EXTRACTA MATHEMATICAE Vol. **13**, Núm. 3, 393 – 397 (1998)

# Central Extensions of Leibniz Algebras *

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(Research announcement presented by Santos González)

AMS *Subject Class.* (1991): 17B55, 17B99, 18G99 *Received May* 6, 1998

Leibniz algebras (introduced by Loday [5], [6]) are modules over a field $K$, equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfaying the Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$, for all $x,y,z \in \mathfrak{g}$. If $[x,x] = 0$, for all $x \in \mathfrak{g}$, then the Leibniz identity is the classical Jacobi identity and the Leibniz algebra is a Lie algebra. A (co)homology theory of Leibniz algebras is constructed by Loday in [5]; we can see another related results in [3], [4], [7].

Let $(e) : 0 \to \mathfrak{n} \stackrel{\kappa}{\to} \mathfrak{g} \stackrel{\pi}{\to} \mathfrak{q} \to 0$ be an extension of Leibniz algebras, then there exists a five term exact and natural sequence [2]

$$(1) \qquad HL_2(\mathfrak{g}) \stackrel{HL_2(\pi)}{\to} HL_2(\mathfrak{q}) \stackrel{\theta_*(e)}{\to} \frac{\mathfrak{n}}{[\mathfrak{n},\mathfrak{g}]} \stackrel{\kappa'}{\to} HL_1(\mathfrak{g}) \stackrel{\pi_{ab}}{\to} HL_1(\mathfrak{q}) \to 0$$

where $HL$ is the Leibniz homology with trivial coefficients. When $(e)$ is a central extension (i.e., $\mathfrak{n} \subseteq Z(\mathfrak{g})$), then the previous sequence is

$$(2) \qquad HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q}) \stackrel{\theta_*(e)}{\to} \mathfrak{n} \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{q}) \to 0.$$

Fix a free presentation $0 \to \mathfrak{r} \to \mathfrak{f} \stackrel{\rho}{\to} \mathfrak{g} \to 0$ of $\mathfrak{g}$, we obtain that the kernel of $HL_2(\pi) : HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q})$ in (1) is $\frac{\mathfrak{r} \bigcap [\mathfrak{s},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]}$, being $0 \to \mathfrak{s} \to \mathfrak{f} \stackrel{\pi\rho}{\to} \mathfrak{q} \to 0$ a free presentation of $\mathfrak{q}$.

If (e) is a central extension, then $[s,f] \subseteq r$ and there exists an epimorphism $C : n \bigotimes g_{ab} \bigoplus g_{ab} \bigotimes n \to \frac{[s,f]}{[r,f]}$, which is induced by commutator maps $c_1 : s \times f \to \frac{[s,f]}{[r,f]}$, $c_1(s,f) = [s,f] + [r,f]$, and $c_2 : f \times s \to \frac{[s,f]}{[r,f]}$, $c_2(f,s) = [f,s] + [r,f]$

*Supported by Xunta de Galicia (Grant: XUGA 37101A97)

since they verify that $c_1(s \times [f,f]) = 0, c_1(r \times f) = 0$ and $c_1(s \times r) = 0$, $c_2([f,f]) \times s) = 0, c_2(f \times r) = 0$ and $c_2(r \times s) = 0$, and then they induce bihomomorphisms of $K$-vector spaces $\overline{c}_1 : \frac{s}{r} \times \frac{f}{r+[f,f]} \cong n \times HL_1(g) \to \frac{[s,f]}{[r,f]}$ and $\overline{c}_2 : \frac{f}{r+[f,f]} \times \frac{s}{r} \cong HL_1(g) \times n \to \frac{[s,f]}{[r,f]}$ which are extended to $c_1$ on the tensor $n \otimes HL_1(g)$ and $c_2$ on the tensor

$HL_1(g) \otimes n$. Let $C = \langle c_1, c_2 \rangle$ be the morphism induced on direct sum by $c_1$ and $c_2$. Now we consider the map $\kappa$ which is the composition of $C$ and the inclusion of $\frac{[s,f]}{[r,f]}$ on $HL_2(g)$. We summarize this reasoning in the following

THEOREM 1. *Let $(e)$ be a central extension of Leibniz algebras. The commutator map $c$ defines a homomorphism of Leibniz algebras $\kappa \colon \mathfrak{n} \otimes HL_1(g) \bigoplus HL_1(g) \otimes \mathfrak{n} \to HL_2(g)$, $\kappa(n_1 \otimes g_1 + [g,g], g_2 + [g,g] \otimes n_2) = [s_1, f_1] + [f_2, s_2] + [\mathfrak{r}, \mathfrak{f}]$, being $\rho(s_i) = s_i + \mathfrak{r} = n_i \in \mathfrak{n}$ and $\rho(f_i) = f_i + \mathfrak{r} = g_i \in \mathfrak{g}$, $i = 1,2$, such that the following sequence is natural and exact*

$$\mathfrak{n} \otimes HL_1(\mathfrak{g}) \bigoplus HL_1(\mathfrak{g}) \otimes \mathfrak{n} \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q}) \to \mathfrak{n} \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{q}) \to 0$$

I like to thank T. Pirashvili for helpful comments about this theorem.

If $\mathfrak{n}$ is an abelian Leibniz algebra, then the extension $(e)$ induces a $\mathfrak{q}$-module structure (representation) on $\mathfrak{n}$ [6], [7]. If $\mathfrak{n}$ is a (right) $\mathfrak{q}$-module, then it is said that $(e)$ is a $\mathfrak{q}$-extension of $\mathfrak{n}$ if the $\mathfrak{q}$-module structure induced by $(e)$ coincides with the previous one. We denote by $\mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$ the set of congruence classes [e] of $\mathfrak{q}$-extensions of $\mathfrak{n}$ [1]. It is verified that $\Delta : \mathrm{Ext}(\mathfrak{q}, \mathfrak{n}) \to HL^2(\mathfrak{q}, \mathfrak{n})$, $\Delta[e] = \xi = \theta^*(e)(1_{\mathfrak{n}})$, is an isomorphism of $K$-vector spaces, being $HL$ the Leibniz homology.

Let $(e)$ be a $\mathfrak{q}$-extension of Leibniz algebras, then there exists a natural and exact sequence [1]

$$0 \to \mathrm{Der}(\mathfrak{q}, \mathfrak{n}) \overset{\mathrm{Der}(\pi)}{\to} \mathrm{Der}(\mathfrak{g}, \mathfrak{n}) \overset{\rho}{\to} \mathrm{Hom}_{\mathfrak{q}}(\mathfrak{n}, \mathfrak{n}) \overset{\theta^*(e)}{\to} HL^2(\mathfrak{q}, \mathfrak{n}) \overset{\pi^*}{\to} HL^2(\mathfrak{g}, \mathfrak{n}). \tag{3}$$

If $(e)$ is a central extension, then

$$0 \to \mathrm{Ext}(HL_1(\mathfrak{q}), \mathfrak{n}) \overset{\Psi}{\to} HL^2(\mathfrak{q}, \mathfrak{n}) \overset{\theta}{\to} \mathrm{Hom}(HL_2(\mathfrak{q}), \mathfrak{n}) \to 0 \tag{4}$$

is a natural and split short exact sequence [1].

By lemma 5.2 in [1] it is verified that

$$\theta_*\Delta[e] = \theta_*\theta^*(e)(1_{\mathfrak{n}}) = \theta_*(e).$$

According to the nature of the homomorphism $\theta_*\Delta[e] : HL_2(\mathfrak{q}) \to \mathfrak{n}$ and using the previous exact sequences, we can characterize several types of central extensions of Leibniz algebras.

Let $(e)$ be a central extension of Leibniz algebras. Keeping in mind the exact sequence (2), $\theta_*\Delta[e] = 0$ if and only if $0 \to \mathfrak{n} \to \mathfrak{g}_{ab} \to \mathfrak{q}_{ab} \to 0$ is exact. On the other hand, the epimorphism $\pi : \mathfrak{g} \to \mathfrak{q}$ induces an epimorphism $\pi_* : [\mathfrak{g}, \mathfrak{g}] \to [\mathfrak{q}, \mathfrak{q}]$ then, by cross lemma [2], we have that $0 \to \mathfrak{n} \to \mathfrak{g}_{ab} \to \mathfrak{q}_{ab} \to 0$ is exact if and only if $n \bigcap [\mathfrak{g}, \mathfrak{g}] = 0$. In this situation we shall say that $(e)$ is a *commutator extension.*

PROPOSITION 1. *The equivalence classes of commutator extensions of* $\mathfrak{q}$ *are classified by* $\mathrm{Ext}(\mathfrak{q}_{ab}, \mathfrak{n})$.

*Proof.* By exactness in sequence (5). ∎

For a central extension $(e)$, $\theta_*\Delta[e]$ is a monomorphism if and only if $ImHL_2(\pi) = 0$ (by exactness in sequence (2)), if and only if $HL_2(\pi) : HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q})$ is the zero map if and only if $0 \to HL_2(\mathfrak{q}) \to \mathfrak{n} \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{q}) \to 0$ is exact. On the other hand, $\theta_*(e)$ factors as an epimorphism $HL_2(\mathfrak{q}) \to \mathfrak{n} \bigcap [\mathfrak{g}, \mathfrak{g}]$ and a monomorphism $\mathfrak{n} \bigcap [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{n}$, so $\theta_*(e)$ is a monomorphism if and only if $HL_2(\mathfrak{q}) \cong \mathfrak{n} \bigcap [\mathfrak{g}, \mathfrak{g}]$. In this situation we shall say that $(e)$ is a *quasi-commutator extension.* Obviously, if $HL_2(\mathfrak{q}) = 0$, then a quasi-commutator extension is a commutator extension.

COROLLARY 1. *Let* $(e)$ *be a central extension with* $\mathfrak{g}$ *a free Leibniz algebra, then* $(e)$ *is a quasi-commutator extension.*

*Proof.* $HL_2(\mathfrak{g}) = 0$ [7], thus $\theta_* : HL_2(\mathfrak{q}) \to \mathfrak{n}$ is a monomorphism (by exactness in sequence (2)). ∎

An example of this situation is the degenerate central extension $0 \to 0 \to \mathfrak{f} \simeq \mathfrak{f} \to 0$, being $\mathfrak{f}$ a free Leibniz algebra.

For a central extension $(e)$, $\theta_*\Delta[e]$ is an epimorphism if and only if $\kappa' : \mathfrak{n} \to \mathfrak{g}_{ab}$ is the zero map (by exactness in sequence (2)) if and only if $\pi_* : HL_1(\mathfrak{g}) \to HL_1(\mathfrak{q})$ is an isomorphism if and only if $\frac{\mathfrak{n}}{\mathfrak{n} \bigcap [\mathfrak{g},\mathfrak{g}]} = 0$ (by cross lemma [2]) if and only if $\mathfrak{n} \subseteq [\mathfrak{g}, \mathfrak{g}]$. In this situation we shall say that $(e)$ is a *stem extension.*

PROPOSITION 2. *Every central extension class of a $K$-vector space (trivial* $\mathfrak{q}$*-module)* $\mathfrak{n}$ *by a Leibniz algebra* $\mathfrak{q}$ *is forward induced from a stem extension if and only if* $\mathrm{Ext}(HL_1(\mathfrak{q}), \mathfrak{n}) = 0$.

*Proof.* Assume $\mathrm{Ext}(HL_1(\mathfrak{q}), \mathfrak{n}) = 0$; pick any central extension class $(e)$, then $\theta_* : HL_2(\mathfrak{q}) \to \mathfrak{n}$ factors as $i\tau : HL_2(\mathfrak{q}) \to \mathfrak{n} \bigcap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}_1 \to \mathfrak{n}$. As $\mathfrak{n}_1$ is a trivial $\mathfrak{q}$-module, then from (5), given $\tau$ there exists a central extension $(e_1) \in HL^2(\mathfrak{q}, \mathfrak{n})$ such that $\theta_*(e_1) = \tau$. Moreover, $(e_1)$ is a stem extension. By naturality of sequence (2) on the forward construction $(e_1) \to i_*(e_1)$ [1], we have that $\theta_* i_*(e_1) = i\tau = \theta_*(e)$, i.e., $i_*(e_1) = (e)$, and so $(e)$ is forward induced by $(e_1)$, which is a stem extension.

Conversely, let $(e) \in \mathrm{Ext}(HL_1(\mathfrak{q}), \mathfrak{n})$, then there exists a morphism of extension $(j, ., 1) : (e_1) \to \psi(e) = ({}^{ab}e)$ [1], where $(e_1)$ is a stem extension. By naturality of sequence (2) on $(e_1) \to \psi(e)$ we have $j\theta_*(e_1) = \theta_*\psi(e) = 0$; as $\theta_*(e_1)$is surjective, $j = 0$, and so $\psi(e) = j_*(e_1) = 0^*(e_1) = 0$. As $\psi$ is injective, $(e) = 0$. ∎

For a central extension $(e)$, $\theta_*\Delta[e]$ is an isomorphism if and only if $\pi_* : HL_1(\mathfrak{g}) \to HL_1(\mathfrak{q})$ is an isomorphism and $\pi_* : HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q})$ is the zero map (by exactness in sequence (2)). In this situation we shall say that $(e)$ is a *stem cover*. Obviously, every stem cover is an example of stem extension and a quasi-commutator extension.

PROPOSITION 3. *Let* $\mathfrak{q}$ *be a perfect Leibniz algebra (i.e.,* $\mathfrak{q} = [\mathfrak{q}, \mathfrak{q}]$*) and let* $(e)$ *be a central extension. Then* $(e)$ *is a stem cover if and only if* $HL_1(\mathfrak{g}) = 0$ *and* $HL_2(\mathfrak{g}) = 0$.

*Proof.* From the exact sequence (3) associated to $(e)$, with $HL_1(\mathfrak{g}) = HL_2(\mathfrak{g}) = 0$, it is easily seen that $\theta_*(e)$ is an isomorphism. The converse is trivial. ∎

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