Composition Operators on Vector-Valued Hardy Spaces

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1. Introduction

If $\phi$ is an analytic self-map of the unit disc $D$, then as a consequence of Littlewood’s subordination theorem the composition transformation $C_\phi$ defined by $C_\phi f = f \circ \phi$ for $f$ holomorphic in $D$ turns out to be a bounded operator on the classical Hardy space $H^p(D)$, $(1 \leq p < \infty)$ and is called composition operator induced by $\phi$ (see Schwartz [5], Nordgren [3] for estimates of the norms of composition operators and Shapiro and Taylor [6] and Cowen and MacCluer [1] for other properties including compactness of these operators on Hardy classes of complex-valued functions). In this paper we attempt to initiate the study of composition operators on a vector-valued Hardy space.

The plan of the rest of the paper is as follows: Next section is preparatory in nature. In this section we collect some known as well as unknown facts about vector-valued Hardy spaces. We also determine generalized reproducing kernels for these spaces and use these kernel functions in the next section as effective tools to study composition operators on vector-valued Hardy spaces. In section 3 we prove that if $\phi: D \to D$ is analytic, then $C_\phi$ is a bounded operator on $H^2(D)$. A necessary and sufficient condition for a bounded operator on $H^2(D)$ to be a composition operator is given. The condition on $\phi$ for which $C_\phi$ is also a composition operator is presented in section 4. In this section we also present characterizations of normal, unitary and co-isometric composition operators on vector-valued Hardy space $H^2(D)$.

2. Background

Let $D = \{z \in \mathbb{C}: |z| < 1\}$ and $(X, \| \cdot \|_X)$ be a complex Banach space. For $1 \leq p < \infty$, the vector-valued Hardy space $H^p_X(D)$ consists of all $f : D \to X$
such that $e^* \circ f$ is analytic in $D$ for every $e^* \in X^*$ and
$$
\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p \, d\theta < \infty.
$$

$H^p_X(D)$ is a Banach space with
$$
\|f\|_p^p = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p \, d\theta.
$$

Throughout this paper we will assume that $(X, < \cdot, >)$ is a separable Hilbert space and so the radial limit $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ exists a.e. [4, Theorem A, page 84]. In this case $H^2_X(D)$ becomes a Hilbert space under the inner product $\langle \cdot, \cdot \rangle$, defined as
$$
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} <f^*(e^{i\theta}), g^*(e^{i\theta})> \, d\theta.
$$

For the sake of convenience, we shall denote $f^*(e^{i\theta})$ simply by $f(e^{i\theta})$. For more details about scalar-valued Hardy spaces we refer to Duren [2], and for vector-valued Hardy spaces consult Rosenblum and Rovnyak [4].

The very first result, which we are listing in the form of a lemma, will be used to find kernel functions for $H^2_X(D)$.

**Lemma 2.1.** If $f \in H^2_X(D)$, then
$$
\|f(z)\|_X \leq \frac{\|f\|_2}{(1-|z|^2)^{1/2}}.
$$

Proof follows from the Hölder’s inequality and the fact that if

$$
f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2_X(D),
$$

then $\|f\|_2^2 = \sum_{n=0}^{\infty} \|a_n\|_X^2$.

Let $\mathbb{N} = \{0, 1, 2, \ldots \}$ and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $X$. For $m, n \in \mathbb{N}$, we define $e_{m,n} : D \to X$ as

$$
e_{m,n}(z) = z^m e_n, \quad \forall z \in D.
$$

Then clearly $\{e_{m,n} : m, n \in \mathbb{N}\}$ is an orthonormal subset of $H^2_X(D)$. Further, if $f \in H^2_X(D)$, then

$$
\langle f, e_{m,n} \rangle = 0 \quad \Rightarrow \quad \frac{1}{2\pi} \int_0^{2\pi} <f(e^{i\theta}), e_{m,n}(e^{i\theta})> \, d\theta = 0
$$

$$
\Rightarrow \quad \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} <f(e^{i\theta}), e_n> \, d\theta = 0
$$
for all \( m, n \in \mathbb{N} \). Therefore \( \langle f(e^{i\theta}), e_n \rangle = 0 \) a.e. for all \( n \in \mathbb{N} \), hence \( f(e^{i\theta}) = 0 \) a.e. Taking into account the properties of the integral de Poisson, we conclude that \( f \equiv 0 \). Hence \( \{e_{m,n} : m, n, \in \mathbb{N}\} \) is a basis for \( H^2_X(D) \).

For each \( z \in D \) and \( j \in \mathbb{N} \), we define \( E_j^2 : H^2_X(D) \to \mathbb{C} \) as follows: \( E_j^2(f) = \langle f(z), e_j \rangle \) for every \( f \in H^2_X(D) \). Then \( E_j^2 \in (H^2_X(D))^* \) and so by Riesz representation theorem, there exists \( k_j^2 \in H^2_X(D) \) such that

\[
E_j^2 f = \langle f, k_j^2 \rangle, \quad \forall f \in H^2_X(D).
\]

We designate \( k_j^2 \)'s as generalized reproducing kernels or simply kernel functions whenever there is no confusion. The span of the set \( \{k_j^2 : (z, j) \in D \times \mathbb{N}\} \) will be denoted by \( \{k_j^2 : (z, j) \in D \times \mathbb{N}\} \). We now evaluate these kernel functions.

By Parseval’s identity,

\[
k_j^2(w) = \sum_{m,n \in \mathbb{N}} \langle k_j^2, e_{m,n} \rangle e_{m,n}(w)
\]

\[
= \sum_{m,n \in \mathbb{N}} \frac{\langle z^m e_n, e_j \rangle}{1 - z w} e_{m,n}(w)
\]

\[
= \frac{e_j}{1 - z w}
\]

and \( \|k_j^2\|_2^2 = \frac{1}{1 - |z|^2} \).

**Lemma 2.2.** \( \{k^2_j : (z, j) \in D \times \mathbb{N}\} \) is dense in \( H^2_X(D) \).

*Proof.* Let \( f \in \{k^2_j : (z, j) \in D \times \mathbb{N}\}^\perp \), the orthogonal complement of \( \{k^2_j : (z, j) \in D \times \mathbb{N}\} \). Then \( \langle f, g \rangle = 0 \) for all \( g \in \{k^2_j : (z, j) \in D \times \mathbb{N}\} \). In particular, \( \langle f, k^2_j \rangle = 0 \) for every \( (z, j) \in D \times \mathbb{N} \) and so \( f \equiv 0 \). This completes the proof. \( \square \)

3. **Composition operators on \( H^2_X(D) \)**

We begin this section by proving that every analytic self-map of the unit disc induces a composition operator on \( H^2_X(D) \).

**Theorem 3.1.** Let \( \phi : D \to D \) be analytic. Then \( C_\phi \) is a composition operator on \( H^2_X(D) \) and

\[
\|C_\phi\|^2 \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|}.
\]
Proof. Since $\phi: D \to D$ is analytic, for any $r_1 < 1$ there exists $r_2 < 1$ such that $|z| \leq r_1 \to |z| \leq r_2$. By [4, Theorem C, p. 89],
\[
\langle f(\phi(r_1 e^{i\theta})), e_j \rangle = \frac{1}{2\pi} \int_0^{2\pi} P(\phi(r_1 e^{i\theta}), r_2 e^{it}) \langle f(r_2 e^{it}), e_j \rangle dt,
\]
where
\[
P(\phi(r_1 e^{i\theta}), r_2 e^{it}) = \text{Re} \left( \frac{r_2 e^{it} + \phi(r_1 e^{i\theta})}{r_2 e^{it} - \phi(r_1 e^{i\theta})} \right)
\]
is the Poisson kernel. Since $x^2$ is a convex function, by Jensen’s inequality, we have
\[
| \langle f(\phi(r_1 e^{i\theta})), e_j \rangle \rangle^2 \leq \frac{1}{2\pi} \int_0^{2\pi} P(\phi(r_1 e^{i\theta}), r_2 e^{it}) | \langle f(r_2 e^{it}), e_j \rangle \rangle^2 dt.
\]
Using Parseval’s identity, we obtain
\[
\| f(\phi(r_1 e^{i\theta})) \|^2_X = \sum_{j \in \mathbb{N}} | \langle f(\phi(r_1 e^{i\theta})), e_j \rangle \rangle^2 \\
\leq \frac{1}{2\pi} \int_0^{2\pi} P(\phi(r_1 e^{i\theta}), r_2 e^{it}) \sum_{j \in \mathbb{N}} | \langle f(r_2 e^{it}), e_j \rangle \rangle^2 dt \\
= \frac{1}{2\pi} \int_0^{2\pi} P(\phi(r_1 e^{i\theta}), r_2 e^{it}) \| f(r_2 e^{it}) \|^2_X dt
\]
Integrating with respect to $\theta$, using Fubini’s theorem to interchange the order of integration in the double integral and well known property of Poisson kernel that
\[
\frac{1}{2\pi} \int_0^{2\pi} P(\phi(r_1 e^{i\theta}), r_2 e^{it}) d\theta = P(\phi(0), r_2 e^{it}) \leq \frac{r_2 + |\phi(0)|}{r_2 - |\phi(0)|},
\]
we get
\[
\frac{1}{2\pi} \int_0^{2\pi} \| f(\phi(r_1 e^{i\theta})) \|^2_X d\theta \leq \frac{r_2 + |\phi(0)|}{r_2 - |\phi(0)|} \frac{1}{2\pi} \int_0^{2\pi} \| f(r_2 e^{it}) \|^2_X dt.
\]
If $r_1 \to 1$, then $r_2 \to 1$, so that
\[
\| C_\phi f \|^2_2 \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \| f \|^2_2.
\]
This implies that
\[
\| C_\phi \|^2 \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|},
\]
hence the theorem. $\blacksquare$
We next present a necessary and sufficient condition for an operator $A$ on $H^2_X(D)$ to be a composition operator.

**Theorem 3.2.** Let $A$ be an operator on $H^2_X(D)$. Then $A$ is a composition operator if and only if for each $z \in D$ there exists unique $w \in D$ such that $A^*k_z^j = k_w^j$ for every $j \in \mathbb{N}$.

**Proof.** If $A = C_\phi$, a composition operator, then

$$
\langle f, A^*k_z^j \rangle = \langle C_\phi f, k_z^j \rangle = E^j_z C_\phi f = \langle f, k_{\phi(z)}^j \rangle,
$$

for every $j \in \mathbb{N}$.

Conversely, suppose that for each $z \in D$ there exists unique $w \in D$ such that $A^*k_z^j = k_w^j$ for every $j \in \mathbb{N}$. Thus, if we define $\phi$ as $\phi(z) = w$, then

$$
\phi(z) = \langle \phi(z) e_1, e_1 \rangle = E^j_w (e_{11})
= \langle e_{11}, A^*k_z^j \rangle = \langle Ae_{11}, k_z^j \rangle
= E^j_z (Ae_{11}) = \langle (Ae_{11})(z), e_1 \rangle.
$$

This proves that $\phi$ is analytic and so, by [4, Theorem C, p. 76], $C_\phi f \in H^2_X(D)$, for every $f \in H^2_X(D)$.

Now

$$
\langle Af, k_z^j \rangle = \langle f, A^*k_z^j \rangle = \langle f, k_{\phi(z)}^j \rangle
= E^j_{\phi(z)}(f) = E^j_z (C_\phi f) = \langle C_\phi f, k_z^j \rangle
$$

for every $(z, j) \in D \times \mathbb{N}$ and every $f \in H^2_X(D)$. Since $[k_z^j: (z, j) \in D \times \mathbb{N}]$ is dense in $H^2_X(D)$, we conclude that $A = C_\phi$.  

As an application of the above theorem, we obtain a lower bound for the norm of a composition operator.

**Corollary 3.3.** If $\phi$ is an analytic self-map of the unit disc, then

$$
\sup_{z \in D} \frac{1 - |z|^2}{1 - |\phi(z)|^2} \leq \|C_\phi\|^2.
$$
Proof. By Theorem 3.2
\[
\frac{1 - |z|^2}{1 - |\phi(z)|^2} = \frac{\|k^j_{\phi(z)}\|_2^2}{\|k^j_z\|_2^2} = \frac{\|C_\phi^* k^j_z\|_2^2}{\|k^j_z\|_2^2} \\
\leq \|C_\phi^*\|^2 = \|C_\phi\|^2
\]
for every \((z, j) \in D \times \mathbb{N}\). This implies that
\[
\sup_{z \in D} \frac{1 - |z|^2}{1 - |\phi(z)|^2} \leq \|C_\phi\|^2.
\]

4. Normal, Unitary and Co-Isometric Composition Operators

In general, the adjoint of a composition operator may or may not be a composition operator. We give a necessary and sufficient condition on \(\phi\) for which \(C_\phi^*\) is also a composition operator. The scalar-valued version of this result was proved by H.J. Schwartz [5] by using the technique of Fourier coefficients. Our method of proof is based on the generalized reproducing kernels.

**Theorem 4.1.** \(C_\phi^*\), the adjoint of \(C_\phi\), is a composition operator if and only if \(\phi(z) = \alpha z\), \(|\alpha| \leq 1\).

**Proof.** We first suppose that \(\phi(z) = \alpha z\), \(|\alpha| \leq 1\). Let \(\psi(z) = \overline{\alpha} z\). Then clearly \(\psi\) is an analytic mapping from \(D\) into itself. We shall show that \(C_\phi^* = C_\psi\). Let \((z, j) \in D \times \mathbb{N}\). Then
\[
(C_\phi^*)^* k^j_z(w) = C_\phi k^j_{\overline{\alpha} z}(w) = k^j_z(\phi(w)) \\
= k^j_{\overline{\alpha} z}(w) = k^j_{\psi(z)}(w)
\]
for every \(w \in D\), i.e., \((C_\phi^*)^* k^j_z = k^j_{\psi(z)}\). Hence, by Theorem 3.2, \(C_\phi^*\) is a composition operator and \(C_\phi^* = C_\psi\).

Conversely, suppose that \(C_\phi^* = C_\psi\) for some \(\psi\). Let \(\phi(z) = \sum_{n=0}^{\infty} a_n z^n\) and \(\psi(z) = \sum_{n=0}^{\infty} b_n z^n\). Since \(C_\phi^* = C_\psi\), we have for any \(j\),
\[
\|C_\phi^* k^j_{\phi(z)}\|_2^2 = \|C_\psi k^j_{\psi(z)}\|_2^2 = \|k^j_{\phi(z)}\|_2^2 = 1 \Rightarrow \frac{1}{1 - |\phi(z)|^2} = 1 \Rightarrow \phi(z) = 0 \Rightarrow a_0 = 0
\]
Similarly, we can show that \(b_0 = 0\). Hence, for any integer \(k\), the first \(k\) Fourier coefficients of \(\phi^k\) and \(\psi^k\) are zero.
Now for $n \geq 1$

\[
\begin{align*}
    b_n &= \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta}) e^{-in\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \langle \psi(e^{i\theta}) e_1, e^{in\theta} e_1 \rangle \, d\theta \\
    &= \langle C_\psi e_{11}, e_{n1} \rangle = \langle e_{11}, C_\phi e_{n1} \rangle \\
    &= \frac{1}{2\pi} \int_0^{2\pi} \langle e^{i\theta} e_1, \phi^n e_1 \rangle \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \phi^n e^{-i\theta} d\theta = a_1 \delta_{n1},
\end{align*}
\]

where $\delta_{n1}$ is the Kronecker delta. Therefore $\psi(z) = a_1 z$.

But $C_\psi^* = C_\phi$, so by first part of the theorem, we have $\phi(z) = a_1 z$.

In the next theorem we present a criterion for the normality of a composition operator on $H^2(X)(D)$.

**Theorem 4.2.** $C_\phi$ is normal if and only if $\phi(z) = az$, $|a| \leq 1$.

**Proof.** Let $\phi(z) = \sum_{n=0}^\infty a_n z^n$. We first suppose that $C_\phi$ is normal. Then $
\|C_\phi f\|_2 = \|C_\phi f\|_2$, for every $f \in H^2(X)(D)$.

In particular, taking $f = k_0^j$, we get $\phi(0) = 0$. Hence for any integer $k$, the first $k$ Fourier coefficients of $\phi^k$ are zero. Since $\{e_{m,n} : m, n \in \mathbb{N}\}$ is an orthonormal basis for $H^2(X)(D)$, by Parseval’s identity, we have

\[
\|C_\phi^* e_{11}\|_2^2 = \sum_{m,n} \langle C_\phi^* e_{11}, e_{m,n} \rangle \langle C_\phi^* e_{11}, e_{m,n} \rangle = \sum_{m,n} \langle C_\phi e_{m,n}, e_{m,n} \rangle^2
\]

\[
= \sum_{m,n} \left| \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} e_1, \phi^m e_n \right| d\theta \right|^2
\]

\[
= \sum_{m,n} \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} \phi^m e_1, e_n \right| d\theta \right|^2
\]

\[
= \sum_{n} \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} \phi^n d\theta \right|^2 = \sum_{n} |a_1 \delta_{n1}|^2 = |a_1|^2.
\]

Also

\[
\|C_\phi e_{11}\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \| e_{11}(\phi(e^{i\theta})) \|^2_\Delta d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{i\theta})|^2 \, d\theta.
\]

Therefore we have

\[
|a_1|^2 = \sum_{n=1}^{\infty} |a_n|^2 \Rightarrow a_n = 0 \text{ for } n \geq 2.
\]
Hence $\phi(z) = \alpha z$.

Conversely, suppose that $\phi(z) = \alpha z$, $|\alpha| \leq 1$. Then by Theorem 4.1, $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha}z$. Therefore

$$C_\phi C_\psi f(z) = C_\phi (f(\psi(z))) = C_\phi (f(\bar{\alpha}z))$$

$$= f(\alpha \bar{\alpha}z) = f(\alpha z)$$

$$= C_\psi f(\phi(z)) = C_\psi C_\phi f(z),$$

for every $f \in H^2_X(D)$ and every $z \in D$. Hence $C_\phi C_\psi = C_\psi C_\phi$ and $C_\phi$ is normal. ■

**Theorem 4.3.** $C_\phi$ is hermitian if and only if $\phi(z) = \alpha z$, where $\alpha \in \mathbb{R}$ and $|\alpha| \leq 1$.

*Proof.* If $C_\phi$ is hermitian, then it is normal and hence, by Theorem 4.2 $\phi(z) = \alpha z$, $|\alpha| \leq 1$. Also, by Theorem 4.1, $C_\phi^* = C_\psi$, where $\phi(z) = \bar{\alpha}z$. But $C_\phi^* = C_\phi$, so $\phi = \psi$, which implies that $\alpha = \bar{\alpha}$, i.e., $\alpha$ is real.

Conversely, we suppose that $\phi(z) = \alpha z$, $\alpha \in \mathbb{R}$ and $|\alpha| \leq 1$. Then, by Theorem 4.1, $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha}z = \alpha z = \phi(z)$. Thus $C_\phi^* = C_\phi$, i.e., $C_\phi$ is hermitian. ■

**Theorem 4.4.** $C_\phi$ is a unitary operator if and only if $\phi(z) = \alpha z$, $|\alpha| = 1$.

*Proof.* We first suppose that $\phi(z) = \alpha z$, $|\alpha| = 1$. Then, by Theorems 4.2 and 4.1, $C_\phi$ is normal and $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha}z$. Therefore,

$$C_\phi C_\phi^* f(z) = f(\psi(\phi(z))) = f(z),$$

for every $f \in H^2_X(D)$ and every $z \in D$, which implies $C_\phi C_\phi^* = I$, the identity operator. Hence, by normality of $C_\phi$, we conclude that $C_\phi$ is unitary.

Conversely, suppose that $C_\phi$ is unitary. Then $C_\phi$ is normal. Hence by Theorem 4.2, $\phi(z) = \alpha z$, $|\alpha| \leq 1$. By Theorem 4.1, $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha}z$, and $C_\phi C_\phi^* = I$ implies that

$$C_\phi C_\phi^* k^j_z = k^j_z,$$

for every $(z, j) \in D \times \mathbb{N}$

$$\Rightarrow k^j_z(\bar{\alpha} \omega w) = k^j_z(w),$$

for every $w \in D$

$$\Rightarrow \bar{\alpha} \omega w = w, $$

for every $w \in D$

$$\Rightarrow |\alpha| = 1$$

Hence $\phi(z) = \alpha z$, $|\alpha| = 1$. ■
THEOREM 4.5. \( C_\phi^* \) is an isometry if and only if \( \phi(z) = \alpha z, |\alpha| = 1 \).

Proof. If \( C_\phi^* \) is an isometry, then \( \|C_\phi^* f\|_2 = \|f\|_2 \), for every \( f \in H_\chi^2(D) \). In particular, taking \( f = k_{\phi(z)}^j \) and using Theorem 3.2, we obtain \( \|k_{\phi(z)}^j\|_2^2 = \|k_z^j\|_2^2 \). So

\[
\frac{1}{1 - |\phi(z)|^2} = \frac{1}{1 - |z|^2}.
\]

This implies that \( \phi(z) = \alpha z, |\alpha| = 1 \).

Conversely, if \( \phi(z) = \alpha z, |\alpha| = 1 \), then, by Theorem 4.1, \( C_\phi^* = C_\psi \), where \( \psi(z) = \bar{\alpha} z \). Thus, if \( f \in H^2_\chi(D) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then

\[
\|C_\phi^* f\|_2^2 = \|C_\psi f\|_2^2 \quad \text{for every } f \in H^2_\chi(D)
\]

\[
= \sum_{n=0}^{\infty} \|a_n\|_\chi^2 = \|f\|_2^2.
\]

Hence \( C_\phi^* \) is an isometry. \( \blacksquare \)

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