Sequentially $m$-Barrelled Algebras

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1. Introduction

The concept of $m$-barrelled algebra was introduced in [5]. Using sequential convergence, we introduce, in this paper, sequentially $m$-barrelled algebras in the same fashion as $s$-barrelled spaces were introduced in [8].

An analogue of the Banach-Steinhaus theorem is proved. As an application, we obtain an interesting result in orthogonal bases, which is the analogue of the isomorphism theorem.

An algebra which is also a locally convex space is called a locally convex algebra if the multiplication in it is jointly continuous. A subset $S$ of an algebra is called $m$-convex if it is convex and idempotent (i.e. $SS \subseteq S$).

A locally convex algebra $E$ is called a locally $m$-convex algebra if it has a neighbourhood basis of 0 consisting of closed, circled and $m$-convex sets [7]. A locally convex algebra $E$ is called an $m$-barrelled algebra if every $m$-barrel (closed, circled, $m$-convex and absorbing set) is a neighbourhood of 0 in $E$ [5].

A locally convex space is called a barrelled space (sequentially barrelled space) if every barrel, i.e. closed, circled, convex, absorbing set, is a neighbourhood of 0 (an $S$-barrel, i.e. sequentially closed, circled, convex, absorbing set, is a sequential neighbourhood of 0 [8]). A mapping $T : E \to F$ ($E$ and $F$ are algebras) is called multiplicative if $T(xy) = T(x)T(y)$. A set $V$ in a topological vector space $X$ is called a sequential neighbourhood of 0 if every sequence in $X$ converging to 0 belongs to $V$ eventually.

A sequence $\{x_i\}$ in a locally convex space $E$ is called a topological basis (or, basis) for $E$ if for each $x$ in $E$, there is a unique sequence $\{\alpha_i\}$ in $K$ such
that

\[ x = \lim_{n} \sum_{i=1}^{n} \alpha_{i} x_{i} \]

in the topology of \( E \) [6]. Each \( \alpha_{i} \), called expansion coefficient, defined by \( \lambda_{i}(x) = \alpha_{i} \), defines a linear functional \( \lambda_{i} \) on \( E \). If each \( \lambda_{i} \) is continuous (sequentially continuous) then \( \{x_{i}\} \) is called a Schauder basis (S-Schauder basis [4]).

Let \( E \) and \( F \) be locally convex spaces. A sequence \( \{x_{i}\} \) in \( E \) is similar to a sequence \( \{y_{i}\} \) in \( F \) if for all sequences \( \{a_{i}\} \subset \mathbb{K} \), \( \sum_{i=1}^{\infty} a_{i} x_{i} \) converges (in \( E \)) iff \( \sum_{i=1}^{\infty} a_{i} y_{i} \) converges (in \( F \)) [2].

A mapping \( T : E \to F \) is called sequential topological isomorphism if it is linear, one-one, onto, sequentially continuous and \( T^{-1} \) is sequentially continuous.

A basis \( \{x_{i}\} \) in a locally convex algebra \( E \) is called orthogonal if \( x_{i} x_{j} = 0 \) for \( i \neq j \) and \( x_{i}^{2} = x_{i} [1] \). In a Hausdorff locally convex algebra (or even in Hausdorff Topological algebra) an orthogonal basis is a Schauder basis [1]. We always consider vector spaces over the field of complex numbers.

2. Sequentially m-barrelled algebras

In this section we introduce the concept of sequentially m-barrelled algebra with two examples and obtain some results.

Definitions 2.1. (a) Let \( E \) be a locally convex algebra. If a subset \( A \) is an S-barrel and idempotent, then it is called a sequentially m-barrel.

(b) If every sequentially m-barrel in \( E \) is a sequential neighbourhood of 0, then \( E \) is called a sequentially m-barrelled algebra.

Remarks 2.2. (a) Every m-barrel is a sequentially m-barrel.

(b) In a metrizable locally convex algebra, the concepts of m-barrelled algebra and sequentially m-barrelled algebra coincide.

Example 2.3. Let \( C(I) \) be the Banach algebra of all continuous functions on \( I = [0, 1] \) with the norm

\[ \|f\| = \sup_{t \in I} \{|f(t)|\} , \quad f \in C(I) \]
Let $E$ be the vector subspace of $C(I)$, consisting of all elements $f \in C(I)$ which vanish in a neighbourhood (depending on $f$) of $t = 0$. Let

$$B = \{ f \in E : |f(1/n)| \leq 1/n \text{ for all } n \in \mathbb{N} \}.$$ 

Then $B$ is a sequentially m-barrel in $E$. But $B$ is not a sequential neighbourhood of 0 in $E$ [3]. Hence $E$ is not a sequentially m-barrelled algebra. However $C(I)$, being a Banach algebra, is sequentially m-barrelled algebra. Since $E$ is an ideal in $C(I)$, it follows that an ideal of a sequentially m-barrelled algebra need not be of the same sort.

**Example 2.4.** If $E$ is an algebra, the family of all circled, convex, absorbing and idempotent sets is a basis of neighbourhoods of 0 for a locally m-convex topology on $E$ which is the strongest locally m-convex topology on $E$. Now let $E$ be the subalgebra of $K[x]$ of all polynomials without constant term. If $\alpha$ is a positive real number, let $V(\alpha)$ be the circled convex envelope of $\{\alpha^m x^m : m \in \mathbb{N}\}$. The family $\{V(\alpha)\}$, with $\alpha$ rational and less than one, is a basis of neighbourhoods of 0 for the strongest locally m-convex topology on $E$. This topology is metrizable. Now, $E$, with this topology, is a sequentially m-barrelled algebra which is not S-barrelled, since it is metrizable but not barrelled [9].

**Open Problem 2.5.** Is there a sequentially m-barrelled algebra which is not m-barrelled?

**Proposition 2.6.** Let $E$ be a sequentially m-barrelled algebra and $F$ a locally m-convex algebra. If $f$ is a multiplicative linear mapping of $E$ into $F$, then $f$ is almost sequentially continuous.

*Proof. Let $V$ be a circled m-convex neighbourhood of 0 in $F$. Then $f^{-1}(V)^S$, the smallest sequentially closed set containing $f^{-1}(V)$, is a sequential m-barrel in $E$ and hence a sequential neighbourhood of 0 in $E$. This proves that $f$ is almost sequentially continuous.*

**Proposition 2.7.** Let $E$ be a sequentially m-barrelled algebra and $F$ a locally convex algebra. If $f$ is a sequentially continuous and almost sequentially open, multiplicative, linear mapping of $E$ into $F$, then $F$ is sequentially m-barrelled.
Proof. Let $B$ be sequential $m$-barrel in $F$. Then $f^{-1}(B)$ is a sequential $m$-barrel in $E$ and hence a sequential neighbourhood of 0 in $E$. Since $f$ is almost sequentially open, it follows that $f\{f^{-1}(B)\}^S$ is a sequential neighbourhood of 0 in $F$. But 
$$f\{f^{-1}(B)\}^S \subseteq B^S = B$$
so that $B$ is a sequential neighbourhood of 0 in $F$. Hence $F$ is a sequentially
$m$-barrelled algebra.

3. Main results

In this section, we obtain an analogue of Banach-Steinhaus theorem for
sets of multiplicative linear mappings on sequentially $m$-barrelled algebras
and we use it to prove an analogue of the isomorphism theorem by using the
orthogonal basis.

Let $E$ and $F$ be locally convex spaces. Then a set $H$ of linear mappings
from $E$ to $F$ is called equi-sequentially continuous if for each neighbourhood $V$ of 0 in $F$, $\cap_{f \in H} f^{-1}(V)$ is a sequential neighbourhood of 0 in $E$.

**Theorem 3.1.** Let $E$ be a sequentially $m$-barrelled algebra and $F$ any
locally $m$-convex algebra. If $H$ is a simply bounded set of sequentially continuous multiplicative linear mappings, then $H$ is equi-sequentially continuous.

**Proof.** Let $V$ be a closed, circled and $m$-convex neighbourhood of 0 in $F$. Then $\cap_{f \in H} f^{-1}(V)$ is a sequentially $m$-barrel in $E$ and hence a sequential neighbourhood of 0 in $E$. Thus $H$ is equi-sequentially continuous. ■

**Corollary 3.2.** Let $E$ and $F$ be as in 3.1. Suppose $\{f_n\}$ is a pointwise bounded sequence of sequentially continuous multiplicative linear mappings from $E$ to $F$. Then $\{f_n\}$ is equi-sequentially continuous.

**Corollary 3.3.** Let $E$ and $F$ be as in 3.1. If $\{f_n\}$ is a sequence of sequentially continuous multiplicative linear mappings from $E$ to $F$ such that it converges pointwise to a mapping $f : E \to F$, then $f$ is linear, multiplicative and sequentially continuous.

As an application of 3.3, we have the following analogue of the isomorphism theorem.
Theorem 3.4. Let \( E \) and \( F \) be sequentially \( m \)-barrelled algebras. Suppose \( \{x_i, \lambda_i\} \) and \( \{y_i, \mu_i\} \) be orthogonal S-Schauder bases in \( E \) and \( F \) respectively. Then \( \{x_i, \lambda_i\} \) is similar to \( \{y_i, \mu_i\} \) if and only if there exists a multiplicative sequentially topological isomorphism \( T : E \to F \) such that \( T(x_i) = y_i \) for all \( i \in \mathbb{N} \).

Proof. If such a \( T \) exists, then for all sequences \( \{a_i\} \subset C, \sum_{i=1}^{\infty} a_i x_i \) converges (in \( E \)) iff 

\[
T \left( \sum_{i=1}^{\infty} a_i x_i \right) = \sum_{i=1}^{\infty} a_i T(x_i) = \sum_{i=1}^{\infty} a_i y_i
\]

converges (in \( F \)). Hence we get similarity. Conversely, we assume that the bases are similar. For each \( x \in E \), \( x = \sum_{i=1}^{\infty} \lambda_i(x)x_i. \)

We define \( T_n \) by 

\[
T_n(x) = \sum_{i=1}^{n} \lambda_i(x)x_i, \quad n \in \mathbb{N},
\]

and \( T \) by 

\[
T(x) = \sum_{i=1}^{\infty} \lambda_i(x)x_i;
\]

\( T \) is well-defined, one-one, onto, each \( T_n \) is sequentially continuous, linear, multiplicative, and \( \{T_n\} \) converges pointwise to \( T \). Hence, by 3.3, \( T \) is sequentially continuous, linear and multiplicative. Similarly \( T^{-1} \) is sequentially continuous. Hence \( T \) is multiplicative sequentially topological isomorphism.

Corollary 3.5. Suppose \( E \) and \( F \) in 3.4 are Hausdorff, and \( \{x_i, \lambda_i\} \) and \( \{y_i, \mu_i\} \) are orthogonal bases in \( E \) and \( F \) respectively. Then the result of 3.4 follows.

Proof. Since \( E \) and \( F \) are Hausdorff, \( \{x_i, \lambda_i\} \) and \( \{y_i, \mu_i\} \) are Schauder bases [1] and hence S-Schauder bases.

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References


