On the Regular Sturm-Liouville Transform

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1. Introduction

The Whittaker-Shannon-Kotel’nikov Sampling Theorem, hereafter WSK Theorem, states that any function \( f \in L^2(\mathbb{R}) \), bandlimited to \([-\pi, \pi]\), i.e. such that the support of its Fourier transform is contained in \([-\pi, \pi]\) (equivalently \( f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega \), where \( \hat{f} \) stands for the Fourier transform of \( f \) ) may be reconstructed from its samples \( \{f(n)\}_{n \in \mathbb{Z}} \) on the integers as

\[
    f(z) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(z - n),
\]

where \( \text{sinc} \) denotes the cardinal sine \( \text{sinc}(z) = \sin \pi z / \pi z \) \([4, 10, 13]\). The choice of the interval \([-\pi, \pi]\) is arbitrary. The same result applies to any compact interval \([-\pi\sigma, \pi\sigma]\) taking the samples in \( \{n/\sigma\} \) and replacing \( \pi \) with \( \pi/\sigma \) in the cardinal sines.

This theorem and its numerous offspring have been proved in many different ways, e.g. using Fourier expansions, the Poisson summation formula, contour integrals, etc. (see, for instance, [4]). But the most elegant proof is probably the one due to Hardy, using that the inverse Fourier transform \( \mathcal{F}^{-1} \) is an isometry from \( L^2[\mathbb{R}] \) onto the Paley-Wiener space \( PW_\pi = \{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \text{supp} \hat{f} \subseteq [-\pi, \pi]\} \). Any value \( f(t_n) \) of \( f \) is the inner product in \( L^2[-\pi, \pi] \) of \( \hat{f} \) and the complex exponential \( e^{-it_n\omega} \). Furthermore, the classical Paley-Wiener Theorem shows that \( PW_\pi \) coincides with the space of entire functions of exponential type at most \( \pi \) whose restriction to the real axis is
square integrable, i.e.

\[ PW_\pi = \{ f \in \mathcal{H}(\mathbb{C}) : |f(z)| \leq Ae^{\pi|z|}, f|_\mathbb{R} \in L^2(\mathbb{R}) \}. \]

The Paley-Wiener space \( PW_\pi \), inverse image space of \( L^2[-\pi, \pi] \) through \( \mathcal{F}^{-1} \), is a reproducing kernel Hilbert space, hereafter RKHS, whose reproducing kernel is the function \( k(z, \omega) = \text{sinc} (z - \overline{\omega}) \), i.e.

\[ f(\omega) = \langle f(z), \text{sinc} (z - \overline{\omega}) \rangle_{PW_\pi}, \quad f \in PW_\pi. \]

The key point in Hardy’s proof is that an expansion converging in \( L^2[-\pi, \pi] \) is transformed by \( \mathcal{F}^{-1} \) into another expansion which converges in the topology of \( PW_\pi \). This implies, in particular, that it converges uniformly on compact sets of the complex plane (to be precise, it converges on horizontal strips of \( \mathbb{C} \)) [10]. Choosing the first expansion in such a way that the coefficients are samples of \( f \) or of some function related to \( f \) (its derivatives, its Hilbert transform, etc.) provides different sampling theorems for functions in \( PW_\pi \). This Fourier duality technique can also be applied to the multidimensional case, or to the so-called multi-band case of functions whose Fourier transform has support on the union of a finite number of disjoint sets of finite Lebesgue measure (see [4] for more details).

One direction in which the WSK Theorem has been generalized is replacing the kernel function, \( e^{i\lambda \omega} \), by a more general kernel \( K(\omega, \lambda) \) leading to the following generalization by Kramer [1, 5]: Let \( K(\omega, \lambda) \) be a function, continuous in \( \lambda \) such that, as a function of \( \omega \), \( K(\omega, \lambda) \in L^2(I) \) for every real number \( \lambda \), where \( I \) is an interval on the real line. Assume that there exists a sequence of real numbers \( \{ \lambda_n \}_{n \in \mathbb{Z}} \) such that \( \{K(\omega, \lambda_n)\}_{n \in \mathbb{Z}} \) is a complete orthogonal sequence of functions of \( L^2(I) \). Then for any \( f \) of the form

\[ f(\lambda) = \int_I F(\omega)K(\omega, \lambda) \, d\omega, \]

where \( F \in L^2(I) \), we have

\[ f(\lambda) = \sum_{n=-\infty}^{\infty} f(\lambda_n)S_n(\lambda), \]

with

\[ S_n(\lambda) = \frac{\int_I K(\omega, \lambda) \overline{K(\omega, \lambda_n)} \, d\omega}{\int_I |K(\omega, \lambda_n)|^2 \, d\omega}. \]
The series (2) converges uniformly wherever \( \| K(\cdot, \lambda) \|_{L^2(I)} \) is bounded.

In particular, if \( I = [-\pi, \pi] \), \( K(\omega, \lambda) = e^{i\lambda \omega} \) and \( \{\lambda_n = n\}_{n \in \mathbb{Z}} \), we get the WSK sampling theorem.

One way to generate kernels \( K(\omega, \lambda) \) and sampling points \( \{\lambda_n\}_{n \in \mathbb{Z}} \) is to consider Sturm-Liouville boundary-value problems \([3, 11, 12]\). The kernel will be the function \( \phi(x, \lambda) \) which generates the eigenfunctions of the problem taking \( \lambda = \lambda_n \), \( n \in \mathbb{N} \). Thus, we obtain the so-called Sturm-Liouville type transform, term first coined in \([14]\).

The aim of this paper is twofold: firstly, to apply the integral transform theory which appears in \([8]\) characterizing the space of output functions from a linear integral transform as a RKHS, and secondly, to obtain a Fourier-type duality to be used in order to obtain the sampling theorem associated with the regular Sturm-Liouville transform. For sampling theorems in the framework of the RKHS see \([7]\).

## 2. Preliminaries

Consider the regular Sturm-Liouville problem

\[
\begin{align*}
-yy'' + q(x)y &= \lambda y, \quad x \in [a, b], \quad q \in C[a, b] \\
y(a) \cos \alpha + y'(a) \sin \alpha &= 0, \\
y(b) \cos \beta + y'(b) \sin \beta &= 0.
\end{align*}
\]

The problem (1)--(3) defines a self-adjoint operator \([2, p. 141]\) with discrete spectrum \([9]\). The eigenvalues \( \{\lambda_n\}_{n=0}^\infty \) are real, and following \([9, pp. 12 and ff.], \) simple and bounded from below. Furthermore, the associate eigenfunctions form an orthogonal basis of \( L^2(a, b) \).

Let \( \phi(x, \lambda) \) and \( \xi(x, t) \) be the solutions of (1) verifying

\[
\begin{align*}
\phi(a, \lambda) &= \sin \alpha, \quad \phi'(a, \lambda) = -\cos \alpha, \\
\xi(b, \lambda) &= \sin \beta, \quad \xi'(b, \lambda) = -\cos \beta.
\end{align*}
\]

The function \( \phi(x, \lambda) \) verifies the boundary condition (2) for all \( \lambda \), and consequently, \( \lambda_n \) will be an eigenvalue if and only if \( \phi(x, \lambda_n) \) fulfills the boundary condition (3). Therefore, \( \{\phi(x, \lambda_n)\}_{n=0}^\infty \) will be the eigenfunctions of the problem (1)--(3).

The wronskian \( W \) of \( \phi \) and \( \xi \) is defined as

\[
W(\phi(\cdot, \lambda), \xi(\cdot, \lambda)) = \begin{vmatrix} \phi(x, \lambda) & \xi(x, \lambda) \\ \phi'(x, \lambda) & \xi'(x, \lambda) \end{vmatrix}.
\]
The following result may be found in [9, pp. 7-11 and 19]

**Lemma 2.1.** \( W(\lambda) = W(\phi(\cdot, \lambda), \xi(\cdot, \lambda)) \) is independent of \( x \in [a, b] \); it is an entire function of order \( 1/2 \), its zeros are real, simple and located at, and only at, the eigenvalues \( \{\lambda_n\}_{n=0}^\infty \). When \( k \to \infty \) we have

\[
\sqrt{\lambda_k} = \frac{k\pi}{b-a} + O\left(\frac{1}{k}\right).
\]

We also have

\[
W(\lambda) = -\cos \beta \phi(b, \lambda) - \sin \beta \phi'(b, \lambda).
\]  

Since \( W(\lambda) \) is an entire function of order \( 1/2 \) with simple zeros in \( \{\lambda_n\}_{n=0}^\infty \), Hadamard’s Factorization Theorem [10] asserts that

\[
W(\lambda) = CP(\lambda)
\]

where \( C \in \mathbb{C} \) and

\[
P(\lambda) = \prod_{n=0}^\infty \left( 1 - \frac{\lambda}{\lambda_n} \right), \quad \text{if} \; 0 \notin \{\lambda_n\}_{n=0}^\infty
\]

\[
P(\lambda) = \lambda \prod_{n=1}^\infty \left( 1 - \frac{\lambda}{\lambda_n} \right), \quad \text{if} \; \lambda_0 = 0.
\]

The function \( \phi(x, \lambda) \) fulfills all the requirements in Kramer’s Theorem. Therefore, the function \( F(\lambda) = \langle f, \phi(\cdot, \lambda) \rangle_{L^2(a,b)} \), with \( f \in L^2(a,b) \), can be recovered through its samples in the eigenvalues of (1)-(3)

\[
F(\lambda) = \sum_{n=0}^\infty F(\lambda_n) S_n(\lambda),
\]

where

\[
S_n(\lambda) = \alpha_n^{-2} \int_a^b \frac{\phi(x, \lambda_n)}{\phi(x, \lambda)} \phi(x, \lambda) \, dx
\]

and the constants \( \alpha_n \) are the normalizing factors for the eigenfunctions of the problem (1)-(3), i.e. \( \alpha_n = \|\phi(\cdot, \lambda_n)\| \). The convergence of the series is absolute and uniform on subsets \( D \subset \mathbb{C} \) where \( \|\phi(\cdot, \lambda)\| \) is bounded.
We define
\[ \rho(\lambda) = \sum_{\lambda_n \leq \lambda} \alpha_n^{-2}. \]
This non-decreasing function will define a positive measure \(d\rho(\lambda)\) on \(\mathbb{R}\) in the Lebesgue–Stieltjes sense. We define
\[ B_{\rho}^{a,b} = \left\{ F : \mathbb{C} \to \mathbb{C} : F(\lambda) = \int_a^b f(x)\phi(x,\lambda) \, dx \text{ with } f \in L^2(a,b) \right\}. \]
We know [9] that \(\phi(x,\lambda) = O(e^{(x-a)\sqrt{|\lambda|}})\) as \(|\lambda| \to \infty\), uniformly in \(x\). Using Cauchy-Schwarz’s inequality in
\[ F(\lambda) = \int_a^b f(x)\phi(x,\lambda) \, dx \]
we obtain the inequality
\[ |F(\lambda)| \leq A e^{(b-a)\sqrt{|\lambda|}}, \]
and therefore, the functions in \(B_{\rho}^{a,b}\) are entire functions of order \(1/2\) and type at most \(b-a\).

**Definition 2.1.** We define the regular Sturm-Liouville transform associated with the problem (1)–(3) as the application \(\tau : L^2(a,b) \to B_{\rho}^{a,b}\) given by
\[ [\tau(f)](\lambda) = F(\lambda) = \int_a^b f(x)\phi(x,\lambda) \, dx, \quad \text{for } f \in L^2(a,b). \]

In the next section we prove that this application \(\tau\) is an isometry mapping the Hilbert space \(L^2(a,b)\) onto \(B_{\rho}^{a,b}\).

3. The space \(B_{\rho}^{a,b}\)

Let \(r\) be the application defined by \(r(F) = F|_{\mathbb{R}}\) for \(F \in B_{\rho}^{a,b}\). We denote by \(T\) the composition \(r \tau\). We have the following result

**Theorem 3.1.** The linear application \(T\) is an isometry from \(L^2(a,b)\) onto \(L^2_\rho(\mathbb{R})\). Furthermore, if \(F = T(f)\) then
\[ f(x) = \int_{-\infty}^{\infty} F(\lambda)\phi(x,\lambda) \, d\rho(\lambda). \] (8)
Proof. Let \( f \in L^2(a,b) \) and \( F(\lambda) = [T(f)](\lambda) \). Since \( \{\phi(x, \lambda_n)\}_{n=0}^{\infty} \) is an orthogonal basis in \( L^2(a,b) \) we have

\[
f(x) = \sum_{n=0}^{\infty} \langle f, \phi(\cdot, \lambda_n) \rangle \frac{\phi(x, \lambda_n)}{\alpha_n^2} = \int_{\mathbb{R}} F(\lambda) \phi(x, \lambda) \, d\rho(\lambda)
\]

with convergence in \( L^2(a,b) \), thus (8) is satisfied.

Furthermore

\[
\int_{\mathbb{R}} |F(\lambda)|^2 \, d\rho(\lambda) = \sum_{n=0}^{\infty} |F(\lambda_n)|^2 \alpha_n^{-2} = \sum_{n=0}^{\infty} |\langle f, \phi(\cdot, \lambda_n) \rangle|^2 \alpha_n^{-2}
= \int_{a}^{b} |f(x)|^2 \, dx,
\]

where we have used Parseval’s equality.

To prove that \( T \) is surjective, let \( F \in L^2_{\rho}(\mathbb{R}) \). Defining

\[
f(x) = \int_{\mathbb{R}} F(\lambda) \phi(x, \lambda) \, d\rho(\lambda) = \sum_{n=0}^{\infty} \frac{F(\lambda_n)}{\alpha_n^2} \phi(x, \lambda_n),
\]

this function belongs to \( L^2(a,b) \) and \( \tau(f) = F \).

The above theorem shows that if \( F(\lambda) \in B^{a,b}_\rho \) then its restriction to the real line belongs to \( L^2_{\rho}(\mathbb{R}) \), and every function in \( L^2_{\rho}(\mathbb{R}) \) can be extended to a function in \( B^{a,b}_\rho \). Thus, \( B^{a,b}_\rho \) is a Hilbert space of entire functions endowed with the inner product

\[
\langle F, G \rangle_{B^{a,b}_\rho} = \int_{\mathbb{R}} F(\lambda) \overline{G(\lambda)} \, d\rho(\lambda) = \sum_{n=0}^{\infty} \frac{F(\lambda_n) \overline{G(\lambda_n)}}{\alpha_n^2}
\]

for any \( F, G \in B^{a,b}_\rho \), or

\[
\langle F, G \rangle_{B^{a,b}_\rho} = \int_{a}^{b} f(x) \overline{g(x)} \, dx,
\]

where \( \tau(f) = F \) and \( \tau(g) = G \). Furthermore, we have found a characterization of the image space \( \tau(L^2(a,b)) = B^{a,b}_\rho \) through the regular Sturm-Liouville transform as

\[
B^{a,b}_\rho = \{ F \in \mathcal{H}(\mathbb{C}), \text{ with } |F(\lambda)| \leq Ae^{(b-a)\sqrt{\lambda}} \text{ and } F|_{\mathbb{R}} \in L^2_{\rho}(\mathbb{R}) \},
\]
with

$$
\| F \|_{\mathcal{B}_\rho^{a,b}}^2 = \int_{\mathbb{R}} |F(\lambda)|^2 d\rho(\lambda) = \sum_{n=0}^{\infty} \frac{|F(\lambda_n)|^2}{\alpha_n^2} = \int_a^b |f(x)|^2 dx,
$$

where $\tau(f) = F$.

The inversion formula (8) is given by means of the $\sigma$-finite, purely atomic measure $d\rho(\lambda)$ whose support is $\{\lambda_n\}$. As we will see in the next section, this inversion formula is important from a theoretical point of view. However, one can obtain an inversion formula involving a continuous measure using other techniques. See [15] for the details.

Now we prove that $\mathcal{B}_\rho^{a,b}$ is a RKHS.

**Theorem 3.2.** $\mathcal{B}_\rho^{a,b}$ is a RKHS space with reproducing kernel

$$
k(\lambda, \mu) \doteq \langle \phi(\cdot, \lambda), \phi(\cdot, \mu) \rangle_{L^2(a,b)}. \tag{9}
$$

**Proof.** Let $F \in \mathcal{B}_\rho^{a,b}$ and $\lambda \in \mathbb{C}$. Defining $l_\lambda F \doteq F(\lambda)$ we have

$$
|l_\lambda F| = |F(\lambda)| = \left| \int_a^b (\tau^{-1}F)(x)\phi(x, \lambda)\,dx \right|.
$$

Applying Cauchy-Schwarz’s inequality we obtain

$$
|l_\lambda F| \leq \| \tau^{-1}F \|_{L^2(a,b)} \| \phi(\cdot, \lambda) \|_{L^2(a,b)} = \| F \|_{\mathcal{B}_\rho^{a,b}} \| \phi(\cdot, \lambda) \|_{L^2(a,b)}.
$$

Thus, $\mathcal{B}_\rho^{a,b}$ is a RKHS space since the point evaluation $l_\lambda$ is a bounded linear functional on $\mathcal{B}_\rho^{a,b}$ for each $\lambda \in \mathbb{C}$ [8, 10]. Taking $f = \tau^{-1}(F) \in L^2(a, b)$, we have

$$
F(\lambda) = \langle f, \overline{\phi(\cdot, \lambda)} \rangle_{L^2(a,b)} = \langle \tau f, \overline{\tau \phi(\cdot, \lambda)} \rangle_{\mathcal{B}_\rho^{a,b}} = \langle F, \overline{\tau \phi(\cdot, \lambda)} \rangle_{\mathcal{B}_\rho^{a,b}},
$$

and therefore,

$$
k(\lambda, \mu) = \langle \phi(\cdot, \lambda), \phi(\cdot, \mu) \rangle_{L^2(a,b)}
$$

is the reproducing kernel of $\mathcal{B}_\rho^{a,b}$. \[\square\]
Since $B^a_b$ is a RKHS, we know that the convergence in the $B^a_b$ norm \( \| \cdot \|_{B^a_b} \) implies pointwise convergence. Furthermore, if \( |k(\lambda, \lambda)| \leq M \), for each \( \lambda \in D \subset \mathbb{C} \), the convergence will be uniform on \( D \).

**Lemma 3.1.** For any compact subset \( \Omega \subset \mathbb{C} \) there exists a constant \( M(\Omega) \) such that
\[
|k(\lambda, \lambda)| \leq M(\Omega), \quad \text{for each } \lambda \in \Omega.
\]

**Proof.** Using \( (9) \) we have
\[
|k(\lambda, \lambda)| = \| \phi(\cdot, \lambda) \|^2_{L^2(a,b)}.
\]
Since \( \| \phi(\cdot, \lambda) \|_{L^2(a,b)} \leq B e^{(b-a)\sqrt{M}} \), we obtain
\[
|k(\lambda, \lambda)| \leq A^2 e^{2(b-a)\sqrt{M}},
\]
and the result follows. Therefore, convergence in $B^a_b$ implies uniform convergence on compact subsets of \( \mathbb{C} \).

4. **Fourier-type duality associated with the regular Sturm-Liouville transform**

The isometry \( \tau \) from $L^2(a,b)$ onto $B^a_b$ enables us to transfer orthogonal and Riesz bases back and forth from one space to the other through \( \tau \) or \( \tau^{-1} \), exactly like in the Fourier setting. For this reason, we say that a Fourier-type duality exists associated with the regular Sturm-Liouville transform.

**Corollary 4.1.** \( \{ \varphi_n(\lambda) \}_{n=0}^{\infty} \equiv \tau(\{ \phi(\cdot, \lambda_n) \}_{n=0}^{\infty}) = \{ k(\lambda, \lambda_n) \}_{n=0}^{\infty} \) is an orthogonal basis of the Hilbert space $B^a_b$.

The following theorem ensures that any function in $B^a_b$ can be recovered through its samples on the eigenvalues of the problem \((1)-(3)\) by means of an interpolatory Lagrange-type series.

**Theorem 4.1.** (Sampling Theorem in $B^a_b$) Any \( F \in B^a_b \) can be expanded as
\[
F(\lambda) = \sum_{n=1}^{\infty} F(\lambda_n) S_n(\lambda), \quad (10)
\]
where
\[ S_n(\lambda) = \frac{P(\lambda)}{(\lambda - \lambda_n)^{P'(\lambda_n)}}, \]
and \( P(\lambda) \) is given by (6) or (7). The convergence is absolute and uniform on compact subsets of \( \mathbb{C} \).

**Proof.** We know that \( \{\varphi_n(\lambda)\}_{n=0}^{\infty} \) is an orthogonal basis of \( B_{\rho}^{a,b} \). Thus, for each \( F \in B_{\rho}^{a,b} \) we have
\[
F(\lambda) = \sum_{n=0}^{\infty} \langle F, \varphi_n \rangle_{H_{\rho}} \frac{\varphi_n(\lambda)}{\| \varphi_n \|_{H_{\rho}}^2} = \sum_{n=0}^{\infty} \langle \tau^{-1}F, \tau^{-1}\varphi_n \rangle_{L^2(a,b)} \frac{\varphi_n(\lambda)}{\| \phi(\cdot, \lambda_n) \|_{L^2(a,b)}^2}.
\]
Since \( \tau^{-1}\varphi_n = \phi(x, \lambda_n) \), then \( \langle \tau^{-1}F, \tau^{-1}\varphi_n \rangle_{L^2(a,b)} = F(\lambda_n) \). The proof will be complete once we identify the sampling functions
\[ S_n(\lambda) = \frac{\varphi_n(\lambda)}{\| \phi(\cdot, \lambda_n) \|_{L^2(a,b)}^2}. \]

For the sake of completeness we include here the proof which appears in [4] or [13].

The functions \( \phi(x, \lambda) \) y \( \phi(x, \lambda_n) \) are solutions of (1). Then,
\[
(\lambda - \lambda_n)\phi(x, \lambda)\phi(x, \lambda_n) = [\phi(x, \lambda)\phi'(x, \lambda_n) - \phi'(x, \lambda)\phi(x, \lambda_n)]'.
\]
Integrating
\[
(\lambda - \lambda_n) \int_a^b \phi(x, \lambda)\phi(x, \lambda_n) \, dx = \phi(b, \lambda)\phi'(b, \lambda_n) - \phi'(b, \lambda)\phi(b, \lambda_n). \tag{11}
\]
If \( \sin \beta \neq 0 \), having in mind that \( \phi(x, \lambda_n) \) verifies the boundary condition (3), using (4) we have
\[
W(\lambda)\phi(b, \lambda_n) = \sin \beta[\phi(b, \lambda)\phi'(b, \lambda_n) - \phi'(b, \lambda)\phi(b, \lambda_n)].
\]
Therefore
\[
\int_a^b \phi(x, \lambda)\phi(x, \lambda_n) \, dx = \frac{W(\lambda)\phi(b, \lambda_n)}{\lambda - \lambda_n} \frac{1}{\sin \beta},
\]
and if $\lambda \to \lambda_n$, 
\[
\|\phi(\cdot, \lambda_n)\|_{L^2(a,b)}^2 = W'(\lambda_n) \frac{\phi(b, \lambda_n)}{\sin \beta}.
\]

Now, using (5), 
\[
\frac{\varphi_n(\lambda)}{\|\phi(\cdot, \lambda_n)\|_{L^2(a,b)}^2} = \frac{P(\lambda)}{(\lambda - \lambda_n)^P(\lambda_n)}.
\] (10)

If $\sin \beta = 0$, by (4) $W(\lambda)\phi'(b, \lambda_n) = -\cos \beta \phi(b, \lambda)\phi'(b, \lambda_n)$, and by (3) we can write (11) as 
\[
\int_a^b \phi(x, \lambda)\phi(x, \lambda_n) \, dx = -\frac{W(\lambda) \phi'(b, \lambda_n)}{\lambda - \lambda_n} \cos \beta.
\]

Proceeding as before we obtain again (12).

The absolute convergence in (10) follows from the unconditional character of any orthonormal basis and the fact that convergence in a RKHS implies pointwise convergence. The uniform convergence is a consequence of Lemma 3.1. 

Let us illustrate all these results with an example, the finite continuous cosine transform:

Consider the regular Sturm-Liouville problem
\[
-\ddot{y} = -\lambda y, \quad x \in [0, \pi], \\
\dot{y}(0) = \dot{y}(\pi) = 0.
\]

In this case, $\phi(x, \lambda) = \cos \sqrt{\lambda} x$ and therefore
\[
[r(f)](\lambda) = F(\lambda) = \int_0^\pi f(x) \cos \sqrt{\lambda} x \, dx, \quad \text{for } f \in L^2(0, \pi).
\]

The eigenvalues are $\lambda_n = n^2$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the eigenfunctions are $\{\cos nx\}_{n \in \mathbb{N}_0}$. As a consequence, 
\[
\rho(\lambda) = \begin{cases} 
\frac{2}{\pi} \left(\sqrt{\lambda} + 1\right) & \text{if } \lambda \geq 0 \\
0 & \text{if } \lambda < 0
\end{cases}
\]

where $[\cdot]$ denotes the integer part of a real number.
The reproducing kernel, \( k(\lambda, \mu) \), is given by

\[
k(\lambda, \mu) = \int_0^\pi \cos \sqrt{\lambda s} \cos \sqrt{\mu s} \, ds,
\]

and therefore

\[
k(\lambda, \mu) = \frac{\sqrt{\lambda} \sin \sqrt{\lambda} \pi \cos \sqrt{\mu} \cos \sqrt{\lambda} \pi \sin \sqrt{\mu} \pi}{\lambda - \mu}.
\]

The functions

\[
\varphi_n(\lambda) = k(\lambda, n^2) = \frac{(-1)^n \sqrt{\lambda} \sin \sqrt{\lambda} \pi}{\lambda - n^2}, \quad n \in \mathbb{N}_0,
\]

constitute an orthogonal basis and the function \( F \) can be recovered through its samples in the points \( n^2 \) as

\[
F(\lambda) = F(0) \frac{\sin \sqrt{\lambda} \pi}{\sqrt{\lambda} \pi} + \frac{2}{\pi} \sum_{n=1}^\infty F(n^2) \frac{(-1)^n \sqrt{\lambda} \sin \sqrt{\lambda} \pi}{\lambda - n^2}.
\]

Since the reproducing kernel \( k \) is equal to

\[
k(\lambda, \mu) = \sum_{n=0}^\infty \frac{1}{\alpha_n^2} \varphi_n(\lambda) \overline{\varphi_n(\mu)},
\]

(see \([8, 10]\)), for \( \lambda, \mu \geq 0 \) we obtain the formula

\[
\frac{\sin \sqrt{\lambda} \pi}{\sqrt{\lambda} \pi} \left( \frac{\sin \sqrt{\mu} \pi}{\sqrt{\mu} \pi} \right) + \frac{2}{\pi} \sum_{n=1}^\infty \frac{\sqrt{\lambda} \sin \sqrt{\lambda} \pi}{\lambda - n^2} \left( \frac{\sqrt{\mu} \sin \sqrt{\mu} \pi}{\mu - n^2} \right)
\]

\[
= \frac{\sqrt{\lambda} \sin \sqrt{\lambda} \pi \cos \sqrt{\mu} \pi - \sqrt{\mu} \cos \sqrt{\lambda} \pi \sin \sqrt{\mu} \pi}{\lambda - \mu}.
\]

5. The Discrete Regular Sturm-Liouville Transform

Let \( l^2_\alpha = \{ \{a_n\}_n \subseteq \mathbb{C}: \sum_{n=0}^\infty \frac{|a_n|^2}{\alpha_n^2} < \infty \} \), endowed with the inner product

\[
\langle \{a_n\}, \{b_n\}\rangle_{l^2_\alpha} = \sum_{n=0}^\infty \frac{a_n \overline{b_n}}{\alpha_n^2}.
\]
Following [10], we can see that \( \{ \lambda_n \} \) is a complete interpolating sequence for \( B^a_b \), i.e. the set of all sequences \( \{ F(\lambda_n) \} \) where \( F \) ranges over \( B^a_b \) coincides with \( l^2_\alpha \), and the interpolation problem

\[
F(\lambda_n) = a_n, \quad n \in \mathbb{N}_0
\]

where \( F \in B^a_b \) has exactly one solution provided \( \{a_n\} \in l^2_\alpha \).

In fact, we have the following result

**Theorem 5.1.** Define \( \gamma : L^2(a,b) \rightarrow l^2_\alpha \) and \( \eta : B^a_b \rightarrow l^2_\alpha \) as \( \gamma(f) = \{ \langle f, \phi(\cdot, \lambda_n) \rangle \}_{L^2(a,b)} \}_{n=0}^{\infty} \), \( \eta(F) = \{ F(\lambda_n) \}_{n=0}^{\infty} \). Then, \( \gamma \) and \( \eta \) are isometric isomorphisms verifying \( \gamma = \eta \tau \).

**Proof.** It will be sufficient to prove that \( \eta \), \( \gamma \) are well defined, are isometries and \( \tau = \eta^{-1} \gamma \). For \( f \in L^2(a,b) \),

\[
f(x) = \sum_{n=0}^{\infty} \langle f, \phi(\cdot, \lambda_n) \rangle_{L^2(a,b)} \frac{\phi(x, \lambda_n)}{\alpha_n^d}.
\]

Using Parseval’s equality

\[
\|f\|^2_{L^2(a,b)} = \sum_{n=0}^{\infty} \frac{\|\langle f, \phi(\cdot, \lambda_n) \rangle_{L^2(a,b)}\|^2}{\alpha_n^d} = \|\{ \langle f, \phi(\cdot, \lambda_n) \rangle_{L^2(a,b)} \}\|^2_{l^2_\alpha},
\]

and \( \gamma \) is an isometry. The classical Riesz-Fischer Theorem assures that \( \gamma \) is surjective, and therefore an isomorphism.

On the other hand, \( \eta \) is a well defined isometry since for each \( F \in B^a_b \)

\[
\|F\|^2_{B^a_b} = \sum_{n=0}^{\infty} \frac{|F(\lambda_n)|^2}{\alpha_n^d} < \infty.
\]

Let \( \{a_n\}_{n=0}^{\infty} \in l^2_\alpha \) and \( f \in L^2(a,b) \) be such that \( f = \gamma^{-1}(a_n) \). Then \( a_n = \langle f, \phi(\cdot, \lambda_n) \rangle_{L^2(a,b)} \) for each \( n \in \mathbb{N}_0 \), and taking \( F = \tau f \) we conclude that \( F \in B^a_b \) and \( F(\lambda_n) = a_n \) for each \( n \in \mathbb{N}_0 \), proving that \( \eta \) is an isomorphism and \( \tau = \eta^{-1} \gamma \).

We may refer to \( \gamma \) as the **discrete regular Sturm-Liouville transform**.

Finally, we can apply this result in connection with the inverse Sturm-Liouville problem. Let \( \{ \lambda_n \}_{n \in \mathbb{N}} \) be a sequence of distinct real positive numbers, and let \( \{ \tau_n \}_{n \in \mathbb{N}} \) and \( \{ \rho_n \}_{n \in \mathbb{N}} \) both belong to \( l^2 \). Let \( a, b \) and \( c \) be
constants, and suppose further that
\[ \sqrt{\lambda_n} = \frac{n \pi}{b - a} + \frac{a}{n} - \frac{b}{n^3} + \frac{\tau_n}{n^3} \quad n \in \mathbb{N}, \]
and that in the sequence
\[ \alpha_n = \frac{2}{b - a} + \frac{c}{n^2} + \frac{\rho_n}{n^3} \]
each \alpha_n is positive. Then, according with an important inverse result due to
Levitan and Gasymov [6], there exists a regular Sturm-Liouville eigenvalue problem having eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}} \), and for which \( \{\alpha_n\}_{n \in \mathbb{N}} \) are the normalizing factors for the eigenfunctions. Using this result, we can obtain the following uniqueness theorem.

**Theorem 5.2.** Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of complex numbers such that \( \sum_{n=1}^{\infty} |a_n|^2 \alpha_n^{-2} < \infty \). There exists a unique entire function \( F \) of order 1/2 and type at most \( b - a \) such that \( F(\lambda_n) = a_n \). Moreover, this function is given by the Lagrange-type interpolatory series
\[ F(\lambda) = \sum_{n=1}^{\infty} a_n \frac{P(\lambda)}{(\lambda - \lambda_n)^{P(\lambda_n)}}. \]

**References**


