Lecturas Matemáticas
Volumen **26** *(2005), páginas 11–26*

An ABS representation theorem for polyhedral sets and its applications

Hamid Esmaeili
Sina University, Hamedan, Iran

Abstract. The ABS methods have been used extensively for solving linear and nonlinear systems of equations. In this paper attempt has been made to find explicitly all solutions of a system of m linear inequalities in n variables, $m \leq n$, with full rank matrix. Having obtaining the result, the problem of finding the least squares point in that polyhedral set is transformed to nonnegative least squares with m variables. Also, those results are applied to the LP problem with $m \leq n$ inequality constraints, obtaining optimality conditions and an explicit representation of all solutions.

Key words and phrases. ABS Algorithm, Polyhedral Set, Least Squares Solution, LP Problem.

2000 Mathematics Subject Classification. Primary 42A40, Secondary 46B42, 46A22.

Resumen. Se buscan explícitamente todas las soluciones de un sistema de m desigualdades lineales en n variables, $m \leq n$, con una matriz de rango completo. Después de obtener este resultado, el problema de hallar el punto de mínimos cuadrados en este conjunto poliédrico en m variables se transforma a uno de mínimos cuadrados no negativos en m variables. Estos resultados se aplican al problema LP con $m \leq n$ restricciones lineales para obtener condiciones de optimalidad y una representación explícita de todas las soluciones.

1. Introduction

Let $A \in R^{m,n}$, $m \leq n$, be a full row rank matrix and $b \in R^m$. Consider the system of linear inequalities

$$Ax \leq b, \quad x \in R^n. \tag{1}$$

Given a general system of inequalities $Ax \leq b$, it is called feasible if there exists a vector $x^* \in R^n$ such that $Ax^* \leq b$, and x^* is a feasible solution, otherwise it is called infeasible. It is clear that system (1) is feasible if A is full rank, since equation $Ax = c, c \in R^n$ is solvable for any $c \leq b$. The intersection of every finite number of half spaces $a_i^T x \leq b_i$, $i = 1, \ldots, m$, is a polyhedron in R^n. Thus, the system (1) denotes a polyhedron in R^n. Our aim is to define an explicit form for points of the polyhedron (1) using the ABS methods.

The ABS methods [1] have been used extensively to solve general linear systems of equations and optimization problems. They are a class of direct iteration type methods that, in a finite number of iterations, either determine the general solution of a system of linear equations or establish its inconsistency. ABS methods for solving a linear system of equations are extensively discussed in monograph [2], while monograph [14] considers their application to several problems in optimization, including an ABS reformulation and generalization of the classical simplex method and Karmarkar interior point method for the LP problem.

ZHANG [11] and ZHANG [15] have applied the ABS methods to get a solution of a linear inequalities system. In particular, ZHANG [12] has shown that the Huang algorithm in the ABS class can be coupled with a Goldfarb-Idnani active set strategy, see [3], to determine, in a finite number of iterations, the least Euclidean norm of an inequalitiy linear system. Here, we find that solution by converting the problem to a nonnegative least squares. SHI [8] has applied the Huang algorithm to generate a sequence of solutions of a linear system such that any limit point is a nonnegative solution.

In this paper attempt has been made to use the ABS methods in order to determine feasible points of the polyhedron defined by (1), providing an explicit form of the feasible points. Hence, we can compute the least squares solutions for (1). Then, we apply that approach to a class of LP problems.

SPEDICATO and ABAFFY [9] have discussed the solution of the linear programming problem with constraints (1). Their method is based on the *LU* implicit algorithm in the ABS class. The problem is solved by getting an explicit form of the points of the polyhedron that is independent of the selection of a particular algorithm in the ABS class. Hence, the present approach is more general in character.

Section 2 recalls the class of ABS methods and provides some of its properties. Furthermore, in this section the Huang algorithm in the ABS class has been presented. In section 3, reasons have been adduced for solving an inequality linear system by providing a representation of the points of a polyhedron type (1). By using this representation theorem, in section 4, computing the least squares solutions of polyhedron (1) is presented. In section 5, the solution of full rank inequality constrained linear programming problems giving the conditions of optimality and unboundedness is considered.

2. ABS Methods for Solving Linear Systems

The ABS methods have been developed by Abaffy, Broyden & Spedicato [1]. Consider the system of linear equations

$$Ax = b, \tag{2}$$

where $A \in R^{m,n}$, $b \in R^m$ and rank $(A) = m$. Let $A = (a_1, \ldots, a_m)^T$, $a_i \in R^n$, $i = 1, \ldots, m$ and $b = (b_1, \ldots, b_m)^T$. Also let $A_i = (a_1, \ldots, a_i)$ and $b^{(i)} = (b_1, \ldots, b_i)^T$.

Give $x_1 \in R^n$ arbitrary and $H_1 \in R^{n,n}$, Spedicato's parameter is arbitrary and nonsingular. It must be borne in mind that any $x \in R^n$ can be represented by $x = x_1 + H_1^T q$ for some $q \in R^n$.

The basic ABS class of methods consists of direct iteration type methods for computing the general solution of (2). In the beginning of the ith iteration, $i \geq 1$, the general solution for the first $i-1$ equations is at hand. It is clear that if x_i is a solution for the first $i-1$ equations and if $H_i \in R^{n,n}$ is such that the columns of H_i^T span the null space of A_{i-1}^T, then

$$x = x_i + H_i^T q,$$

with arbitrary $q \in R^n$, forms the general solution of the first $i-1$ equations. That is, with

$$H_i A_{i-1} = 0,$$

the following result is obtained:

$$A_{i-1}^T x = b^{(i-1)}.$$

Now, since H_i^T is a spanning matrix for null (A_{i-1}^T), by assumption (one that is trivially valid for $i = 1$), then if it is assumed that

$$p_i = H_i^T z_i,$$

with arbitrary $z_i \in R^n$,Broyden's parameter, such that

$$a_i^T H_i^T z_i \neq 0,$$

then $A_{i-1}^T p_i = 0$ and

$$x(\alpha) = x_i - \alpha p_i,$$

for any scalar α, solves the first $i-1$ equations. We can set $\alpha = \alpha_i$ so that $x_{i+1} = x(\alpha_i)$ solves the ith equation as well. By assuming

$$\alpha_i = \frac{a_i^T x_i - b_i}{a_i^T p_i},$$

then

$$x_{i+1} = x_i - \alpha_i p_i$$

is a solution for the first i equations. Now, to complete the ABS step, H_i must be updated to H_{i+1} so that $H_{i+1} A_i = 0$. It is sufficient to let

$$H_{i+1} = H_i - u_i v_i^T \tag{3}$$

and to select u_i, v_i so that $H_{i+1} a_j = 0$, $j = 1, \ldots, i$. The updating formula (3) for H_i is a rank-one correction to H_i. The ABS methods of the unscaled or basic class define $u_i = H_i a_i$ and $v_i = H_i^T w_i / w_i^T H_i a_i$, where w_i, Abaffy's parameter, is an arbitrary vector satisfying

$$w_i^T H_i a_i \neq 0.$$

Thus, the updating formula can be written as below:

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}. \tag{4}$$

The matrix H_i is generally known as the *Abaffian*. ZHANG [13] has shown that every $n \times n$ matrix is an Abaffian matrix. At this point the general steps of an ABS algorithm are given [1, 2]. In the algorithm below, r_{i+1} denotes the rank of A_i and hence the rank of H_i equals $n - r_{i+1} + 1$.

The Basic ABS Algorithm for Solving General Linear Systems

(1) Choose $x_1 \in R^n$, arbitrary, and $H_1 \in R^{n,n}$, arbitrary and nonsingular. Let $i = 1$ and $r_i = 0$.
(2) Compute $t_i = a_i^T x_i - b_i$ and $s_i = H_i a_i$.
(3) *If* ($s_i = 0$ and $t_i = 0$) *then* let $x_{i+1} = x_i$, $H_{i+1} = H_i$, $r_{i+1} = r_i$ and *go to* step (7). *If* ($s_i = 0$ and $t_i \neq 0$) *then* Stop.

(4) Compute the search direction $p_i = H_i^T z_i$, where $z_i \in R^n$ is an arbitrary vector satisfying $z_i^T s_i = a_i^T p_i \neq 0$. Compute

$$\alpha_i = \frac{t_i}{a_i^T p_i}$$

and let

$$x_{i+1} = x_i - \alpha_i p_i.$$

(5) Update H_i to H_{i+1} by

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$$

where $w_i \in R^n$ is an arbitrary vector satisfying $w_i^T s_i \neq 0$.

(6) Let $r_{i+1} = r_i + 1$.

(7) *If* $i = m$ *then* Stop *else* let $i = i + 1$ and *go to* step (2).

Remark 2.1.

- If $s_i = 0$ and $t_i = 0$ then the ith equation is redundant.
- If $s_i = 0$ and $t_i \neq 0$ then the ith equation and hence the system is incompatible.
- If the system (2) is compatible then the general solution is given by

$$x = x_{m+1} + H_{m+1}^T q, \tag{5}$$

where $q \in R^n$ is arbitrary.

Below is listed certain properties of the ABS methods [2]. For simplicity, it is assumed that rank $(A_i) = i$.

- $H_i a_i \neq 0$ if and only if a_i is linearly independent of $a_1, \ldots, a_{i-1}$.
- Every row of H_i corresponding to a nonzero component of w_i is linearly dependent on other rows.
- The search directions $p_1, \ldots, p_i$ are linearly independent.
- The matrix

$$L_i = A_i^T P_i,$$

where $P_i = (p_1, \ldots, p_i)$, is a nonsingular lower triangular matrix.
- The set of directions $p_1, \ldots, p_i$ together with independent columns of H_{i+1}^T form a basis for R^n.
- The matrix $W_i = (w_1, \ldots, w_i)$ has full column rank and

$$\text{null}\,(H_{i+1}^T) = \text{range}\,(W_i), \qquad \text{null}\,(H_{i+1}) = \text{range}\,(A_i).$$

- If $s_i \neq 0$ then

$$\text{rank}\,(H_{i+1}) = \text{rank}\,(H_i) - 1.$$

- The updating formula H_i can be written as:

$$H_{i+1} = H_1 - H_1 A_i (W_i^T H_1 A_i)^{-1} W_i^T H_1, \tag{6}$$

where $W_i^T H_1 A_i$ is strongly nonsingular (the determinants of all of its forward principal submatrices are nonzero).

Remark 2.2. Using the second property, GU [4], SPEDICATO & ZHU [10] modified the ABS algorithm such that the Abaffian matrices H_i are rectangular with full row rank.

Huang Algorithm. Huang algorithm [5] is one of the most important algorithms in ABS class. It corresponds to

$$H_1 = I, \quad z_i = w_i = a_i$$

for parameters of ABS algorithm. Thus it can be shown that (see [2])

- Abaffian matrices H_i are symetric and projection to the orthogonal complement range space of the matrix A_{i-1}.
- The updating formula (6) can be written as

$$H_i = I - A_i A_i^+,$$

where A_i^+ is the Moore-Penrose pseudo inverse of the matrix A_i.
- Vectors $p_i,\ i = 1, \ldots, m$, are corresponding to vectors obtained by Gram-Schmidt orthogonalization process applied to rows of A.
- If x_1 is a multiplier of a_1 then, x_{m+1} is the least squares solution of (2).

In the next section, representation of the general solution of an inequality linear system using the modified ABS algorithm has been deduced.

3. Solving Certain Linear Inequalities System

Consider the inequality linear system

$$Ax \leq b, \tag{7}$$

where $A = (a_1, \ldots, a_m)^T \in R^{m,n}$, $b \in R^m$ and $m \leq n$. Suppose rank $(A) = m$. The aim of this paper is to determine the general solution of (7) using the ABS methods. In doing so, consider the linear system

$$Ax = y, \quad x \in R^n, \tag{8}$$

where $y = (y_1, \ldots, y_m)^T \in R^m$ is a parameter vector. Note that x is a solution to (7) if and only if it satisfies (8) for some $y \leq b$. Thus, the solutions of (7) can be obtained by solving (8) for all parameter vectors y such that $y \leq b$.

Now consider a parameter vector y such that $y \leq b$. The system (8) can be solved by using the ABS methods. For this task, take $x_1 \in R^n$ and $H_1 \in R^{n,n}$ arbitrary and nonsingular. From the ABS properties, x_{i+1}, for all i, $1 \leq i \leq m$, satisfies the first i equations of (8). Since, $y_j \leq b_j$, for $j \leq i$, then x_{i+1} satisfies the first i inequalities of (7) too. Therefore, x_{m+1} is a solution to inequalities (7). Note that x_{m+1} is a function of the parameter vector y, i.e. $x_{m+1} = x_{m+1}(y)$. Thus by varying y, $y \leq b$, an infinite number of solutions to (7) of the form $x_{m+1} = x_{m+1}(y)$ can be obtained. However, an explicit form of x_{m+1} as a function of the parameter vector y is not obvious.

At this juncture the intention is to obtain an explicit form for $x_{m+1} = x_{m+1}(y)$. Let $A_i = (a_1, \ldots, a_i)$ and $W_i = (w_1, \ldots, w_i)$. The matrix $(W_i^T H_1 A_i)^{-1} W_i^T H_1$ is called a W_i-left inverse of A_i and is denoted by $A_{W_i}^{-1}$. It is obvious from the ABS properties, that $Q_i = W_i^T H_1 A_i$ is strongly nonsingular and also that

$$H_{i+1} = H_1 - H_1 A_i (W_i^T H_1 A_i)^{-1} W_i^T H_1 = H_1 - H_1 A_i A_{W_i}^{-1}. \tag{9}$$

The proof to the following theorem can be found in [7].

Theorem. *Suppose that A is a $m \times n$ matrix with full row rank. For the system of equations (8) we can have*

(a) *The vector $A_{W_i}^{-T} y^{(i)}$, where $y^{(i)} = (y_1, \ldots, y_i)^T$, satisfies the first i equations of (8).*

(b) *If $w_j = z_j$, for all j, $1 \leq j \leq i$, then the general solution for the first i equations of (8) is given by*

$$x = A_{W_i}^{-T} y^{(i)} + \left(I - A_{W_i}^{-T} A_i^T\right) x_1 + H_{i+1}^T q, \tag{10}$$

where q is an arbitrary vector. ☑

If H_1 is an arbitrary nonsingular matrix, then from (9) we have that: $H_1^{-1} H_{i+1} = I - A_i A_{W_i}^{-1}$. Hence $A_{W_i}^{-T} y^{(i)} + \left(I - A_{W_i}^{-T} A_i^T\right) x_1 = A_{W_i}^{-T} y^{(i)} +$

$H_{i+1}^T H_1^{-T} x_1$. Therefore, by setting

$$x_{i+1} = A_{W_i}^{-T} y^{(i)} + H_{i+1}^T H_1^{-T} x_1,$$

and noticing that x_{i+1} is a solution of the first i equations of (8), then (10) is just another form of (5).

Remark 3.1. The matrix $A_{W_i}^{-T}$ can be computed by the recurrence

$$A_{W_1}^{-T} = H_1^T w_1 / a_1^T H_1^T w_1,$$

$$A_{W_i}^{-T} = \left[\left(I - H_i^T w_i a_i^T / a_i^T H_i^T w_i \right) A_{W_{i-1}}^{-T} \qquad H_i^T w_i / a_i^T H_i^T w_i \right]$$

or, if it is supposed that $w_i = z_i$, for all i, by the recurrence

$$A_{W_1}^{-T} = p_1 / a_1^T p_1,$$

$$A_{W_i}^{-T} = \left[\left(I - p_i a_i^T / a_i^T p_i \right) A_{W_{i-1}}^{-T} \qquad p_i / a_i^T p_i \right].$$

(For a proof see [7]). Thus, the explicit form of the particular solution $x_{i+1} = x_{i+1}\left(y^{(i)}\right)$ of the first i equations of (8) is

$$x_{i+1}\left(y^{(i)}\right) = A_{W_i}^{-T} y^{(i)} + H_{i+1}^T H_1^{-T} x_1,$$

where $y^{(i)} = (y_1, \ldots, y_i)^T$.

In this manner the following theorem is proved.

Theorem 3.1. *If we take $w_i = z_i$, for all i, in the ABS algorithm, then the general solution of the inequality system* (7) *is*

$$x = x_{m+1} + H_{m+1}^T q, \quad q \in R^{n-m}$$

where $y \in R^m,\ y \leq b$, is an arbitrary vector and

$$x_{m+1} = A_{W_m}^{-T} y + H_{m+1}^T H_1^{-T} x_1. \quad ☑$$

The vector $y \in R^m$ is called a feasible parameter vector for (7) if $y \leq b$. In what follows, a relation between the coefficient matrix A, the matrix L in the implicit transformation of A by the ABS algorithm, and the feasible parameter vector y is obtained.

From the ABS algorithm we have

$$x_{i+1} = x_i - \frac{a_i^T x_i - y_i}{\delta_i} p_i,$$

where $\delta_i = a_i^T p_i = s_i^T z_i \neq 0$. Since z_i is arbitrary, we can assume $\delta_i > 0$ without loss of generality. Let $\alpha_i = (a_i^T x_i - y_i)/\delta_i$. Then $y_i = a_i^T x_i - \alpha_i \delta_i$. But $y_i \leq b_i$, thus $\alpha_i \geq (a_i^T x_i - b_i)/\delta_i$. Therefore, y_i has the form $y_i = a_i^T x_i - \alpha_i \delta_i$, where $\delta_i = a_i^T p_i > 0$ and $\alpha_i \geq (a_i^T x_i - b_i)/\delta_i$ is arbitrary. Consequently, since

$$x_i = x_1 - \sum_{j=1}^{i-1} \alpha_j p_j$$

the following formula results

$$y_i = a_i^T x_i - \alpha_i \delta_i = a_i^T x_i - \alpha_i a_i^T p_i = a_i^T (x_i - \alpha_i p_i) = a_i^T \Big(x_1 - \sum_{j=1}^{i} \alpha_j p_j \Big).$$

Since for $j > i$, $a_i^T p_j = 0$, thus $y_i = a_i^T (x_1 - P\alpha)$ where $P = (p_1, \ldots, p_m)$ and $\alpha = (\alpha_1, \ldots, \alpha_m)^T$. Therefore, every feasible vector y can be written as

$$y = A(x_1 - P\alpha) = Ax_1 - L\alpha,$$

where $L = AP$ is a nonsingular lower triangular matrix and $\alpha \in R^m$ satisfies the inequality system

$$L\alpha \geq Ax_1 - b. \tag{11}$$

At this point it is proved that (11) is not only a necessary but is also a sufficient condition for y to be a feasible parameter vector.

Theorem 3.2. *Let $x_1 \in R^n$ be an arbitrary initial point for an ABS algorithm with $w_i = z_i$, for all i, and let α be a vector with components α_i as set in the i-th iteration of the ABS algorithm. The vector y is feasible for the inequality system* (7) *if and only if*

$$y = A(x_1 - P\alpha) = Ax_1 - L\alpha \tag{12}$$

and (11) *holds.*

Proof: it is proved above that if y is a feasible parameter vector then it has the form (12). Conversely, assume that for some x_1 we have:

$$y = Ax_1 - L\alpha, \quad L\alpha \geq Ax_1 - b.$$

Then

$$y = Ax_1 - L\alpha = Ax_1 - AP\alpha = A(x_1 - P\alpha),$$

implies that $x = x_1 - P\alpha$ satisfies the system (8). On the other hand,

$$y = Ax_1 - L\alpha \leq Ax_1 - Ax_1 + b = b.$$

Therefore, y is a feasible parameter vector for the inequalities (7). ☑

Remark 3.2.

(1) Every feasible parameter vector y from a feasible parameter vector y^* can be deduced. To show this, suppose that y^* is a feasible parameter vector ($y^* = Ax_1 - L\alpha^*$, where $L\alpha^* \geq Ax_1 - b$). If $y = y^* - L\beta$, where $L\beta \geq y^* - b$, then y is a feasible parameter vector since firstly, $b \geq y^* - L\beta = y$ and secondly, the system $Ax = y$ is compatible becuse $y^* \in \text{range}\,(A)$ by (12) and $L\beta = AP\beta \in \text{range}\,(A)$. Conversely, if y is a feasible parameter vector, then

$$y = Ax_1 - L\alpha = y^* + L\alpha^* - L\alpha = y^* - L(\alpha - \alpha^*) = y^* - L\beta,$$

for $\beta = \alpha - \alpha^*$, and

$$L\beta = L\alpha - L\alpha^* \geq Ax_1 - b - L\alpha^* = y^* - b.$$

(2) If $b \geq 0$, then $y^* = 0$ is a feasible parameter vector. In this case, the feasible parameter vectors y are in the form of $y = L\beta$, where $L\beta \leq b$.

Now, let $H = H_{m+1}$. By Theorem 3.2 and the property

$$\text{null}\,(A) = \text{range}\,(H^T)\,,$$

the following theorem can be proved.

Theorem 3.3. Let $A_{W_m}^{-T}$ be the right W_m-inverse of A obtained from the application of an ABS algorithm with $w_i = z_i$ for all i, to A and H be the resulted Abaffian. Then

$$\{x \in R^n \mid Ax \leq b\} = \{A_{W_m}^{-T} b - A_{W_m}^{-T} u - H^T q \mid q \in R^{n-m},\ u \in R_+^m\}. \quad (13)$$

Proof: In the first place it is shown that

$$\{x \in R^n \mid Ax \leq b\} = \{x_1 - P\alpha - H^T q \mid q \in R^{n-m},\ \alpha \in R^m,\ L\alpha \geq Ax_1 - b\}.$$

To continue, suppose $x \in R^n$ has the form $x = x_1 - P\alpha - H^T q$, where $q \in R^{n-m}$ and $\alpha \in R^m$ is such that $L\alpha \geq Ax_1 - b$. Then $Ax = Ax_1 - L\alpha \leq b$.

Conversely, assume x^* satisfies $Ax \leq b$. Let $y^* = Ax^*$. Thus the system $Ax = y^*$ is compatible. Hence, by using the ABS algorithm, it may be written $x^* = x_1 - P\alpha^* - H^T q^*$, for some $\alpha^* \in R^m$ and $q^* \in R^{n-m}$. Thus, $y^* = Ax^* = Ax_1 - L\alpha^*$. Since $y^* \leq b$, then α^* satisfies $Ax_1 - L\alpha^* \leq b$.

Now, let $w_i = z_i$, for all i. Then

$$\{x \in R^n \mid Ax \le b\}$$
$$= \{x_1 - P\alpha - H^T q \mid q \in R^{n-m},\ \alpha \in R^m,\ L\alpha \ge Ax_1 - b\}$$
$$= \{x_1 - P\alpha - H^T q \mid q \in R^{n-m},\ \alpha \in R^m,\ L\alpha = AP\alpha = Ax_1 - b + u,\ u \in R^m_+\}$$
$$= \{x_1 - P\alpha - H^T q \mid q \in R^{n-m},\ \alpha \in R^m,\ P\alpha = A^{-T}_{W_m}(Ax_1 - b + u) + (I - A^{-T}_{W_m}A)x_1 + H^T q,\ u \in R^m_+\}$$
$$= \{x_1 - A^{-T}_{W_m}(Ax_1 - b + u) - (I - A^{-T}_{W_m}A)x_1 - H^T q \mid q \in R^{n-m},\ u \in R^m_+\}$$
$$= \{A^{-T}_{W_m}b - A^{-T}_{W_m}u - H^T q \mid q \in R^{n-m},\ u \in R^m_+\}.$$

☑

4. Least Squares Solution for Linear Inequalities System

In this section, the results obtained previously are applied for finding the least squares point of a polyhedral set as (1):

$$\min_{Ax \le b} \| x \|_2 . \tag{14}$$

Consider Huang algorithm on ABS class in which, $H_1 = I$ and $W_m = A^T$. Therefore,

$$A^{-1}_{W_m} = \left(AA^T\right)^{-1} A = \left(A^T\right)^+ = (A^+)^T$$

and

$$H = I - A^T \left(A^T\right)^+ = I - \left(A^+A\right)^T = I - A^+A = P_{N(A)},$$

where, A^+ is the Moore-Penrose pseudo-inverse of A and $P_{N(A)}$ the matrix projection into its null space. Then, from (13) it is implied that

$$\{x \in R^n \mid Ax \le b\} = \{A^+b - A^+u - (I - A^+A)q \mid q \in R^{n-m},\ u \in R^m_+\}.$$

It has been shown below that (14) is equivalent to the nonnegative least squares problem

$$\min_{u \ge 0} \ \| \bar{x} - A^+u \|_2, \tag{15}$$

where, $\bar{x} = A^+b$. To do so, let $y(u) = \bar{x} - A^+u$. Since the Abaffian matrix H is symmetric, hence the optimal value of the least squares problem

$$\min_{q} \ \| y(u) - H^T q \|_2 \tag{16}$$

will be $\| (I - H^+H)\, y(u) \|_2$. On the other hand, $P_{N(H)} = I - H^+H$ is the projection matrix into the null space of H. By the ABS properties, null$\,(H) =$ range$\,(A^T)$ is obtained. Therefore, $P_{N(H)} = P_{R(A^T)} = A^+A$ and the optimal value of (16) is $\| A^+Ay(u) \|_2$.

Hence the following results are obtained:

$$\begin{aligned}\min_{Ax\le b} \ \| x \|_2 &= \min_{u\ge 0} \ \min_{q} \ \| \bar{x} - A^+u - H^Tq \|_2 \\ &= \min_{u\ge 0} \ \| A^+A(\bar{x} - A^+u) \|_2 \\ &= \min_{u\ge 0} \ \| A^+AA^+(b-u) \|_2 \\ &= \min_{u\ge 0} \ \| A^+(b-u) \|_2 \\ &= \min_{u\ge 0} \ \| \bar{x} - A^+u \|_2 \ .\end{aligned}$$

The following theorem is thus proved.

Theorem 4.1. *If A is a $m \times n$ matrix with full row rank then by using Huang algorithm in the ABS class, we have*

$$\min_{u\ge 0,q} \ \| \bar{x} - A^+u - H^Tq \|_2 = \min_{Ax\le b} \ \| x \|_2 = \min_{u\ge 0} \ \| \bar{x} - A^+u \|_2 \ .$$

☑

From the above theorem, solving (14), with n variables, is equivalent to solving the nonnegative least squares problem (15), with m variables. The following algorithm may thus be used for solving (15) [6].

(1) set $N = \phi$, $M = \{1, \ldots, m\}$, and $u = 0$.
(2) compute the m-vector $w = A^{+T}(\bar{x} - A^+u)$.
(3) If the set M is empty or $w_j \le 0$ for all $j \in M$, go to Step (12).
(4) Find such an index $t \in M$ that $w_t = \max_{j\in M} \{w_j\}$.
(5) Move the index t from set M to set N.
(6) Let B denote the $n \times m$ matrix defined by $Be_j = 0$ for $j \in M$, and $Be_j = A^+e_j$ for $j \in N$. Compute the m-vector z as a solution of the least squares problem $Bz = \bar{x}$. Define $z_j = 0$ for $j \in M$.
(7) If $z_j > 0$ for all $j \in N$, set $u = z$ and go to Step (2).

(8) Find such an index $r \in N$ that

$$\frac{u_r}{u_r - z_r} = \min_{j \in N} \left\{ \frac{u_j}{u_j - z_j} \mid z_j \le 0 \right\}.$$

(9) Set $\alpha = u_r/(u_r - z_r)$.
(10) Set $u = u + \alpha(z - u)$.
(11) Move from set N to set M all indices $j \in N$ for which $u_j = 0$. Go to Step (6).
(12) *Comment*: The computation is completed.

It is proved that the above algorithm is finite and the number of iterations is typically $O(m/2)$.

5. ABS Solution of a Certain Linear Programming Problem

Now consider the linear programming problem

$$\max \ c^T x \quad : \quad Ax \le b \tag{17}$$

where $A \in R^{m,n}$, $m \le n$, rank$\,(A) = m$, and $b \in R^m$. For simplicity, take $M = A_{W_m}^{-T}$ and $x^* = Mb$. From Theorem 3.3, the above problem is equivalent to

$$\min \ (c^T M u + c^T H^T q) \quad : \quad u \in R_+^m, \ q \in R^{n-m}. \tag{18}$$

Now, let $\bar{c} = M^T c$ and

$$\begin{aligned} I^+ &= \{i \mid \bar{c}_i > 0\}, \\ I^- &= \{i \mid \bar{c}_i < 0\}, \\ I^0 &= \{i \mid \bar{c}_i = 0\}. \end{aligned} \tag{19}$$

Theorem 5.1. *Let the ABS algorithm be applied to A as in Theorem 3.3. Then*

(a) *If $Hc \ne 0$, then problem (17) is unbounded with no solution.*
(b) *If $Hc = 0$ and $I^- \ne \phi$, then problem (17) is unbounded with no solution.*
(c) *If $Hc = 0$ and $I^- = \phi$, then there are an infinite number of optimal solutions for (17) in the form of*

$$x = x^* - H^T q - \sum_{j \in I^0} u_j M e_j,$$

where $q \in R^{n-m}$ and $u_j \geq 0$, $j \in I^0$, are arbitrary and e_j is the j-th unit vector in R^m.

Proof:

(a) Considering (18), as Hc is not a zero vector, it is obvious that q can be set appropriately to obtain an arbitrarily small value for the objective function.

Using the notation of (19), problem (17) can be rewritten as

$$\min\ z = \sum_{j\in I^-} \bar{c}_j u_j + \sum_{j\in I^+} \bar{c}_j u_j + \sum_{j\in I^0} 0u_j \quad : \quad u_j \geq 0 \quad \forall j.$$

(b) Since $I^- \neq \phi$ then $k \in I^-$ exists. Letting $u = te_k$, then $z \to -\infty$, when $t \to \infty$. Hence (17) is unbounded and has no solution.

(c) In this case, $z = \sum_{j\in I^+} \bar{c}_j u_j + \sum_{j\in I^0} 0u_j$. The minimizers of z are written as $u = \sum_{j\in I^0} u_j e_j$, where $u_j \geq 0$, $j \in I^0$, is arbitrary. Thus, the optimal solutions to (17) have the form

$$x = x^* - H^T q - \sum_{j\in I^0} u_j M e_j,$$

where $q \in R^{n-m}$ and $u_j \geq 0$, $j \in I^0$, are arbitrary. ☑

Remark 5.1. According to the ABS properties, $HA^T = 0$ and hence range$\,(A^T) =$ null$\,(H)$. The condition $Hc = 0$ is then equivalent to the Kuhn-Tucker condition $c = A^T\lambda$, for some λ. Since A^T has full column rank, λ is unique. On the other hand, the vector $\bar{c}$ satisfies the system $A^T\lambda = c$, because, the rows of $A_{W_m}^{-1}$ being linearly independent, the solution of $A^T\lambda = c$ is equivalent to the solution of

$$A_{W_m}^{-1} A^T \lambda = A_{W_m}^{-1} c$$

and hence $\lambda = A_{W_m}^{-1} c = \bar{c}$. Thus, $\bar{c}$ has the same sign as the Lagrange multipliers, component-wise. If $\bar{c} \ngeq 0$ then the problem is unbounded and has no solution. In this case, it can be realized that $\lambda_i < 0$, for all $i \in I^-$, and $\lambda_i \geq 0$, for all $i \notin I^-$. If $\bar{c} \geq 0$ then the problem has infinitely many optimal solutions. Here, as it is expected for optimality, $\lambda_i \geq 0$, for all i.

Using the above theorem, the following algorithm is suggested for solving the linear programming problem (17).

An ABS Algorithm for Solving the Linear Programming Problem (17):

(1) Applying an ABS algorithm to the coefficient matrix A and computing $M = A_{W_m}^{-T}$ and $H = H_{m+1}$.

(2) *If* $Hc \neq 0$ *then* Stop (the problem is unbounded and hence has no solution).

(3) Let $\bar{c} = M^T c$ and form the following sets

$$I^+ = \{i \mid \bar{c}_i > 0\},$$

$$I^- = \{i \mid \bar{c}_i < 0\},$$

$$I^0 = \{i \mid \bar{c}_i = 0\}.$$

(4) *If* $I^- \neq \phi$ *then* Stop (the problem is unbounded and hence has no solution).

(5) $(I^- = \phi)$ Compute

$$x^* = Mb.$$

The optimal solutions have the form

$$x = x^* - H^T q - \sum_{j \in I^0} u_j M e_j,$$

where $q \in R^{n-m}$, $u_j \geq 0$, $j \in I^0$, are arbitrary numbers and e_j denotes the j-th unit vector in R^m. Stop.

Conclusion

In this paper a representation theorem on the basis of ABS methods has been implemented to obtain the points of a polyhedral set. By making use of the above theorem it has been shown that the problem of finding the least squares points in the polyhedral set is equivalent to a nonnegative least squares problem. Similarly, by applying the above theorem certain typical LP problems have been considered and dwelt upon in detail to obtain optimality conditions and an explicit representation of all solutions.

References

[1] J. ABAFFY, C.G̃. BROYDEN & E. SPEDICATO, *A class of direct methods for linear equations*, Numer. Math. **45**,(1984), 361–376.

[2] J. ABAFFY & E. SPEDICATO, *ABS Projection Algorithms: Mathematical Techniques for Linear and Nonlinear Equations*, Ellis Horwood, Chichester, 1989.

[3] D. GOLDFARB & A. IDNANI, *A numerically stable dual method for solving strictly convex quadratic programming*, Math.Prog. **27**,(1983), 1–33.
[4] S. GU, *The simplified ABS algorithm*, Report 1/98, University of Bergamo, 1998.
[5] H. Y. HUANG, *A direct method for the general solution of a system of linear equations*, JOTA 16, 1975, 429–445.
[6] C. L. LOWSON & R. J. HANSON, *Solving Least Squares Problems*, Prentice-Hall, Inc., 1974.
[7] S. LI, *On a variational characterization of the ABS algorithms*, Communication at the Second International Conference on ABS Algorithms, Beijing, June 1995.
[8] G. SHI, *An ABS algorithm for generating nonnegative solutions of linear systems*, Proceedings of the First International Conference on ABS Algorithms, Luoyang, September 1991, E. SPEDICATO (editor), University of Bergamo, 54–57.
[9] E. SPEDICATO & J. ABAFFY, *On the use of the ABS algorithms for some linear programming problems*, Preprint, University of Bergamo, 1987.
[10] E. SPEDICATO & M. ZHU, *Reformulation of the ABS algorithm via full rank Abaffian*, Proceedings EAMA Conference, Seville, 396–403, 1997.
[11] L. ZHANG, *A method for finding a feasible point of inequalities*, Proceedings of the First International Conference on ABS Algorithms, Luoyang, September 1991, E. SPEDICATO (editor), University of Bergamo, 131–137.
[12] L. ZHANG, *An algorithm for the least Euclidean norm solution of a linear system of inequalities via the Huang ABS algorithm and the Goldfarb-Idnani strategy*, Report 2/95, University of Bergamo, 1995.
[13] L. ZHANG, *The ABS Algorithm with singular initial matrix and its application to linear programming*, Report 9/95, University of Bergamo, 1995.
[14] L. ZHANG, X. XIA AND E. FENG, *Introduction to ABS methods for optimization*, Dalian Technology Univeersity Press, Dalian, 1998 (in Chinese).
[15] J. ZHAO, *ABS algorithms for solving linear inequalities*, Report 21/91, University of Bergamo, 1991.

(Recibido en octubre de 2004. Aceptado para publicación en junio de 2005)

DEPARTMENT OF MATHEMATICS
BU–ALI SINA UNIVERSITY
HAMEDAN, IRAN
e–mail: Esmaeili@basu.ac.ir