alpha = \{ \ (C, D) \mid C \cap D \neq \emptyset \ \} \cup \{ \ (C, D) \mid C, D \ are \ infinite \ \}.$$

Then α satisfies $(L1)$ to $(L5)$. The topological representation of (A, α) is $Spec(A) = \beta\mathbb{N}$, the Stone-$\breve{C}$ech compactification of $\mathbb{N}$, with the relation $R^{\alpha} = \{ (\mathcal{U},\mathcal{U}) \mid \mathcal{U} \in \beta\mathbb{N} \} \cup \{ (\mathcal{U},\mathcal{G}) \mid \mathcal{U}\ \textit{and}\ \mathcal{G}\ \textit{are nonprincipal ultrafilters} \}$. In this case the quotient $\beta\mathbb{N}/R^{\alpha}$ is (homeomorphic to) the Alexandroff compactification of $\mathbb{N}$.

Now, if BR_1 denotes the classic category: *Boolean rings with unit-Boolean ring morphisms wich preserve the unit* and **ST** denotes the classic category: *Stone spaces-continuous functions*, then **BRLR** and **RHQS** can be viewed as extensions of BR_1 and **ST** respectively, such that they are equivalent when the functors $\mathbb{S}$ and $\mathbb{A}$ are restricted to them. That is,

Proposition 2.19. The categories **BRLR** and **RHQS** contain a subcategory isomorphic to BR_1 and a subcategory isomorphic to **ST**, respectively. Furthermore, these subcategories are equivalent.

Proof. Let A be a Boolean ring with unit. Define in A the relation α_A as follows: for all $c, d \in A$,

$$c\alpha_A d \Longleftrightarrow c \wedge d \neq 0.$$

Then α_A is a link relation (it is moreover the "smallest" link relation that we can define in A, that is, if α is another link relation on A, then $\alpha_A \subseteq \alpha$). On the other hand, if $f : A \longrightarrow A'$ is a Boolean ring morphism which preserves the unit and $f(c)\alpha_{A'}f(d)$, e.g., $f(c)f(d) \neq 0$, then $f(cd) \neq 0$ which implies $cd \neq 0$, that is, $c\alpha_A d$. Hence, f is a morphism of **BRLR** and the functor I described in the diagram of Figure 4 is well-defined.

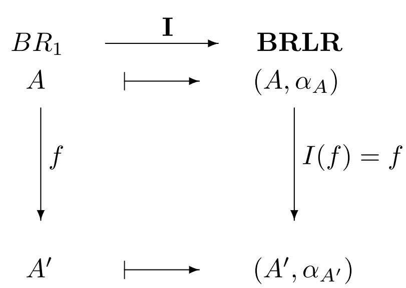

Figure 4

In the opposite direction, we have the forgetful functor **O** defined in the diagram of Figure 5.

$$
\begin{array}{ccc}
BR_1 & \xleftarrow{\quad \mathbf{O} \quad} & \mathbf{ABRL} \\
A & \longleftarrow\!\shortmid & (A, \alpha) \\
\Big\downarrow O(f) = f & & \Big\downarrow f \\
A' & \longleftarrow\!\shortmid & (A', \alpha')
\end{array}
$$

Figure 5

It is clear that the functors **I** and **O** establish an isomorphism between the categories BR_1 and $I(BR_1)$ (the last one is a subcategory of **BRLR**). On the other hand, if X is a Stone space, it is immediate that the equality relation on X is a closed equivalence relation and that if $f : X \longrightarrow Y$ is a continuous function between Stone spaces, then f preserves the equality relation. So, the functor F defined by the diagram of Figure 6, is well-defined.

$$
\begin{array}{ccc}
\mathbf{ST} & \xrightarrow{\quad \mathbf{F} \quad} & \mathbf{RHQS} \\
X & \longmapsto & (X, =) \\
\Big\downarrow f & & \Big\downarrow F(f) = f \\
Y & \longmapsto & (Y, =)
\end{array}
$$

Figure 6

In the opposite direction we have another forgetful functor **O'** defined by the diagram of Figure 7.

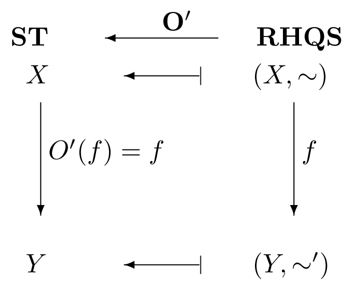

Figure 7

With these functors, **F(ST)** is a subcategory of **RHQS**, isomorphic to **ST**.

Now, let (A, α_A) be an object of $\mathbf{I}(BR_1)$. We have that R^{α_A} is exactly the equality relation on $Spec(A)$. In fact, if $U, G \in Spec(A)$ then $UR^{\alpha_A}G$ means that, for all $u \in U$ and for all $g \in G$, $ug \neq 0$, which implies $U = G$. In this manner $(Spec(A), R^{\alpha_A}) = (Spec(A), =)$ is an object of **F(ST)**. In a similar way, if $(X, =)$ is an object of $F(ST)$ then the relation $R_=$ is precisely $\alpha_{\mathbb{A}(X)}$, since if $C, D \in \mathbb{A}(X)$ then $CR_=D$ means that $C \cap D \neq \emptyset$, therefore $(\mathbb{A}(X), R_=) = (\mathbb{A}(X), \alpha_{\mathbb{A}(X)})$ an object of $\mathbf{I}(BR_1)$. Therefore the categories $\mathbf{I}(BR_1)$ and **F(ST)** are equivalent. □

Remark Observe that from Proposition 2.11, Corollary 2.16 and Proposition 2.19, two extensions of the Stone duality are established.

And last but not least, we want to point out that Hausdorff quotients of the Cantor space form a wide family of topological spaces: these are precisely the compact metric spaces and includes, for example, all **continua** (metric, compact and connected spaces). Moreover, it is well known that the Boolean ring that corresponds to the Cantor space is the unique (up isomorphisms) Boolean ring with unit, countable and without atoms [1], [2] or [4]. We will call this ring the **Cantor's ring** and we will denote it by $\mathbb{K}$.

We define the following full subcategories of **BRLR** and **RHQS**:

Definition 2.20. We denote by **CRLR** (Cantor's ring with a link relation) the full subcategory of **BRLR** whose objects are the pairs $(\mathbb{K}, \alpha)$.

Definition 2.21. We denote by **RHQC** (representations of Hausdorff quotients of Cantor space) the full subcategory of **RHQS** whose objects are the pairs $(\Sigma^{\mathbb{N}}, \sim)$ where $\Sigma^{\mathbb{N}}$ is the Cantor space.

The following result is immediate.

Corollary 2.22. The categories **CRLR** and **RHQC** are equivalent.

In the diagram of Figure 8 the main results established up to now are illustrated and summarized. In particular, this diagram shows:

1. The adjunction established by the functors $\mathbb{S}$ and $\mathbb{A}$, (Proposition 2.5).
2. The equivalence established by the restrictions of the functors $\mathbb{S}$ and $\mathbb{A}$ to the subcategories $\mathbf{BR\mathcal{L}_1R}$ and **RHPS** (Proposition 2.11).
3. The equivalence established by the restrictions of the functors $\mathbb{S}$ and $\mathbb{A}$ to the subcategories **BRLR** and **RHQS** (Corollary 2.16).
4. The equivalence established by the restrictions of the functors $\mathbb{S}$ and $\mathbb{A}$ to the subcategories $I(BR_1)$ and $F(ST)$, (Proposition 2.19).
5. The equivalence established by the restrictions of the functors $\mathbb{S}$ and $\mathbb{A}$ to the subcategories **CRLR** and **RHQC**, (Corollary 2.22).
6. The isomorphism established by the functors **I** and **O**, (Proposition 2.19).
7. The isomorphism established by the functors **F** and **O'**, (Proposition 2.19).
8. The equivalence established by the Stone's Representation Theorem for Boolean rings with unit (Stone duality).

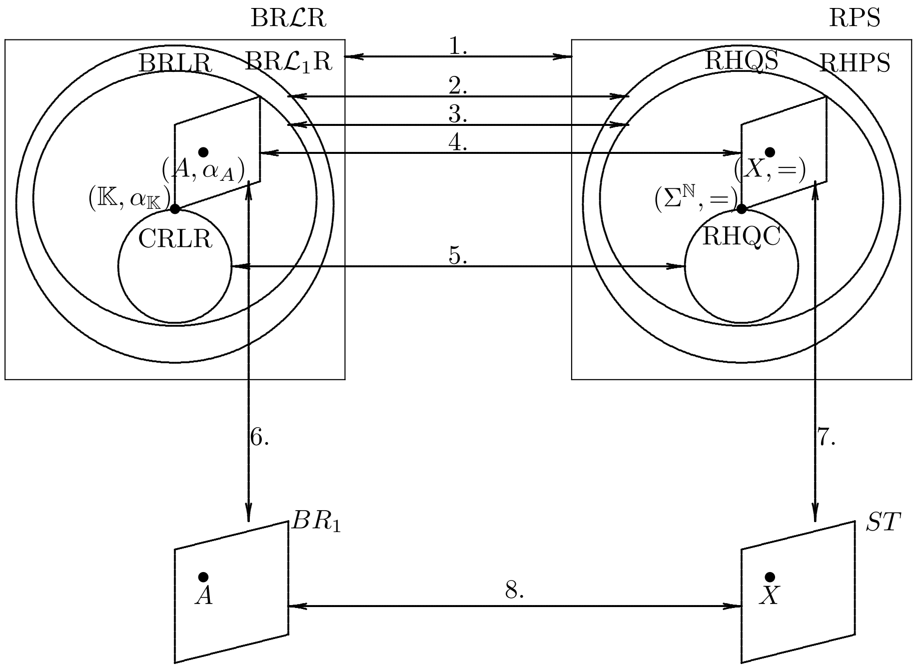

Figure 8

3. Connectedness

In this section using results of previous section, an algebraic interpretation of the topological notion of connectedness is presented.

The following notion appears in [1] and will be used in Lemma 3.1: let A be a Boolean ring, $t \in A$, $t \neq 0$ and $n \in \mathbb{Z}^+$. We will call $C \subseteq A$ a **n-partition** of t if: (i) $|C| = n$; (ii) $(\forall c \in C)(c < t)$; (iii) $\bigvee_{c \in C} c = t$; (iv) $0 \notin C$; (v) $(\forall c, d \in C)(c \neq d \Longrightarrow cd = 0)$.

Lemma 3.1. Let (A, α) be an object of **BRLR**. The following statements are equivalent:

- (i) every 2-partition $C = \{a, b\}$ of the ring's unit, satisfies $a\alpha b$.
- (ii) If $C = \{b_1, \cdots, b_m\}$ is a m-partition of 1 ($m \in \mathbb{N}$, $m \geq 2$), then for any b_i, b_j, $1 \leq i, j \leq m$, there exist $b_{i_1}, b_{i_2}, \cdots, b_{i_k}$ belonging to C, such that $b_{i_1}\alpha b_{i_2}$, $b_{i_2}\alpha b_{i_3}, \cdots, b_{i_{k-1}}\alpha b_{i_k}$, $b_{i_1} = b_i$ and $b_{i_k} = b_j$ (that is, the graph $(C, \alpha|_C)$ is path connected).
- (iii) For all $a \in A \smallsetminus \{0, 1\}$, we have $a\alpha a'$.

Proof.
$(i) \Longrightarrow (ii)$: By induction on m: for $m = 2$, (ii) coincides with (i). Suppose that the assertion is valid for some $m \in \mathbb{N}$, $m \geq 2$. Let $C = \{b_1, \cdots, b_m, b_{m+1}\}$ be a $m+1$-partition of 1. It suffices to prove that for every $b_j \in C$ there exists a path that connects b_j with b_1. Obviously b_1 connects with himself (α is reflexive in $A \setminus \{0,1\}$). If $C = \{b_1, b_2, b_3\}$ is a 3-partition, then $\{b_1, b_2 \vee b_3\}$ is a 2-partition and therefore $b_1 \alpha b_2 \vee b_3$, which implies (applying $(L4)$) that $b_1 \alpha b_2$ or $b_1 \alpha b_3$. If $b_1 \alpha b_2$, we consider the 2-partition $\{b_3, b_1 \vee b_2\}$, from which $b_3 \alpha\, b_1 \vee b_2$ and, $b_3 \alpha b_1$ or $b_3 \alpha b_2$. In any case, we obtain that b_1 can be connected with b_2 and with b_3. Similarly, if $b_1 \alpha b_3$, consider the 2-partition $\{b_2, b_1 \vee b_3\}$ and obtain the same conclusion.

Now, if $m > 3$ then $\{b_1, b_2, \cdots, b_{m-1}, b_m \vee b_{m+1}\}$ is a m-partition of 1, and by the inductive hypothesis, there exists a path from b_1 to b_2, from b_1 to b_3, $\cdots$, from b_1 to b_{m-1}. Similarly $\{b_1, b_m, b_3, , \cdots, b_2 \vee b_{m+1}\}$ and $\{b_1, b_{m+1}, b_3, \cdots, b_2 \vee b_m\}$ are m-partitions too and again, there exist paths from b_1 to b_m and from b_1 to b_{m+1}. In total, there exists a path from b_1 to b_j for all $j = 1, 2, \cdots, m+1$.

$(ii) \Longrightarrow (iii)$: if $a \in A \setminus \{0,1\}$, then $C = \{a, a'\}$ is a 2-partition of 1 and by hypothesis there is a path which connects a with a', therefore necessarily $a \alpha a'$.

$(iii) \Longrightarrow (i)$: let $C = \{a, b\}$ a 2-partition of 1. Clearly $a \in A \setminus \{0,1\}$ and $b = a'$, then $a \alpha a' = b$. □

Definition 3.2 (Connectedness in the category BRLR)**.** An object (A, α) of **BRLR** will be called **connected** if it satisfies any one of the three conditions of the previous lemma.

Lemma 3.3. Let X be a Stone space and let $X/\sim$ be a Hausdorff quotient of X. Then $X/\sim$ is disconnected if and only if there exist C and D clopen subsets of X, disjoint, nonempty, such that $C \cup D = X$ and $C \neg R_\sim D$.

Proof. If $X/\sim$ is disconnected let $A \cup B = X/\sim$ be a disconnection of $X/\sim$. If $j : X \longrightarrow X/\sim$ is the quotient map, then it is enough to take $C = j^{-1}(A)$ and $D = j^{-1}(B)$. Conversely, if there exist C and D clopen subsets of X satisfying the required conditions, then $j(C) \cup j(D) = X/\sim$ is a disconnection of $X/\sim$. □

We have a translation from topological connectedness to algebraical connectedness and conversely:

Proposition 3.4. Let (A, α) be an object of **BRLR** and $(X, \sim)$ be an object of **RHQS**. Then

(*i*) $X/\sim$ is connected if and only if $(\mathbb{A}(X), R_\sim)$ is connected.
(*ii*) (A, α) is connected if and only if $Spec(A)/R^\alpha$ is connected.

Proof. (*i*) It follows by contradiction and by means of Lemma 3.3.

(*ii*) It's an immediate consequence of (*i*) and the isomorphism (see Proposition 2.14):

$$\begin{aligned} \mathbb{D} : (A, \alpha) &\longrightarrow (\mathbb{A}(Spec(A)), R_{R^\alpha}) \qquad (*) \\ c &\longmapsto \mathbb{D}(c) := \{\ U \in Spec(A) \mid c \in U\ \}. \end{aligned}$$

□

The following corollary establishes an algebraic characterization of continua:

Corollary 3.5. Every continuum can be represented (algebraically) as a connected object of **CRLR**. Conversely, every connected object of **CRLR** represents a continuum

Proof. X is a continuum if and only if it is homeomorphic to a Hausdorff connected quotient of the Cantor space, $X/\sim$, which (using Proposition 3.4. (i)) is equivalent to a connected object $(\mathbb{A}(X), R_\sim)$ of **CRLR**. Conversely, from Proposition 3.4. (*ii*), we have that if $(\mathbb{K}, \alpha)$ is a connected object of **CRLR**, then $Spec(\mathbb{K})/R^\alpha$ is a continuum. □

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