Boundary of Polyhedral Spaces: An Alternative Proof

Libor Veselý

Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini, 50, 20133-Milano, Italy

(Research paper presented by P.L. Papini)

AMS Subject Class. (1991): 46B20, 46B04, 52B99 Received September 25, 1998

A Banach space space $X$ is called polyhedral if the unit ball of each of its finite-dimensional (equivalently: two-dimensional [6]) subspaces is a polytope. Polyhedral spaces were studied by various authors; most of the structural results are due to V. Fonf. We refer the reader to the surveys [1], [2] for other definitions of polyhedrality, main properties and bibliography. In this paper we present a short alternative proof of the basic result on the structure of the unit ball of a polyhedral space (Theorem 1) and a related Theorem 2.

Let us start with some definitions. Throughout the paper, $X$ denotes an infinite-dimensional real Banach space with closed unit ball $B_X$, unit sphere $S_X$ and density character $\text{dens } X$ (i.e. the minimal cardinality of a dense subset of $X$).

We shall say that a set $F \subset S_X$ is a true face of $B_X$ if there exists a closed hyperplane $H \subset X$ supporting $B_X$ such that $F = H \cap B_X$ and $\text{int}_H F$ (the relative topological interior of $F$ in $H$) is nonempty. A set $B \subset S_X$, is called boundary for $X$ if for each $x \in S_X$ there exists $f \in B$ such that $f(x) = 1$. (In [5], $B$ is called “James boundary”.)

The following theorem is a slight reformulation of Theorem 1 from [3].

**THEOREM 1.** Let $X$ be a polyhedral Banach space. Then the sphere $S_X$ is covered by the true faces of $B_X$. Hence the set $B_0 = \{ f \in S_X : f^{-1}(1) \cap B_X \text{ is a true face of } B_X \}$ is a boundary for $X$. In particular, $B_0$ is countable whenever $X$ is separable.

The original proof in [3] is rather technical. About ten years later, V. Fonf considerably simplified the proof in an unpublished manuscript (see also [4]).
Our proof is quite different from those by Fonf. It is less elementary, since
it uses results about generic differentiability of convex functions, but simpler
than the proof in [3]. For separable $X$, our proof uses only the classical
Mazur’s theorem about generic Gâteaux differentiability of continuous convex
functions. Even in view of [4], we consider our proof geometric and maybe
interesting.

Let us remark the following

FACT. Since each relative interior point of a true face has a unique sup-
porting functional of norm one, the boundary $B_0$ from Theorem 1 is minimal
in the sense that it is contained in each boundary of the polyhedral space.

Moreover, in separable case, $B_{X^*}$ is the norm-closed convex hull of $B_0$, as
follows from the following result by Rodé [8]. (For a simpler proof of similar
nature see [5]; a different and more geometric proof has been found recently
by V. Fonf, J. Lindenstrauss and R. R. Phelps.)

THEOREM. (Rodé’s Theorem [8]) Let $B \subset S_{X^*}$ be a separable boundary
for $X$. Then $B_{X^*} = \overline{\text{conv}} B$ (the norm-closure of $\text{conv} B$).

We shall show by a separable reduction argument that, for polyhedral
spaces, the separability assumption is not necessary. We shall prove the fol-
lowing theorem.

THEOREM 2. Let $X$ be a polyhedral Banach space, and $B_0$ be the bound-
dary for $X$ from Theorem 1. Then $B_{X^*} = \overline{\text{conv}} B_0$ and $\text{card} B_0 = \text{dens} X = \text{dens} X^*$. (Consequently, $B_{X^*} = \overline{\text{conv}} B$ whenever $B$ is a boundary for $X$.)

The algebraic interior of a set $A \subset X$ is the set $a\text{-int} A$ of all points
$x \in A$ such that $x \in \text{int}_L (C \cap L)$ whenever $L \subset X$ is a line that contains $x$.
Obviously, $\text{int} A$ is always contained in $a\text{-int} A$. The following lemma about
$F_\sigma$-sets is well known for closed sets. The first part of it was suggested to the
author by L. Zajiček.

LEMMA 1. Let $A$ be an $F_\sigma$-set in $X$. Then $\text{int} A \neq \emptyset$ if and only if $a\text{-int} A \neq \emptyset$. If, moreover, $A$ is also convex, then $\text{int} A = a\text{-int} A$.

Proof. Suppose $0 \in a\text{-int} A$ and $A = \bigcup A_n$ where $(A_n)$ is a sequence of
closed sets. For every $v \in S_X$ there exists $t > 0$ such that the segment $[0, tv]$ is
covered by $A$. The Baire theorem implies that some $A_n$ contains a nontrivial
subsegment of $[0, tv]$. Consequently,

\[ S_X = \bigcup \{ S(n, \alpha, \beta) : n \in \mathbb{N}, \quad 0 < \alpha < \beta, \quad \alpha, \beta \text{ rational} \}, \]

where $S(n, \alpha, \beta) = \{ v \in S_X : [\alpha v, \beta v] \subset A_n \}$.

Since the sets $S(n, \alpha, \beta)$ are easily seen to be closed and they are countably many, another application of the Baire category theorem implies that some $S(\overline{\pi}, \overline{\pi}, \overline{\beta})$ has nonempty interior in $S_X$. Thus $A_{\overline{\pi}}$ (and hence $A$) contains the nonempty open set

\[ \bigcup \{ (\overline{\alpha}v, \overline{\beta}v) : v \in \text{int}_{S_X} S(\overline{\pi}, \overline{\pi}, \overline{\beta}) \}. \]

The assertion concerning convex sets follows from the Hahn-Banach theorem (indeed, if $A$ is convex and int $A$ is nonempty, no boundary point of $A$ can belong to a-int $A$ because it is a support point).

If $A \subset Y$ and $Y$ is an affine set in $X$, we denote by a-int$_Y A$ the relative algebraic interior of $A$ in $Y$:

\[ \text{a-int}_Y A = \{ x \in A : x \in \text{int}_L (A \cap L) \text{ whenever } L \text{ is a line and } x \in L \subset Y \}. \]

**Remark.** (a) Lemma 1 clearly implies: if $A$ is a set of the first category in a Banach space, then a-int $A$ is empty. (Indeed, $A$ is contained in an $F_\sigma$-set with empty interior.)

(b) The equality int $A = \text{a-int} A$ does not hold in general. Consider the origin in $X = \mathbb{R}^2$ and the set $A = \{(x, y) : y \geq x^2\} \cup \{(x, y) : y \leq 0\}$.

c) Lemma 1 remains valid if we replace $X$ by a closed affine subspace of a Banach space (and consider relative interior and relative algebraic interior).

**Lemma 2.** Let $X$ be polyhedral, $x_0 \in S_X$. Then the following assertions are equivalent.

(i) $x_0$ is interior point of a true face of $B_X$;

(ii) $B_X$ is Fréchet smooth in $x_0$;

(iii) $B_X$ is Gâteaux smooth in $x_0$.

**Proof.** The implications (i)⇒(ii)⇒(iii) are obvious. Suppose (iii) holds. Then $B_X$ has a unique supporting hyperplane $Y$ at $x_0$. For any two-dimensional subspace $Z \subset X$ that contains $x_0$, the line $Y \cap Z$ is the unique supporting line of the polygon $B_X \cap Z$ at $x_0$, hence the line intersects the polygon in a nontrivial line segment that contains $x_0$ as its (relative) interior point. Consequently, $x_0 \in \text{int}_Y (Y \cap B_X)$. Then Lemma 1 implies that $Y \cap B_X$ is a true face and (i) holds. \[ \square \]
Proof of Theorem 1. Let $Q$ be the set of the points from $S_X$ that are not contained in the union of all true faces.

Fix a point $u \in Q$ and a functional $f \in S_X^*$ with $f(u) = 1$. Let $Z = f^{-1}(0)$ and let $\pi: X \to Z$ be the linear projection along $u$, i.e. $\pi(z + tu) = z$ whenever $z \in Z$, $t \in \mathbb{R}$. It is easy to see that $\pi$ is a homeomorphism of an open neighborhood $G$ of $u$ in $S_X$ onto $G_0 := Z \cap \text{int} \left( \frac{1}{2} B_X \right)$. Define $p: G \to G_0$ by $p(x) = \pi(x)$. Then for each $z \in G_0$ we have

$$p^{-1}(z) = z + \varphi(z)u$$

where $\varphi: G_0 \to \mathbb{R}$ is continuous and concave. Let $Q_0 = p(Q \cap G)$.

Claim: the point $u_0 = p(u)$ belongs to $\text{a-int}_Z Q_0$.

Let $z$ be an arbitrary nonzero vector from $Z$. Since the unit ball of $\text{span}\{u, z\}$ is a polygon that contains $u$ as a boundary point, the boundary of this polygon contains two non-overlapping nondegenerate segments $[v_1, u]$ and $[u, v_2]$ with $v_1, v_2 \in G$. It is easy to see that the segment $p([v_1, u] \cup [u, v_2]) = [p(v_1), p(v_2)]$ is parallel to $z$ and contains $u_0$ as an interior point. Now it is not difficult to see that $(v_1, u] \cup [u, v_2] \subset Q$. Indeed, if some point $y \in (v_1, u]$ belonged to a true face, the hyperplane that defines this face would support $B_X$ at $y$ and hence also at each point of $[v_1, u]$. But this is impossible since $u \in Q$. (Similarly for $y \in (u, v_2]$.) This implies that $(p(v_1), p(v_2)) \subset Q_0$. The claim is proved.

Lemma 2 implies that no point of $Q$ is a point of Gâteaux differentiability of $B_X$; hence $Q_0$ contains only points of Gâteaux nondifferentiability of $\varphi$.

(a) If $X$ is separable, $\varphi$ is generically Gâteaux differentiable on $G_0$ by Mazur’s theorem ([7], [5]). By Remark (a), we must have $\text{a-int}_Z Q_0 = \emptyset$. But this contradicts our Claim. Thus Theorem 1 holds for separable spaces.

(b) If $X$ is not separable, then each separable subspace of $X$ has a countable boundary by (a), and hence, by Rode’s theorem, a separable dual. Thus $\varphi$ is generically Fréchet differentiable on $G_0$ (cf. [7]). By Remark (a), we get again $\text{a-int}_Z Q_0 = \emptyset$, a contradiction with our Claim.

Proof of Theorem 2. Suppose that $\text{dist}(f, \text{conv} B_0) > \varepsilon$ for some $f \in S_X^*$ and some $\varepsilon > 0$. Then, for every $g \in \text{conv} B_0$ there exists $z_g \in S_X$ such that $|(f - g)(z_g)| > \varepsilon$.

Let us perform the following inductive procedure. For a set $H \subset X^*$ and a subspace $L \subset X$, we denote by $H|_L$ the set $\{h|_L : h \in H\}$ of all restrictions to $L$ of elements of $H$.

1) Let $\{x_i\}^\infty_{i=1} \subset S_X$ be such that $f(x_i) \to 1$. Put $Y_1 = \text{span} \{x_i\}^\infty_{i=1}$. Since $B_0|_{Y_1}$ is obviously a boundary for $Y_1$, by Theorem 1 and Fact, there exists a
countable set $B_1 \subset B_0$ such that $B_1|Y_1$ is a boundary for $Y_1$. Let $D_1$ be a countable dense subset of $\text{conv } B_1$.

2) Suppose we already have separable subspaces $Y_1 \subset \cdots \subset Y_n$, countable subsets $B_1 \subset \cdots \subset B_n$ of $B_0$, and countable dense sets $D_k$ in $\text{conv } B_k$ for $k = 1, \ldots, n$. Put $Y_{n+1} = \overline{\text{span}}(Y_n \cup \{z_g : g \in D_n\})$. As above, take a countable set $B_{n+1} \subset B_0$ such that $B_{n+1} \supset B_n$ and $B_{n+1}|Y_{n+1}$ is a boundary for $Y_{n+1}$. Let $D_{n+1}$ be any countable dense subset of $\text{conv } B_{n+1}$.

Let us put $Y = \bigcup_{n=1}^{\infty} Y_n$, $A = \bigcup_{n=1}^{\infty} B_n$ and $D = \bigcup_{n=1}^{\infty} D_n$. Then $Y$ is separable, $A$ is countable, and $D$ is a countable dense subset of $\text{conv } A$.

We claim that $A_Y$ is a boundary for $Y$. Indeed, since $Y$ is polyhedral, by Theorem 1 each true face $F$ of $B_Y$ contains in its relative interior a point $y$ that belongs to some $Y_n$. By our construction, there exists $h \in B_n \subset A$ such that $h(y) = 1$. Thus the face $F$ is all contained in $h^{-1}(1)$.

Since for each $g \in D$ the point $z_g$ belongs to $S_Y$, we have

$$\text{dist}(f|_Y, \text{conv } A_Y) = \text{dist}(f|_Y, D|_Y) = \inf_{g \in D} |(f-g)(z_g)| \geq \varepsilon.$$ 

This contradiction with Rödö’s theorem proves that $B_X$ is the closed convex hull of $B_0$. Consequently, we have card $B_0 \leq \text{dens } X \leq \text{dens } X^* \leq \text{card } B_0$ (the first inequality follows from Theorem 1, and the second one holds for any normed space). $lacksquare$

References


