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Conservation Laws and Symmetry in Economic Growth Models: a Geometrical Approach

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1. Introduction

The fundamental laws of Mechanics and Economic Dynamics are essentially the same, and they can be reduced to compute the extremals of a suitable functional. In the past four decades a lot of effort has been done in order to introduce topological and geometrical techniques in Mechanics, the result being the now so-called Geometric or Symplectic Mechanics. In particular, the knowledge of symmetries and their associated conserved quantities has been a useful tool to obtain information about the integrability of the equations of motion. In some cases, one can collect together all the symmetries and obtain a Lie group for which there exists an appropiate momentum map which provides a reduction of the dynamics (see for instance Abraham and Marsden (1978)).

A similar effort for Economic theories is not accomplished yet. However, important results have been obtained in this direction (see for instance Chiang (1992), Magill (1970), Sato and Ramachandran (1990) and references therein).

In many economical models determined by a Lagrangian function and several constraints is very difficult to solve the equations of motion. Therefore it is useful to find conservations laws, that is, functions which are constants along the temporal evolution of the system, in order to know relevant aspects of the initial dynamical system. The study of economical conservation laws is not new. For instance, the first reference to an economic conservation law appears in the famous paper of Ramsey (1928) where he obtained the optimal saving rule. But, it was Samuelson (1970) who explicitely introduced the concept of conservation law in economy. After him, this concept was used by Weitzmann (1976), Sato (1975, 1981), Mimura and Nôno (1997), Kataoka and Hashimoto (1995) and others (see the book of Sato and Ramachadran (1990) for a complete list of references).

The aim of the present paper is twofold. By the one hand, we present a classification of infinitesimal symmetries for Lagrangian systems, and the corresponding Noether theorems. The derivation of the results is made by using the symplectic techniques. Some of the results were previously obtained by other authors (see Prince (1985) for instance), and an exhaustive presentation can be found in de León and Martín de Diego (1995, 1996). Let us note that these results are true even if the Lagrangian function is singular, which is usually the case in economic models. On the other hand, we apply our methods to derive some well-known conservation laws, in particular the income-wealth conservation law obtained by Weitzman (1976) and the Samuelson's first law (see Samuelson (1970)).

An important remark is the following. In many economic models, we have a Lagrangian function subjected to some constraint functions. The geometrical model is just a vakonomic mechanical system. Therefore, it seems that such a kind of dynamical systems deserves a careful study. We have recently started a program to develop a geometrical setting for vakonomic mechanics with the hope to obtain some applications to economic systems (see de León, Marrero and Martín de Diego (1998)).

2. Lagrangian mechanical systems

The configuration space of a mechanical system is an n-dimensional manifold Q, so that the evolution space of a dynamical system described by a time-dependent Lagrangian system is $\mathbb{R} \times TQ$, where TQ is the tangent bundle of Q (see Abraham and Marsden (1978), de León and Rodrigues (1989)).

We denote by $(t, q^i, \dot{q}^i)$, $1 \leq i \leq n$ the bundle coordinates on $\mathbb{R} \times TQ$, induced from local coordinates (q^i) on Q. If $\pi : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times Q$ is the canonical projection then we have $\pi(t, q^i, \dot{q}^i) = (t, q^i)$.

Let d_T be a differential operator which maps each function $f : \mathbb{R} \times Q \longrightarrow \mathbb{R}$ into a function $d_T f$ on $\mathbb{R} \times TQ$ locally defined by

$$d_T f = \frac{\partial f}{\partial t} + \dot{q}^i \frac{\partial f}{\partial q^i} .$$

The geometry of the evolution space $\mathbb{R} \times TQ$ is characterized by two geometric objects:
a $(1,1)$-tensor field

$$S = \frac{\partial}{\partial \dot{q}^i} \otimes dq^i,$$

and the Liouville vector field

$$\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

From both we construct a new $(1,1)$-tensor field: $\tilde{S} = S - \Delta \otimes dt$.

Let X be a vector field on $\mathbb{R} \times Q$, with local expression

$$X = \tau \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i}.$$

Its natural lift to $\mathbb{R} \times TQ$ is the vector field X^1 defined by

$$X^1 = \tau \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + \left(d_T(X)^i - \dot{q}^i d_T(\tau)\right) \frac{\partial}{\partial \dot{q}^i}.$$

A vector field ξ on $\mathbb{R}\times TQ$ is said to be a second order differential equation (SODE, for simplicity) if $S(\xi) = \Delta$ and $\tilde{S}(\xi) = 0$. Therefore, a SODE ξ is locally expressed as

$$\xi = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \xi^i \frac{\partial}{\partial \dot{q}^i},$$

where $\xi^i = \xi^i(t, q^j, \dot{q}^j)$. A curve $\sigma : \mathbb{R} \longrightarrow Q$ is said to be a solution of a SODE ξ if its natural prolongation to $\mathbb{R} \times TQ$ given by $\sigma^1(t) = (t, q^i(t), dq^i/dt)$ is an integral curve of ξ, or, equivalently, σ is a solution of the following system of non-autonomous differential equations of second order:

$$\frac{d^2q^i}{dt^2} = \xi^i\left(t, q^j(t), \frac{dq^j}{dt}\right), \quad 1 \leq i \leq n.$$

Let $\mathbb{L} : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ be a non-autonomous (or time-dependent) Lagrangian with energy

$$E_{\mathbb{L}} = \dot{q}^i \frac{\partial \mathbb{L}}{\partial \dot{q}^i} - \mathbb{L}.$$

(Notice that $E_{\mathbb{L}}$ can be globally defined by $E_{\mathbb{L}} = \Delta(\mathbb{L}) - \mathbb{L}$). We construct from $\mathbb{L}$:

(i) the *Poincaré-Cartan 1-form*

$$\Theta_{\mathbb{L}} = \tilde{S}^*(d\mathbb{L}) = \frac{\partial \mathbb{L}}{\partial \dot{q}^i}(dq^i - \dot{q}^i dt) + \mathbb{L}dt,$$

where $\tilde{S}^*$ is the transpose operator of $\tilde{S}$,

(ii) and the *Poincaré-Cartan 2-form*

$$\Omega_{\mathbb{L}} = -d\,\Theta_{\mathbb{L}}.$$

$E_{\mathbb{L}}$ is a function defined on $\mathbb{R} \times TQ$, and $\Theta_{\mathbb{L}}$ and $\Omega_{\mathbb{L}}$ are forms on $\mathbb{R} \times TQ$.

To find (SODE) solutions ξ of the following intrinsic equations

$$i_\xi \Omega_{\mathbb{L}} = 0, \quad dt(\xi) = 1, \tag{1}$$

is equivalent to solve the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathbb{L}}{\partial q^i} = 0, \quad 1 \leq i \leq n.$$

If the Lagrangian $\mathbb{L}$ is regular, that is, the Hessian matrix $(\partial^2 \mathbb{L}/\partial \dot{q}^i \partial \dot{q}^j)$ is non-singular, then the pair $(\Omega_{\mathbb{L}}, dt)$ is a cosymplectic structure, and, in such a case, equations (1) have a unique solution $\xi_{\mathbb{L}}$ which is automatically a SODE. However, in many economic problems we will treat with Lagrangians which are singular. In that case, the existence of a solution ξ of equations (1) is not guaranteed, and if any solution exists, it will not be necessarily a SODE.

Observe that, in general, for time-dependent Lagrangians the energy $E_{\mathbb{L}}$ is not conserved. In fact, we have

$$\frac{dE_{\mathbb{L}}}{dt} = -\frac{\partial \mathbb{L}}{\partial t}. \tag{2}$$

The above equation is an immediate consequence of the equations of the motion (1).

3. Symmetries and conservation laws of time-dependent Lagrangian systems

Let ξ be a SODE on the evolution space $\mathbb{R} \times TQ$.

DEFINITION 3.1. A differentiable function $f : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ is called a conservation law or a constant of the motion of ξ if $\xi(f) = 0$.

Therefore, if $\gamma : \mathbb{R} \longrightarrow \mathbb{R} \times TQ$ is a integral curve of ξ, then $f \circ \gamma$ is a constant function.

Next, assume that $\mathbb{L}$ is a Lagrangian function on $\mathbb{R} \times TQ$.

DEFINITION 3.2. A vector field X on $\mathbb{R} \times Q$ is said to be an infinitesimal symmetry of $\mathbb{L}$ if

$$X^1(\mathbb{L}) = -d_T(\tau)\mathbb{L},$$

where $\tau = dt(X)$.

In coordinates, a vector field $X = \tau\partial/\partial t + X^i\partial/\partial q^i$ is an infinitesimal symmetry of $\mathbb{L}$ if and only if it satisfies the following condition:

$$\tau\frac{\partial\mathbb{L}}{\partial t} + X^i\frac{\partial\mathbb{L}}{\partial q^i} + \left(d_T X^i - \dot{q}^i d_T(\tau)\right)\frac{\partial\mathbb{L}}{\partial\dot{q}^i} = -d_T(\tau)\mathbb{L}.$$

PROPOSITION 3.3. *If X is an infinitesimal symmetry of $\mathbb{L}$, then $\Theta_{\mathbb{L}}(X^1)$ is a constant of the motion of any solution ξ of equations* (1).

Proof. From the local expression of the Poincaré-Cartan 1-form $\Theta_{\mathbb{L}}$, we have

$$\mathcal{L}_{X^1}\Theta_{\mathbb{L}} = \lambda_i(dq^i - \dot{q}^i dt),$$

for some $\lambda_i \in C^\infty(\mathbb{R} \times TQ)$. Here $\mathcal{L}$ denotes the Lie derivative (see Abraham and Marsden (1978)). Therefore, for a solution ξ of equations (1) we obtain

$$i_\xi\mathcal{L}_{X^1}\Theta_{\mathbb{L}} = \lambda_i(dq^i - \dot{q}^i dt)(\xi) = 0,$$

since ξ is a SODE. Thus, we finally have

$$\xi(\Theta_{\mathbb{L}}(X^1)) = 0.$$

∎

In coordinates, the constant of the motion obtained from an infinitesimal symmetry X is:

$$\Theta_{\mathbb{L}}(X^1) = \frac{\partial\mathbb{L}}{\partial\dot{q}^i}(X^i - \dot{q}^i\tau) + \mathbb{L}\tau.$$

There is a more general class of symmetries which permit to obtain additional conservation laws.

DEFINITION 3.4. (i) A vector field X on $\mathbb{R} \times Q$ is called a Noether symmetry if

$$\mathcal{L}_{X^1}\Theta_{\mathbb{L}} = df,$$

for some function f on $\mathbb{R} \times TQ$.
(ii) A vector field $\tilde{X}$ on $\mathbb{R} \times TQ$ is called a Cartan symmetry if

$$\mathcal{L}_{\tilde{X}}\Theta_{\mathbb{L}} = df,$$

for some function f on $\mathbb{R} \times TQ$.

In coordinates, a vector field $X = \tau\partial/\partial t + X^i\partial/\partial q^i$ is a Noether symmetry if it verifies the following condition:

$$\tau\frac{\partial \mathbb{L}}{\partial t} + X^i\frac{\partial \mathbb{L}}{\partial q^i} + \left(d_T(X^i) - \dot{q}^i d_T(\tau)\right)\frac{\partial \mathbb{L}}{\partial \dot{q}^i} = -d_T(\tau)\mathbb{L} + d_T(f).$$

A Cartan symmetry is also characterized as follows:

$$i_{\tilde{X}}\Omega_{\mathbb{L}} = d(\Theta_{\mathbb{L}}(\tilde{X}) - f).$$

Indeed, we have

$$df = \mathcal{L}_{\tilde{X}}\Theta_{\mathbb{L}} = i_{\tilde{X}}d\Theta_{\mathbb{L}} + di_{\tilde{X}}\Theta_{\mathbb{L}} = -i_{\tilde{X}}\Omega_{\mathbb{L}} + d(\Theta_{\mathbb{L}}(\tilde{X})),$$

and the result follows.

PROPOSITION 3.5. *If X is a Noether symmetry of $\mathbb{L}$, then $f - \Theta_{\mathbb{L}}(X^1)$ is a constant of the motion of any solution ξ of equations* (1).

Proof. If X is a Noether symmetry, we have

$$df = \mathcal{L}_{X^1}\Theta_{\mathbb{L}} = i_{X^1}d\Theta_{\mathbb{L}} + di_{X^1}\Theta_{\mathbb{L}},$$

and, therefore, we deduce that

$$d\left(f - \Theta_{\mathbb{L}}(X^1)\right) = -i_{X^1}\Omega_{\mathbb{L}}.$$

Thus, for any solution ξ of equations (1) we obtain

$$\xi\left(f - \Theta_{\mathbb{L}}(X^1)\right) = d\left(f - \Theta_{\mathbb{L}}(X^1)\right)(\xi) = -i_{X^1}\Omega_{\mathbb{L}}(\xi) = 0.$$

Hence, $g = f - \Theta_{\mathbb{L}}(X^1)$ is a constant of the motion of any solution ξ. ∎

In coordinates, the conservation law obtained from a Noether symmetry X is

$$f - \Theta_{\mathbb{L}}(X^1) = f - \frac{\partial \mathbb{L}}{\partial \dot{q}^i}(X^i - \dot{q}^i \tau) - \mathbb{L}\tau.$$

Next, we will obtain the relation between Cartan symmetries and conservation laws.

THEOREM 3.6. (Noether theorem and its converse) *If $\tilde{X}$ is a Cartan symmetry, then $F = \Theta_{\mathbb{L}}(\tilde{X}) - f$ is a constant of the motion of any solution ξ of equations* (1). *Conversely, if F is a constant of the motion of any solution ξ, then there exists a vector field Z on $\mathbb{R} \times TQ$ such that*

$$i_Z \Omega_{\mathbb{L}} = dF.$$

Proof. Suppose that X is a Cartan symmetry. Then:

$$i_{\tilde{X}} \Omega_{\mathbb{L}} = d(\Theta_{\mathbb{L}}(\tilde{X}) - f) = dF.$$

Therefore, if ξ is any solution of equations (1), we have

$$0 = -i_{\tilde{X}} i_{\xi} \Omega_{\mathbb{L}} = i_{\xi} i_{\tilde{X}} \Omega_{\mathbb{L}} = i_{\xi} dF = \xi(F).$$

Conversely, if F is a conservation law, the equation

$$i_Y \Omega_{\mathbb{L}} = dF$$

has a globally defined solution $\tilde{X}$, and then $\xi(F) = 0$. Therefore we deduce that F is a conservation law. ∎

4. VAKONOMIC MECHANICS AND SINGULAR LAGRANGIAN SYSTEMS

Recently, vakonomic mechanics has received much attention (see Arnold (1988), Lewis and Murray (1995)). In vakonomic mechanics, a motion is an extremal of the functional

$$c \mapsto \int_0^1 \mathbb{L}(\dot{c}(t))\, dt,$$

where c is a curve joinning two fixed points and satisfying the constraints given by a submanifold M of TQ, and the permitted variations also satisfy them. Thus, using the Lagrange Multipliers Theorem in an infinite dimensional context we deduce (see Arnold (1988), Lewis and Murray (1995), de

León, Marrero and Martín de Diego (1998)) that c is a motion if and only if there exist n functions $\lambda^1, \dots, \lambda^m$ such that

$$\frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathbb{L}}{\partial q^i} = -\lambda^A \left(\frac{d}{dt}\left(\frac{\partial \Phi_A}{\partial \dot{q}^i}\right) - \frac{\partial \Phi_A}{\partial q^i}\right) - \frac{d\lambda^A}{dt}\frac{\partial \Phi_A}{\partial \dot{q}^i}, \quad 1 \leq i \leq n, \tag{3}$$

where Φ_A are the functions defining the constraint submanifold M.

An alternative approach to vakonomic mechanics is the following. We can prove that a curve c is a solution of the vakonomic equations if and only if there exist local functions $\lambda^1, \dots, \lambda^n$ on TQ such that c is an extremal for the extended Lagrangian

$$\mathcal{L} = \mathbb{L} + \lambda^A \Phi_A,$$

that is, $c(t) = (q^i(t))$ satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i} = 0, \quad 1 \leq i \leq n.$$

(see Arnold (1988), Lewis and Murray (1995), de León, Marrero and Martín de Diego (1998) for details).

A geometrical formulation for vakonomic mechanics similar to that for non-holonomic mechanics was recently started in de León, Marrero and Martín de Diego (1998). However, we can use the unconstrained formulation developed in Sections 2 and 3 taking into account that the extended Lagrangian $\mathcal{L}$ could be singular. But the above results are still valid in this case!

5. A model of economic growth of a nation

Assume that the economic growth of a nation is determined by adequate choices of consumption and investment.

Suppose that consumption is expressed as a function

$$C = F(K, \dot{K}; L, \dot{L}),$$

where $K = (K_1, K_2, \dots, K_n)$ is the vector of capital goods and L is the labor input ($L > 0$), exogenously given by $\dot{L} = \lambda L$. Assume that $\partial^2 F/\partial \dot{K}_i \partial \dot{K}_j \neq 0$ (F is non-linear) and F is an homogeneous function of first degree.

Consumption per capita is given by

$$c = \frac{C}{L} = F\left(\frac{K_i}{L}, \frac{\dot{K}_i}{L}; 1, \frac{\dot{L}}{L}\right) = F(k_i, \dot{k}_i + \lambda k_i; 1; \lambda) = f(k_i, \dot{k}_i + \lambda k_i; \lambda),$$

since

$$\dot{k}_i = \frac{d}{dt}\left(\frac{K_i}{L}\right) = \frac{\dot{K}_i L - \dot{L} K_i}{L^2} = \frac{\dot{K}_i}{L} - \lambda k_i.$$

The society's objective is assumed to maximize the discounted future value of consumption per capita; i.e.

$$\max \int_0^\infty e^{-\rho t} f(k_i, \dot{k}_i + \lambda k_i; \lambda)\, dt.$$

The Lagrangian function is then

$$\mathbb{L}(t, k_i, \dot{k}_i) = e^{-\rho t} f(k_i, \dot{k}_i + \lambda k_i; \lambda),$$

and it is defined on $\mathbb{R} \times TQ$ where $Q = \mathbb{R}^n$ with coordinates (k_i).

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{k}_i}\right) - \rho \frac{\partial f}{\partial \dot{k}_i} - \frac{\partial f}{\partial k_i} = 0, \quad 1 \leq i \leq n.$$

By applying equation (2) we obtain that

$$\frac{d}{dt}(E_{\mathbb{L}}) = -\frac{\partial \mathbb{L}}{\partial t},$$

which gives

$$\frac{d}{dt}\left(e^{-\rho t} \dot{k}_i(t) \frac{\partial f}{\partial \dot{k}_i} - e^{-\rho t} f(k_i(t), \dot{k}_i(t) + \lambda k_i(t), \lambda)\right)$$
$$= -\rho e^{-\rho t} f(k_i(t), \dot{k}_i(t) + \lambda k_i(t), \lambda).$$

Integrating we have

$$e^{-\rho t}\left(\dot{k}_i(t) \frac{\partial f}{\partial \dot{k}_i} - f(k_i(t), \dot{k}_i(t) + \lambda k_i(t), \lambda)\right)$$
$$= -\rho \int_t^{+\infty} e^{-\rho s} f(k_i(s), \dot{k}_i(s) + \lambda k_i(s), \lambda)\, ds,$$

or, equivalently,

$$-\dot{k}_i(t) \frac{\partial f}{\partial \dot{k}_i} + f(k_i(t), \dot{k}_i(t) + \lambda k_i(t), \lambda)$$
$$= \rho \int_t^{+\infty} e^{\rho(t-s)} f(k_i(s), \dot{k}_i(s) + \lambda k_i(s), \lambda)\, ds.$$

This is precisely the income-wealth conservation law obtained by Weitzman (1976) (see also Sato (1994)):

"Income"$=\rho\cdot$"wealth".

6. A neoclassical von Neumann model

Following Kataoka and Hashimoto (1995) we consider the following generalization of the Samuelson's initial model (see Samuelson (1970)):

$$\max \int_0^T e^{-rt} p^i(t) \dot{K}_i(t)\, dt,$$

subjected to the constraint

$$F(K_i(t), \dot{K}_i(t)) = 0, \tag{4}$$

where $1 \leq i \leq n$ and $\partial F/\partial K_i > 0$, $\partial F/\partial \dot{K}_i < 0$, with given initial data $K_i(0) = K_{i0}$ and final time T. The function F is a C^1-neoclassical transformation funtion and it is linear homogeneous. The coordinates (K_i) are the capital stocks, $(\dot{K}_i)$ the capital formations, p^i the prices of the goods and r a constant discount rate.

The problem is mathematically described by a Lagrangian function:

$$\mathbb{L}(t, K_i, \dot{K}_i) = e^{-rt} p^i(t) \dot{K}_i$$

and the constraint $F = 0$. According to the precedent section, this constrained problem is equivalent to the unconstrained time-dependent Lagrangian system given by the Lagrangian function

$$\mathcal{L}(t, K_i, \lambda, \dot{K}_i, \dot{\lambda}) = e^{-rt} p^i(t) \dot{K}_i + \lambda F,$$

defined on $\mathbb{R} \times T(Q \times \mathbb{R})$.

The Euler-Lagrange equations are

$$e^{-rt}\left(-rp^i(t) + \dot{p}^i(t)\right) + \dot{\lambda}\frac{\partial F}{\partial \dot{K}_i} + \lambda \frac{\partial^2 F}{\partial \dot{K}_i \partial K_j} \dot{K}_j + \lambda \frac{\partial^2 F}{\partial \dot{K}_i \partial \dot{K}_j} \ddot{K}_j - \lambda \frac{\partial F}{\partial K_i} = 0$$

$$F(K_i, \dot{K}_i) = 0$$

It is not easy to solve these equations, so we will discuss the existence of symmetries and conservation laws for explicit values of the prices p^i, $1 \leq i \leq n$. As we know, the existence of such symmetries facilitates the integration of the equations.

(i) $p^i(t) = C^i e^{rt}$, with $(C^1, \ldots, C^n) \in \mathbb{R}^n$.
In this case, we have

$$\mathcal{L} = C^i \dot{K}_i + \lambda F,$$

which is autonomous. Therefore, the vector field $\partial/\partial t$ is an infinitesimal symmetry of $\mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial t} = 0.$$

By applying Proposition 3.3, we obtain the following conserved quantity:

$$\Theta_{\mathcal{L}}(\frac{\partial}{\partial t}) = -\left(C^i + \lambda \frac{\partial F}{\partial \dot{K}_i}\right)\dot{K}_i + \left(C^i \dot{K}_i + \lambda F\right) = \lambda K_i \frac{\partial F}{\partial K_i},$$

since F is homogeneous.

Observe that the above formula is precisely the Samuelson's first law (Samuelson (1970)).

In addition, the vector field $K_i \partial/\partial K_i$ is a Noether symmetry. Indeed, we have

$$K_i \frac{\partial \mathcal{L}}{\partial K_i} + \dot{K}_i \frac{\partial \mathcal{L}}{\partial \dot{K}_i} = \mathcal{L} = C^i \dot{K}_i + \lambda F = d_T(C^i K_i)$$

since $\mathcal{L}$ is homogeneous and $F = 0$ along the solutions. According to Proposition 3.5, we derive the corresponding conservation law:

$$C^i K_i - \Theta_L \left(K_i \frac{\partial}{\partial K_i} + \dot{K}_i \frac{\partial}{\partial \dot{K}_i}\right) = -\lambda K_i \frac{\partial F}{\partial \dot{K}_i}.$$

(ii) $p^i(t) = C^i e^{rt} + D^i t e^{rt}$, with $(C^1, \dots, C^n, D^1, \dots, D^n) \in \mathbb{R}^{2n}$.
In such a case, the Lagrangian is:

$$\mathcal{L} = (C^i + D^i t)\dot{K}_i + \lambda F$$

which admits $\partial/\partial t$ as a Noether symmetry. Indeed, we have

$$\frac{\partial \mathcal{L}}{\partial t} = D^i \dot{K}_i = d_T(D^i K_i).$$

Therefore, the associated conservation law is:

$$K_i \left(D^i - \lambda \frac{\partial F}{\partial \dot{K}_i}\right).$$

(iii) $p^i(t) = C^i e^{\beta t}$, with $(C^1, \dots, C^n, \beta) \in \mathbb{R}^{n+1}$.
The Lagrangian becomes

$$\mathcal{L} = C^i e^{-(r-\beta)t} \dot{K}_i + \lambda F$$

so that the vector field

$$X = \frac{\partial}{\partial t} + (r-\beta)K_i \frac{\partial}{\partial K_i}$$

is an infinitesimal symmetry of $\mathcal{L}$. Indeed, we have

$$\frac{\partial \mathcal{L}}{\partial t} + (r-\beta)K_i \frac{\partial \mathcal{L}}{\partial K_i} + (r-\beta)\dot{K}_i \frac{\partial \mathcal{L}}{\partial \dot{K}_i} = 0,$$

and the conserved quantity is:

$$(r-\beta)C^i e^{-(r-\beta)t} K_i + \lambda \frac{\partial F}{\partial \dot{K}_i} \left((r-\beta)K_i - \dot{K}_i \right).$$

(iv) $p^i(t) = C^i e^{\beta t} + D^i e^{rt}$, with $(C^1, \ldots, C^n, D^1, \ldots, D^n, \beta) \in \mathbb{R}^{2n+1}$. In this case, the Lagrangian becomes

$$\mathcal{L} = (C^i e^{-(r-\beta)t} + D^i)\dot{K}_i + \lambda F.$$

Thus, the vector field

$$X = \frac{\partial}{\partial t} + (r-\beta)K_i \frac{\partial}{\partial K_i}$$

is a Noether symmetry. In fact, we have

$$\frac{\partial \mathcal{L}}{\partial t} + (r-\beta)K_i \frac{\partial \mathcal{L}}{\partial K_i} + (r-\beta)\dot{K}_i \frac{\partial \mathcal{L}}{\partial \dot{K}_i} = d_T((r-\beta)D^i K_i),$$

and the associated conserved quantity is exactly the same as in the precedent case.

7. Conclusion

We have shown that the well-known methods to obtain conservation laws from infinitesimal symmetries in Geometric Mechanics can be applied to optimal economic growth problems. We have considered different types of infinitesinal symmetries: Lagrangian, Noether and Cartan symmetries, and all of them have been related with conservation laws via Noether theorems. To our knowledge the study of Cartan symmetries (also called hidden symmetries) is a new concept in the economics literature, and it would be interesting in further research. Finally, we have applied the results to two models: a model of economic growth of a nation, and a neoclassical von Neumann model.

The so-called vakonomic mechanics has proved to be the appropiate geometrical setting for economic models given by a Lagrangian subjected to constraints. A further development of vakonomic mechanics would permit to go deeply into this direction. This is one of our research interest in the next future.

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