This is a reprint of
Lecturas Matemáticas
Volumen **25** *(2004), páginas 149–155*

Data dependence for Ishikawa iteration

Ştefan M. Şoltuz
Cluj-Napoca, Rumania

Lecturas Matemáticas
Volumen **25** *(2004), páginas 149–155*

Data dependence for Ishikawa iteration

Ştefan M. Şoltuz
Cluj-Napoca, Romania

Abstract. A convergence result and a data dependence for Ishikawa iteration are established dealing with contractions.

Key words and phrases. Ishikawa iteration, fixed points.

2000 Mathematics Subject Classification. Primary: 47H10.

Resumen. Se demuestra un resultado sobre la convergencia y la dependencia para la iteración de Ishikawa en el caso de contracciones.

1. Introduction

Let X be a Banach space and $B \subset X$ be a nonempty convex closed and bounded set. Let $T, S : B \to B$ be two maps. For given $x_1 \in B$ and $u_1 \in B$, we consider the Ishikawa iteration (see [4]) for T and S :

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,\ y_n = (1-\beta_n)x_n + \beta_n T x_n\ ; \qquad (1)$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n S v_n,\ v_n = (1-\beta_n)u_n + \beta_n S u_n\ , \qquad (2)$$

where $(\alpha_n)_n,\ (\beta_n)_n \subset (0,1)$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The map T is a *contraction* if there exists $k \in (0,1)$ such that

$$\|Tx - Ty\| \leq k \|x - y\| \ , \tag{3}$$

for all $x, y \in X$. In this note we prove a data dependence result for Ishikawa iteration.

The following proposition is in [7]. The sequence $(a_n)_n$ there appearing, is very famous, being present in most papers on Mann and Ishikawa iterations. As far as we know, there is not so far a different proof of such proposition than that in [7]. Here we give another proof.

Proposition 1.1. *Let $(a_n)_n$ be a nonnegative sequence satisfying the inequality*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \varepsilon, \tag{4}$$

where $\lambda_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\varepsilon > 0$ is fixed. Then, the following holds:

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \varepsilon \ . \tag{5}$$

Proof. (I) We first assume that $a_1 \leq \varepsilon$. Then we have $a_2 \leq (1-\lambda_1)a_1 + \lambda_1 \varepsilon \leq \varepsilon$, and assuming that $a_n \leq \varepsilon$, we prove that $a_{n+1} \leq \varepsilon$. Indeed,

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \varepsilon \leq (1 - \lambda_n)\varepsilon + \lambda_n \varepsilon = \varepsilon \ ,$$

i.e., for all $n \in N$, $a_n \leq \varepsilon$. Hence, the conclusion holds in this case.

(II) Now we assume $a_1 > \varepsilon$. Then

$$a_2 \leq (1 - \lambda_1)a_1 + \lambda_1 \varepsilon = a_1 - \lambda_1 a_1 + \lambda_1 \varepsilon \leq a_1.$$

Now, two cases are always possible:

(1^o) There exists n_0 such that $a_{n_0} \leq \varepsilon$. Then from (I) (with a_{n_0} instead of a_1), we get the conclusion.

(2^o) For all $n \in \mathbb{N}$ we have $a_n > \varepsilon$. By iterating the same argument in (II) we obtain

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq ... > \varepsilon \ ,$$

so that if we assume $\limsup_{n\to\infty} a_n > \varepsilon$ then, for some $p \in \mathbb{N}$,

$$\varepsilon + \frac{\varepsilon}{p} < a_n \ , \quad n \in \mathbb{N} \ .$$

Thus

$$\lambda_n \varepsilon \leq \lambda_n a_n \frac{p}{p+1} \ , \quad n \in \mathbb{N} \ ,$$

and (4) yields

$$\begin{aligned} a_{n+1} &\leq (1-\lambda_n)a_n + \lambda_n \varepsilon \\ &\leq (1-\lambda_n)a_n + \lambda_n a_n \frac{p}{p+1} = (1 - \frac{\lambda_n}{p+1})a_n \ , \end{aligned}$$

so that

$$a_{n+1} \leq \prod_{k=1}^{n} (1 - \frac{\lambda_k}{p+1}) \ a_1.$$

Since for all $x \in [0,1]$ we have $(1-x) \leq \exp(-x)$, it follows that

$$0 \leq a_{n+1} \leq \exp\left(- \frac{1}{p+1} \sum_{n=1}^{\infty} \lambda_n \right) a_1 \to 0 \ , \quad (n \to \infty).$$

Hence we get $\lim_{n\to\infty} a_n = 0$, in contradiction with $\varepsilon < a_n$, for all $n \in \mathbb{N}$. The proof is complete. ☑

From the argument in the above proof we can observe that:

Remark 1.2. *If* $(\beta_n)_n$ *is sequence such that* $\beta_n \in (0,1]$, *for all* $n \in \mathbb{N}$, *and if* $\sum_{n=1}^{\infty} \beta_n = \infty$ *then* $\prod_{n=1}^{\infty} (1-\beta_n) = 0$.

2. A convergence result

We are now able to establish a convergence result.

Theorem 2.1. *Let X be a Banach space and $B \subset X$ be a nonempty convex, closed and bounded set. Let $T : B \to B$ be a contractive map. Then iteration* (1) *converges to a unique fixed point of T.*

Proof. The existence and uniqueness of the fixed point follow from the Picard–Banach theorem. Let $x^* = Tx^*$ be such fixed point. Then we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|Ty_n - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n k \|y_n - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n k(1-\beta_n)\|x_n - x^*\| \\
&\qquad\qquad + \alpha_n k \beta_n \|Tx_n - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n k(1-\beta_n)\|x_n - x^*\| \\
&\qquad\qquad + \alpha_n k^2 \beta_n \|x_n - x^*\| \\
&= (1-\alpha_n(1-k(1-\beta_n)-k^2\beta_n))\|x_n - x^*\|
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\le \big(1-\alpha_n(1-k(1-\beta_n)-k^2\beta_n)\big)\|x_n - x^*\|, \\
\|x_n - x^*\| &\le \big(1-\alpha_{n-1}(1-k(1-\beta_{n-1})-k^2\beta_{n-1})\big)\|x_{n-1} - x^*\|, \\
&\dots \\
\|x_2 - x^*\| &\le \big(1-\alpha_1(1-k(1-\beta_1)-k^2\beta_1)\big)\|x_1 - x^*\| \ .
\end{aligned}
$$

From this we obtain

$$
\|x_{n+1} - x^*\| \le \left[\prod_{k=1}^{n} (1-\alpha_n(1-k))\right]\|x_1 - x^*\|.
$$

But $\sum_{n=1}^{\infty} \alpha_n(\alpha_n(1-k(1-\beta_n)-k^2\beta_n)) = \infty$, and from Remark 2.1 we have

$$\prod_{k=1}^{n} (1-\alpha_n(1-k)) \to 0.$$

Hence $\lim_{n\to\infty} \|x_n - x^*\| = 0$. ☑

3. Data dependence

We are now able to establish the following data dependence result.

Theorem 3.1. *Let X be a Banach space and $B \subset X$ be a nonempty, convex, closed and bounded set. Let $\varepsilon > 0$ be a fixed number. If S and T are contractions and if*

$$\|Tz - Sz\| \le \varepsilon, \quad z \in B$$

holds, then we have

$$\|x^* - u^*\| \le \frac{\varepsilon}{1-k},$$

for $x^* = Tx^*, u^* = Su^*$.

Proof. Theorem 2.1 grants the existence of x^* and u^*. From (1) and (2) we have $x_{n+1} - u_{n+1} = (1-\alpha_n)(x_n - u_n) + \alpha_n(Ty_n - Sv_n)$. Thus,

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \|(1-\alpha_n)(x_n - u_n) + \alpha_n(Ty_n - Sv_n)\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n \|Ty_n - Sy_n + Sy_n - Sv_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n \|Ty_n - Sy_n\| + \alpha_n \|Sy_n - Sv_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\varepsilon + \alpha_n k \|y_n - v_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\varepsilon + \alpha_n k(1-\beta_n)\|x_n - u_n\| \\
&\qquad + \alpha_n k\beta_n \|Tx_n - Su_n\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\left\|x_n-u_n\right\|+\alpha_n\varepsilon+\alpha_n k(1-\beta_n)\left\|x_n-u_n\right\| \\
&\qquad\qquad +\alpha_n k\beta_n\left(\left\|Sx_n-Su_n\right\|+\left\|Tx_n-Sx_n\right\|\right) \\
&\leq (1-\alpha_n)\left\|x_n-u_n\right\|+\alpha_n\varepsilon+\alpha_n k(1-\beta_n)\left\|x_n-u_n\right\| \\
&\qquad\qquad +\alpha_n k^2\beta_n\left\|x_n-u_n\right\|+\alpha_n k\beta_n\varepsilon \\
&\leq \left(1-\alpha_n(1-k(1-\beta_n)-k^2\beta_n)\right)\left\|x_n-u_n\right\|+\alpha_n\varepsilon(1+k\beta_n) \\
&\leq \left(1-\alpha_n(1-k(1-\beta_n)-k^2\beta_n)\right)\left\|x_n-u_n\right\| \\
&\qquad\qquad +\alpha_n\left(1-k(1-\beta_n)-k^2\beta_n\right)\frac{\varepsilon(1+k\beta_n)}{1-k(1-\beta_n)-k^2\beta_n} \\
&\qquad\qquad +\alpha_n\left(1-k(1-\beta_n)-k^2\beta_n\right)\frac{\varepsilon(1+k\beta_n)}{1-k(1-\beta_n)-k^2\beta_n} \\
&\leq \left(1-\alpha_n(1-k(1-\beta_n)-k^2\beta_n)\right)\left\|x_n-u_n\right\| \\
&\qquad\qquad +\alpha_n\left(1-k(1-\beta_n)-k^2\beta_n\right)\ \frac{\varepsilon}{1-k}
\end{aligned}
$$

We have used that $\frac{\varepsilon(1+k\beta_n)}{1-k(1-\beta_n)-k^2\beta_n}=\frac{\varepsilon}{1-k}$. Thus, let

$$\lambda_n := \alpha_n(1-k(1-\beta_n)-k^2\beta_n)\in(0,1), \quad a_n := \left\|x_n-u_n\right\| .$$

From Proposition 1.1 it follows that

$$0\leq \limsup_{n\to\infty} a_n\leq \frac{\varepsilon}{1-k}.$$

But $\limsup\limits_{n\to\infty} a_n=\limsup\limits_{n\to\infty}\left\|x_n-u_n\right\|$, and from Theorem 2.1 we know that $\lim\limits_{n\to\infty} x_n=x^*$, $\lim\limits_{n\to\infty} u_n=u^*$. Then we have

$$\left\|x^*-u^*\right\|\leq\frac{\varepsilon}{1-k}. \quad ☑$$

The result can be improved by assuming that only S is a contraction and that $\lim\limits_{n\to\infty} x_n=x^*$. For $\beta_n=0$, $n\in\mathbb{N}$, the above result can be found in [10]. Theorem 2.1 is not new. We can recognize it in classical analysis books, where for the proof, the Picard–Banach iteration is used instead of the Ishikawa iteration. Normally the Picard–Banach iteration converges geometrically to the fixed point of a contraction. The Ishikawa iteration is much more slower in convergence. This fact does not change the data dependence result.

References

[1] S. S. Chang, *On Chidume's open questions and approximate solution of multivalued strongly accretive mapping equations in Banach spaces*, J. Math. Anal. Appl. **216** (1997), 94–111.

[2] S. S. Chang, Y. J. Cho, B. S. Lee, J. S. Jung, S. M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl. **224** (1998), 149–165.

[3] Gun Feng, *Iteration processes for approximating fixed points of operators of monotone type*, Proc. Amer. Math. Soc. **129** (2001), 2293–2300.

[4] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[5] L.–S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114-125.

[6] W. R. Mann, *Mean value in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[7] J. A. Park, *Mann-iteration for strictly pseudocontractive maps*, J. Korean Math. Soc. **31** (1994), 333–337.

[8] Ştefan M. Şoltuz, *Some sequences supplied by inequalities and their applications*, Revue d'analyse numérique et de théorie de l'approximation, Tome **29,** No. 2, (2000), 207–212.

[9] Ştefan M. Şoltuz, *Three proofs for the convergence of a sequence*, Octogon Math. Magazine **9** (2001):1, 503–505.

[10] Ştefan M. Şoltuz, *Data dependence for Mann iteration*, Octogon Math. Magazine **9** (2001):2, 825–828.

[11] X. Weng, *Fixed point iteration for local strictly pseudocontractive mapping*, Proc. Amer. Math. Soc. **113** (1991), 727–731.

[12] Haiyun Zhou, Yuting Jia, *Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption*, Proc. Amer. Math. Soc. **125** (1997), 1705–1709.

(Recibido en diciembre de 2001. Aceptado para publicación junio 2003)

Ştefan M. Şoltuz
str. Avram Iancu 13
3400 Cluj- Napoca, Romania
e-mail: `soltuzul@yahoo.com, soltuz@itwm.fhg.de`