Matched Pairs and Extensions of Lie Bialgebras

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1. Introduction

A Lie bialgebra is a Lie algebra also equipped with a Lie coalgebra compatible structure ([2], [3]). A Lie bialgebra morphism is a Lie algebra morphism which is also a Lie coalgebra morphism. Let $\mathfrak{g}$ and $A$ be finite dimensional Lie bialgebras over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A Lie bialgebra $\widehat{\mathfrak{g}}$ is called an extension of $\mathfrak{g}$ by $A$ if there exists an exact sequence $0 \to A \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0$ where $i$ and $\pi$ are Lie bialgebra morphisms. Two extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by $A$ are called equivalent if there exists a Lie bialgebra morphism $\rho : \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ such that the following diagram commutes:

![Diagram](https://via.placeholder.com/150)

We denote by $\text{Ext}_{\text{big}}(\mathfrak{g}, A)$ the set of all inequivalent Lie bialgebra extensions of $\mathfrak{g}$ by $A$.

For an arbitrary commutative Lie bialgebra $A$ and a general Lie bialgebra $\mathfrak{g}$ an explicit description of $\text{Ext}_{\text{big}}(\mathfrak{g}, A)$ can be found in ([1]). If $A \neq \mathbb{K}$ then the set $\text{Ext}_{\text{big}}(\mathfrak{g}, A)$ is not in general a group: it is described by a non-abelian cohomology of Lie bialgebras.

In this work we suppose that $\mathfrak{g}$ is a co-commutative Lie bialgebra to avoid non-abelian cohomology. The set $\text{Ext}_{\text{big}}(\mathfrak{g}, A)$ naturally admits an abelian group structure and it is isomorphic to the second cohomology group of a
differential complex constructed out of $g$ and $A$. More precisely, the data of an element of $\text{Ext}_{\text{big}}(g, A)$ induces a matched structure on the pair $(g, A)$. For such a fixed structure we give an explicit description of $\text{Ext}_{\text{big}}(g, A)$ and define a cohomology of the matched pair $(g, A)$ such that its second group is isomorphic to $\text{Ext}_{\text{big}}(g, A)$.

For a particular matched structure on $(g, g^*)$ we realize the double $D = g \bowtie g^*$ of $g$ as the co-central trivial extension of $g$ by $g^*$.

In the entire sequel $(g, A)$ denote an abelian pair of Lie bialgebras i.e. $g$ is a co-commutative Lie bialgebra and $A$ is a commutative Lie bialgebra. All vector spaces over $K = \mathbb{R}$ or $\mathbb{C}$ considered here are finite dimensional.

**2. DESCRIPTION OF $\text{Ext}_{\text{big}}(g, A)$ FOR A FIXED MATCHED PAIR**

Let $(E): 0 \to A \xrightarrow{i} \widehat{g} \xrightarrow{r} g \to 0$ be an extension of Lie bialgebras. By definition we obtain the following two Lie algebra extensions:

$$(E_1): 0 \to A \xrightarrow{i} \widehat{g} \xrightarrow{r} g \to 0,$$

$$(E_2): 0 \to g^* \xrightarrow{r^*} \widehat{g}^* \xrightarrow{i^*} A^* \to 0,$$

where $g^*$ (resp. $A^*$) is the dual Lie algebra of $g$ (resp. $A$) and $i^*$ (resp. $\pi^*$) denotes the transpose of $i$ (resp. $\pi$).

Extension $(E_1)$ induces a (left) $g$-module structure $\varphi: g \to \text{End}(A)$ and similarly extension $(E_2)$ determines an $A^*$-module structure $F: A^* \to \text{End}(g^*)$ on $g^*$. Using the fact that $\widehat{g}$ is a Lie bialgebra we obtain the following compatibilities between $\varphi$ and $F$ (see ([1])): $\forall \alpha, \beta \in A^*, \forall x, y \in g$,

$$F(\alpha)^*(\{x, y\}) = [F(\alpha)^*(x), y] + [x, F(\alpha)^*(y)]$$

$$+ F(\alpha \circ \varphi(x))^*(y) - F(\alpha \circ \varphi(y))^*(x), \quad (2.1)$$

$$[\alpha, \beta] \circ \varphi(x) = [\alpha \circ \varphi(x), \beta] + [\alpha, \beta \circ \varphi(x)]$$

$$+ \beta \circ \varphi(F(\alpha)^*x) - \alpha \circ \varphi(F(\alpha)^*y). \quad (2.2)$$

**DEFINITION.** ([3]) A matched pair structure on $(g, A)$ is the data of a $g$-module structure $\varphi$ on $A$ and an $A^*$-module structure $F$ on $g^*$ satisfying the identities (2.1) and (2.2).

If $(E)'$ is a Lie bialgebra extension equivalent to the given extension $(E)$, then the induced Lie algebra extensions $(E_1)'$ and $(E_2)'$ are equivalent to $(E_1)$ and $(E_2)$ respectively. As a consequence, extension $(E)'$ induces the same matched structure $(\varphi, F)'$ on $(g, A)$. Therefore, we can conclude the following:
PROPOSITION. The data of an equivalence class of Lie bialgebra extensions induces a matched pair structure on the pair \((g, A)\).

Let \((\varphi, F)\) be a fixed (but arbitrary) matched pair structure on \((g, A)\). Our aim is to describe the set \(\text{Ext}_{\text{bli}}(g, A)\) of Lie bialgebra inequivalent extensions of \(g\) by \(A\) inducing the given matched structure on the pair \((g, A)\).

Let \(0 \to A \xrightarrow{i} \widehat{g} \xrightarrow{\pi} g \to 0\) be an extension of \(g\) by \(A\). By choosing a linear section \(s\) of \(\pi\) we identify the vector spaces \(\widehat{g}\) and \(g \times A\); \((x, a) \equiv s(x) + i(a)\), where \(x \in g\) and \(a \in A\). The Lie bracket on \(\widehat{g} = g \times A\) is given by: \(\forall x, y \in g, \forall a, b \in A\),

\[
([x, a], (y, b)) = ([x, y], \varphi(x)(b) - \varphi(y)(a) + \gamma(x, y)),
\]

where \(\gamma : g \times g \to A\), \(\gamma(x, y) = [s(x), s(y)] - s([x, y])\), is a 2-cocycle, \(\gamma \in Z^2_{\text{alg}}(g, A)\), of the Lie algebra \(g\) with values in the \(g\)-module \(A\). The fact that \(i\) and \(\pi\) are Lie bialgebra morphisms implies that the Lie bracket on \(\widehat{g}^* \cong g^* \times A^*\) is necessarily of the following form: \(\forall \xi, \eta \in g^*, \forall \alpha, \beta \in A^*\),

\[
([\xi, \alpha], (\eta, \beta)) = (F(\alpha)(\eta) - F(\beta)(\xi)) + \Omega(\alpha, \beta), [\alpha, \beta]),
\]

where \(\Omega : A^* \times A^* \to g^*\) is a 2-cocycle, \(\Omega \in Z^2_{\text{alg}}(A^*, g^*)\), of the Lie algebra \(A^*\) with values in the \(A^*\)-module \(g^*\). The compatibility between these two brackets on the Lie bialgebra \(\widehat{g}\) implies the following condition \((1)\): \(\forall \alpha, \beta \in A^*, \forall x, y \in g\),

\[
\langle \Omega(\alpha, \beta), [x, y] \rangle + \langle \gamma(x, y), [\alpha, \beta] \rangle =
\]

\[
(F(\alpha)(\beta \circ \gamma(x)) - F(\beta)(\alpha \circ \gamma(x)) + \Omega(\alpha, \beta \circ \varphi(x)) - \Omega(\beta, \alpha \circ \varphi(x)), y)
\]

\[
- (F(\alpha)(\beta \circ \gamma(y)) - F(\beta)(\alpha \circ \gamma(y)) + \Omega(\alpha, \beta \circ \varphi(y)) - \Omega(\beta, \alpha \circ \varphi(y)), x)
\]

\(\gamma : g \to L(g, A)\) is given by \(\langle \gamma(x), y \rangle = \gamma(x, y)\), where \(L(g, A)\) is the vector space of linear maps of \(g\) in \(A\).

DEFINITION. We say that \(\gamma \in Z^2_{\text{alg}}(g, A)\) and \(\Omega \in Z^2_{\text{alg}}(A^*, g^*)\) are compatible if the identity \((2.5)\) is satisfied. We denote by \(Z^2_{\text{alg}}(g, A) \times Z^2_{\text{alg}}(A^*, g^*)\) the subspace of \(Z^2_{\text{alg}}(g, A) \times Z^2_{\text{alg}}(A^*, g^*)\) consisting of all compatible cocycles.

So we have established that to any extension \(\widehat{g}\) of \(g\) by \(A\) we can associate an element \((\gamma, \Omega)\) of \(Z^2_{\text{alg}}(g, A) \times Z^2_{\text{alg}}(A^*, g^*)\). Conversely, the data of such a pair gives an extension \(g \times A\) of \(g\) by \(A\) defined by the formulae \((2.3)\) and \((2.4)\).
A change of the chosen section \( s \) to another section \( s' = s + i \circ \theta \) of \( \pi \) with \( \theta \in L(g, A) \) transforms the pair \((\gamma, \Omega)\) to \((\gamma + \delta \theta, \Omega - \partial \theta^*)\). Here \( \delta \theta \) denotes the coboundary operator of the 1-cochain \( \theta : g \rightarrow A \) of the Lie algebra \( g \) with values in the \( g \)-module \( A \) and \( \partial \theta^* \) is the coboundary of the 1-cochain \( \theta^* : A^* \rightarrow g^* \) of the Lie algebra \( A^* \) with values in the \( A^* \)-module \( g^* \).

The map \((g \times A, s) \rightarrow (g \times A, s'), (x, a) \rightarrow (x, a + \theta(x))\) between trivializations of \( g \) defined by \( s \) and \( s' \) respectively is an equivalence of extensions. An isomorphism \( \rho \) defining an extension equivalence between \( g_1 \) and \( g_2 \) of \( g \) by \( A \) trivialized by \((g \times A, \gamma, \Omega)\) and \((g \times A, \gamma', \Omega')\) respectively is always of the above form. We deduce that \( g_1 \) and \( g_2 \) are equivalent if and only if there exists \( \theta \in L(g, A) \) such that \( \gamma' = \gamma + \delta \theta \) and \( \Omega' = \Omega - \partial \theta^* \).

So we have established the following result:

**Theorem.** There is a bijective correspondence between \( \text{Ext}_{\text{big}}(g, A) \) and the quotient \( B(g, A) \) of \( Z^2_{\text{sig}}(g, A) \times Z^2_{\text{sig}}(A^*, g^*) \) by \( \{ (\delta \varphi, - \partial \varphi^*) \mid \varphi \in L(g, A) \} \).

The set \( B(g, A) \) is an abelian group for the natural addition:

\[
((\gamma, \Omega)) + ((\gamma', \Omega')) = ((\gamma + \gamma', \Omega + \Omega'))
\]

where double parentheses denote equivalent classes in \( B(g, A) \).

3. \( B(g, A) \) IS A SECOND COHOMOLOGY GROUP

Let \((\varphi, F)\) be a fixed matched pair structure on \((g, A)\). The \( g \)-module structure \( \varphi \) on \( A \) naturally induces a \( g \)-module structure, which we also denote by \( \varphi \), on \( \wedge^q A \) for every integer \( q \geq 2 \):

\[
\varphi(x)(a_1 \wedge a_2 \wedge \ldots \wedge a_q) = \sum_{k=1}^{q} a_1 \wedge a_2 \wedge \ldots \wedge a_{k-1} \wedge \varphi(x)(a_k) \wedge a_{k+1} \wedge \ldots \wedge a_q.
\]

Similarly, \( \wedge^p g^* \) is endowed via \( F \) with \( A^* \)-module structure for every integer \( p \geq 2 \). We denote by \( \partial \) (resp. \( \delta \)) the coboundary operator of the Lie algebra \( A^* \) (resp. \( g \)) with values in the \( A^* \) (resp. \( g \)) module \( \wedge^q A \) (resp. \( \wedge^q g^* \)). An element \( \omega \) of \( \wedge^p g^* \otimes \wedge^q A \) is a \( p \)-cochain of the Lie algebra \( g \) with values in the \( g \)-module \( \wedge^q A \). This element \( \omega \) is also regarded (via its transpose) as a \( q \)-cochain of the Lie algebra \( A^* \) with values in the \( A^* \)-module \( \wedge^p g^* \). In this identification, the following result is an immediate fact by definition of \( \delta \) and \( \partial \).
Lemma.
\[ Z^2_{alg}(g, A) \times_c Z^2_{alg}(A^*, g^*) = \{(\gamma, \Omega) \in Z^2_{alg}(g, A) \times Z^2_{alg}(A^*, g^*) \mid \delta \Omega + \partial \gamma = 0 \}. \]

As \((g, A)\) is a matched pair of Lie bialgebras we obtain the following differential double complex:

\begin{align*}
\delta & \downarrow & \delta & \downarrow \\
\partial & \rightarrow & \wedge^p g^* \otimes \wedge^q A & \rightarrow & \partial & \rightarrow & \wedge^p g^* \otimes \wedge^{q+1} A & \rightarrow & \partial & \rightarrow \\
\delta & \downarrow & \delta & \downarrow \\
\partial & \rightarrow & \wedge^{p+1} g^* \otimes \wedge^q A & \rightarrow & \partial & \rightarrow & \wedge^{p+1} g^* \otimes \wedge^{q+1} A & \rightarrow & \partial & \rightarrow \\
\delta & \downarrow & \delta & \downarrow \\
\end{align*}

A. Masuoka ([4]) obtained this double complex from a double complex constructed on universal enveloping algebras of \(g\) and \(A\) which also form a matched pair of Hopf algebras.

Let us denote by \((T, D)\) the total differential complex associated to our double complex: \(T^n = \bigoplus_{p, q \geq 1} \wedge^p g^* \otimes \wedge^q A\) and \(D|_{\wedge^p g^* \otimes \wedge^q A} = \delta + (-1)^p \partial\).

Definition. The cohomology \(H^\bullet(g, A)\) of a matched pair \((g, A)\) is the cohomology of the total complex \((T, D)\) restricted to the intersection of the kernels of all vertical and horizontal operators.

As a consequence of the preceding lemma and theorem, we conclude the following:

Theorem. Set \(\mathcal{B}(g, A)\) is the second cohomology group \(H^2_{big}(g, A)\) of the matched pair \((g, A)\).

This result is analogous to the work of W. M. Singer ([5]) on Hopf algebras. Let \(H_1\) be a commutative Hopf algebra and \(H_2\) a co-commutative Hopf algebra and let \(\text{Ext}(H_2, H_1)\) denote the set of all inequivalent Hopf algebra extensions of \(H_2\) by \(H_1\). An element of \(\text{Ext}(H_2, H_1)\) determines a matched structure on the pair \((H_2, H_1)\). For such a fixed structure on the pair \((H_2, H_1)\), the set \(\text{Ext}(H_2, H_1)\) is an abelian group isomorphic to the second cohomology group of a differential complex constructed out of the given matched pair \((H_2, H_1)\), similarly as for Lie bialgebras.
4. THE DOUBLE OF A CO-COMMUTATIVE LIE BIALGEBRA.

The double $\mathcal{D} = \mathfrak{h} \bowtie \mathfrak{h}^*$ of a Lie bialgebra $\mathfrak{h}$ is the vector space $\mathcal{D} = \mathfrak{h} \oplus \mathfrak{h}^*$ endowed with the following Lie bracket and Lie co-bracket:

$$[[x, \xi], [y, \eta]]_{\mathfrak{h} \oplus \mathfrak{h}^*} = ([x, y]_{\mathfrak{h}}, \text{coad}_{\eta} x - \text{coad}_{\xi} y, [\xi, \eta]_{\mathfrak{h}^*} + \text{coad}_x \eta - \text{coad}_y \xi),$$

$$[[\xi, x], [\eta, y]]_{\mathfrak{h}^* \oplus \mathfrak{h}} = ([\xi, \eta]_{\mathfrak{h}^*}, -[x, y]_{\mathfrak{h}}).$$

In the following coad$_x x$ denotes the coadjoint action of $\eta \in \mathfrak{h}^*$ on $x \in \mathfrak{h}$ $\cong (\mathfrak{h}^*)^*$ ($\mathfrak{h}$ is finite dimensional) and coad$_x \eta$ is the coadjoint action of $x \in \mathfrak{h}$ on $\eta \in \mathfrak{h}^*$. Endowed with the preceeding structures the double $\mathcal{D} = \mathfrak{h} \bowtie \mathfrak{h}^*$ is in fact a Lie bialgebra.

Let $\mathfrak{g}$ be a co-commutative Lie bialgebra. The commutative (not co-commutative) Lie bialgebra $A = \mathfrak{g}^*$ is a $\mathfrak{g}$-module for the coadjoint action ($\theta = \text{coad}$) and $\mathfrak{g}^*$ is considered as a trivial $A^* = \mathfrak{g}$-module i.e. $F = 0$. With these data the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a matched pair of Lie bialgebras; condition (2.1) is trivially verified and condition (2.2) is reduced to a Jacobi identity in $\mathfrak{g}$. The brackets (2.3) and (2.4) of an extension $\mathfrak{g} \oplus \mathfrak{g}^*$ of $\mathfrak{g}$ by $\mathfrak{g}^*$ described by the zero class $((0, 0)) \in \text{H}_\text{cyc}^2 (\mathfrak{g}, \mathfrak{g}^*)$ are given by:

$$[[x, \xi], [y, \eta]]_{\mathfrak{g} \oplus \mathfrak{g}^*} = ([x, y], \text{coad}_{\eta} (\xi - \varphi(y)) - \text{coad}_{\xi} (\eta - \varphi(x)) + \varphi([x, y])),$$

$$[[\xi, x], [\eta, y]]_{\mathfrak{g}^* \oplus \mathfrak{g}} = (-\varphi([x, y]), [x, y]),$$

where $\varphi$ is an element of $L(\mathfrak{g}, \mathfrak{g}^*)$. If $\varphi = 0$ then we obtain the double $\mathfrak{g} \bowtie \mathfrak{g}^*$ with the opposite bracket on its dual $\mathfrak{g}^* \oplus \mathfrak{g}$.

A co-central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by $A$ is an extension of $\mathfrak{g}$ by $A$ such that the dual extension $\widehat{\mathfrak{g}}^*$ is a central extension of $A^*$ by $\mathfrak{g}^*$. This holds since here $F = 0$.

**Prosposition.** The double $\mathfrak{g} \bowtie \mathfrak{g}^*$ of a co-commutative Lie bialgebra is the trivial co-central extension of $\mathfrak{g}$ by the $\mathfrak{g}$-module $\mathfrak{g}^*$ for the coadjoint action, except for one sign.

**Remark.** The double $\mathfrak{h} \bowtie \mathfrak{h}^*$ of an arbitrary Lie bialgebra $\mathfrak{h}$ is not an extension of $\mathfrak{h}$ by $\mathfrak{h}^*$; the natural projection $\mathfrak{h} \bowtie \mathfrak{h}^* \rightarrow \mathfrak{h}$ is not a Lie algebra morphism.
LIE BIALGEBRAS

REFERENCES
