EXTRACTA MATHEMATICAE Vol. **13**, Núm. 1, 69 – 72 (1998)

A Note on the L^1-Convergence of a Superadditive Bisexual Galton-Watson Process

M. González and M. Molina

Dpto. de Matemáticas, Univ. de Extremadura, 06071-*Badajoz, Spain*

AMS *Subject Class.* (1991): 60J80

Received February 18, 1998

1. Introduction

Introduced by [3] as a modification, with sexual reproduction, of the standard Galton-Watson process, the bisexual Galton-Watson process (BGWP) is a two type branching model $\{(F_n, M_n)\colon n = 1, 2, \dots\}$ defined in the following recursive manner:
$Z_0 = N \geq 1$,

$$(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (F_{ni}, M_{ni}), \quad Z_{n+1} = L(F_{n+1}, M_{n+1}), \quad n = 0, 1, \dots$$

with the empty sum defined to be (0,0), where F_{ni} and M_{ni}, respectively, represent the number of females and males produced by the i-th mating unit in the nth generation, being $\{(F_{ni}, M_{ni})\colon i = 1, 2 \dots, n = 0, 1, \dots\}$ a sequence of integer-valued, independent and identically distributed random variables, and the mating function $L\colon \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, is assumed to be monotonic non-decreasing in each argument, integer-valued for integer-valued arguments and such that $L(x, y) \leq xy$. Thus, F_n and M_n represent, respectively, the number of females and males in the nth generation, which form $Z_n = L(F_n, M_n)$ mating units. These mating units reproduce, independently, through the same offspring distribution for each generation. We assume that the offspring distribution and the mating function are such that $Pr(Z_n \to 0) + Pr(Z_n \to \infty) = 1$ holds. It is easy to verify that the sequences $\{(F_n, M_n)\}$ and $\{Z_n\}$, are Markov chains with stationary transition probabilities.

A BGWP is said to be superadditive if the mating function verifies:

$$L(x_1 + x_2, y_1 + y_2) \geq L(x_1, y_1) + L(x_2, y_2), \quad x_i, y_i \in \mathbb{R}^+, \quad i = 1, 2$$

In the last years, the BGWP has received some attention in the scientific literature. Investigating the extinction problem, [2] introduced the concept of average reproduction mean per mating unit, defined as

$$r_k := k^{-1}E[Z_{n+1}|Z_n = k], \quad k = 1, 2, \dots ,$$

and subsequently, [4] defined, for superadditive BGWPs, the asymptotic growth rate $r := \lim_{k\to\infty} r_k$, and proved that:

1. $r = \sup_{k>0} r_k = \lim_{k\to\infty} k^{-1}L(kE[F_{01}], kE[M_{01}])$.
2. If for $j = 1, 2, \dots$, q_j denotes the extinction probability when $Z_0 = j$, then $q_j = 1$ for all $j = 1, 2, \dots$ if and only if $r \leq 1$.

Thus, for $r > 1$, we know that if Z_0 is large enough, there is a positive probability of survival, and for this situation will be important to study the limit behaviour of the process. [1], using the mating function $L(x, y) = x \min\{1, y\}$ and considering the Daley's scheme, namely he supposes $T_{ni} = F_{ni} + M_{ni}$, $i = 1, \dots , Z_n$, where $\{T_{ni}\}$ are independent and identically distributed random variables, and each offspring is female with probability α $(0 < \alpha < 1)$ or male with probability $1 - \alpha$, has provided some asymptotic results. Recently, [5], [6], [7] have studied the limit behaviour for any superadditive mating function. In particular, in [5], section 4, a sufficient condition for the L^1-convergence, under a $Z \log^+ Z$ condition, to a non-degenerate variable, of the sequences $\{r^{-n}Z_n\}$, $\{r^{-n}F_n\}$ and $\{r^{-n}M_n\}$, was given. The purpose of this note is provided a necessary and sufficient condition for the L^1-convergence, using the classical condition $Z \log^+ Z$. As consequence, we obtain the L^1-convergence for some historical superadditive BGWPs.

2. Convergence under the $Z \log^+ Z$ condition

We consider a superadditive BGWP with $r > 1$. Let $W_n = r^{-n}Z_n$, $F_n^* = r^{-n}F_n$, $D_n^* = F_n^* - r^{-1}W_nE[F_{01}]$ and $\epsilon_n = r - r_n$, $n = 1, 2, \dots$

Theorem. *Suppose that*

1. $\{\epsilon_n\}$ *is monotonic non-decreasing and such that* $\sum_{n=1}^{\infty} \epsilon_n/n < +\infty$.
2. $D_n^* \geq 0$ *for all* n.

Then $\{F_n^*\}$ converges in L^1 to a non-degenerate in 0 random variable if and only if $E[F_{01}\log^+ F_{01}] < +\infty$.

Proof. $D_n^* \geq 0$ implies that $D_n = F_n^* - r^{-1}W_nE[F_{01}I_{\{F_{01}\leq r^n\}}] \geq 0$, consequently taking into account i) the sufficient condition is proved as in [5], theorem 4.1.

We now prove the necessary condition.

If $\{F_n^*\}$ converges to a non-degenerate in 0 limit W, then $Pr(W > 0) > 0$ and hence $Pr(W^* > 0) > 0$ where $W^* = \inf_n W_n$.

On the other hand $\sum_{n=1}^{\infty}(F_{n+1}^* - F_n^* + D_n) < +\infty$ a.s. and $\sum_{n=1}^{\infty}(F_{n+1}^* - F_n^*) < +\infty$ a.s. Therefore it is derived that $\sum_{n=1}^{\infty} D_n < +\infty$ a.s. and taking into account that $D_n \geq 0$ and $D_n^* \geq 0$ for all n, we have that

$$W^* \sum_{n=0}^{\infty} E[F_{01}I_{\{F_{01}>r^n\}}] \leq \sum_{n=0}^{\infty} W_nE[F_{01}I_{\{F_{01}>r^n\}}] < +\infty \quad a.s.$$

Then $\sum_{n=0}^{\infty} E[F_{01}I_{\{F_{01}>r^n\}}] < +\infty$ which is equivalent to $E[F_{01}\log^+ F_{01}] < +\infty$ and this completes the proof. ∎

Remark. By symmetry, it is clear that we can rewrite the theorem, to provide a necessary and sufficient condition which guarantees the L^1-convergence to a non-degenerate in 0 limit of $\{M_n^*\}$, with $M_n^* = r^{-n}M_n$, assuming i) and $D_n^{**} = M_n^* - r^{-1}W_nE[M_{01}] \geq 0$ for all n.

Remark. Can be verified that the condition $L(x,y) \leq xrE^{-1}[F_{01}]$ (respectively $L(x,y) \leq yrE^{-1}[M_{01}]$) is sufficient to $D_n^* \geq 0$ (respectively $D_n^{**} \geq 0$) for all n.

COROLLARY. *For a BGWP with mating function $L(x,y) = x\min\{1,y\}$ and such that the Daley's scheme holds, it is verified that $\{F_n^*\}$ converges in L^1 to a non-degenerate in 0 random variable if and only if $E[F_{01}\log^+ F_{01}] < +\infty$.*

Proof. If Daley's scheme holds, it is deduced that $\sum_{n=1}^{\infty}\epsilon_n/n < +\infty$, and will be sufficient to verify that $D_n^* \geq 0$ for all n

$$\begin{aligned} D_n^* &= F_n^* - r^{-1}W_nE[F_{01}] = F_n^* - r^{-(n+1)}F_n\min\{1,M_n\}E[F_{01}] \\ &= F_n^*(1-\min\{1,M_n\}) \geq 0 \quad \text{for all } n \end{aligned}$$

where we have used that for the mating function considered $r = E[F_{01}]$. ∎

COROLLARY. *For a BGWP with mating function $L(x,y) = \min\{x,y\}$ and such that the Daley's scheme holds, we have*

1. *If $E[F_{01} \log^+ F_{01}] < +\infty$ (iff $E[M_{01} \log^+ M_{01}] < +\infty$), then F_n^* (equivalently M_n^*) converges in L^1 to a non-negative and finite random variable W.*

2. *If W is non-degenerate in 0, then it is verified that $E[F_{01} \log^+ F_{01}] < +\infty$ (iff $E[M_{01} \log^+ M_{01}] < +\infty$).*

Proof. If Daley's scheme is verified then $E[F_{01} \log^+ F_{01}] < +\infty$ is equivalent to $E[M_{01} \log^+ M_{01}] < +\infty$ (see [1]). Consequently, we may suppose, without loss of generality, that $r = \min\{E[F_{01}], E[M_{01}]\} = E[F_{01}]$. Therefore

$$D_n^* = F_n^* - r^{-1} W_n E[F_{01}] = F_n^* - \min\{F_n^*, M_n^*\} \geq 0 \quad \text{for all } n$$

According to [5], theorem 4.1, the first assertion of the corollary is proved.

The second one is derived from the necessary condition of the theorem, considering that for all n, either $D_n^* \geq 0$ or $D_n^{**} \geq 0$. ∎

REFERENCES

[1] BAGLEY, J.H., On the asymptotic properties of a supercritical bisexual branching process, *J. Appl. Prob.,* **23** (1986), 820–826.

[2] BRUSS, F.T., A note on extinction criteria for bisexual Galton-Watson branching processes, *J. Appl. Prob.,* **21** (1984), 915–919.

[3] DALEY, D.J., Extinction condition for certain bisexual Galton-Watson branching processes, *Z. Wahrscheinlichkeitsth,* **9** (1968), 315–322.

[4] DALEY, D.J., HULL, D.M., TAYLOR, J.M., Bisexual Galton-Watson branching processes with superadditive mating functions, *J. Appl. Prob.,* **23** (1986), 585–600.

[5] GONZÁLEZ, M., MOLINA, M., On the limit behaviour of a superadditive bisexual Galton-Watson branching process, *J. Appl. Prob.,* **33** (1996), 960–967.

[6] GONZÁLEZ, M., MOLINA, M., On the L^2-convergence of a superadditive bisexual Galton-Watson branching process, *J. Appl. Prob.,* **34** (1997), 575–582.

[7] GONZÁLEZ, M., MOLINA, M., Some theoretical results on the progeny of a superadditive bisexual Galton-Watson branching process, *Serdica Math. J.,* **23** (1997), 15–24.