On the Spectra of Elements in Certain Algebras of Vector Valued Functions and Sequences

S. DIEROLF AND KHIN AYE AYE

FB IV-Mathematik, Universität Trier, D 54286-Trier, Germany

University of Yangon, Yangon, Myanmar

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1. TERMINOLOGY AND NOTATIONS.

By an algebra $A$ we will always mean an associative algebra over the field $\mathbb{C}$ of complex numbers.

$\sigma(A) := \{ \chi : A \rightarrow \mathbb{C} \text{ linear, multiplicative, } \neq 0 \}$ denotes the set of all characters on $A$ and for $x \in A$ let $\sigma_A(x)$ denote the spectrum of $x$ with respect to $A$. An element $x \in A$ is called quasiinvertible element in $A$, if there is a (so-called quasiinverse) element $y \in A$ such that $xy = yx = x + y$ and the quasiinverse element is uniquely determined. Let $Q(A)$ denote the set of quasiinvertible elements in $A$, and $q : Q(A) \rightarrow Q(A)$ the map which assigns to each $x \in Q(A)$ its quasiinverse element; we will call it the quasiinversion in $A$.

If $A$ has a unit element $e$, then $e - Q(A) = G(A)$ (group of invertible elements in $A$) and for each $x \in Q(A)$ one has $(e - x)^{-1} = e - q(x)$. Let $A_e := A \times \mathbb{C}$ denote the algebra which arises from $A$ by the formal adjunction of a unit element. Then for $(x, \lambda) \in A_e$ we have $(x, \lambda) \in Q(A_e) \iff \lambda \neq 1$ and $\frac{1}{1-\lambda} x \in Q(A)$, and if $(x, \lambda) \in Q(A_e)$, then its quasiinverse element in $A_e$ is equal to $\left(\frac{1}{1-\lambda} q\left(\frac{1}{1-\lambda} x\right), \frac{1}{1-\lambda}\right)$. An algebra $A$ provided with a locally convex topology is called a locally convex algebra, if multiplication is jointly continuous. A locally convex algebra $A$ is called locally $m$-convex, if its $0$-nbhd-filter has a basis of sets $U$ satisfying $U^2 \subset U$.

LEMMA. Let $A$ be an algebra and $I \subset A$ a proper ideal. Then

$\sigma_I(x) \cup \{0\} = \sigma_A(x), \; \forall x \in I.$

If $I$ does not contain a unit element, then

$\sigma_I(x) = \sigma_A(x), \; \forall x \in I.$
Proof. Clearly, as $I$ is a proper ideal in $A$, $\sigma_A(x)$ contains 0 for every $x \in I$. Now let $\lambda \in \mathbb{C} \setminus \{0\}$ and $x \in I$. If $\frac{1}{\lambda}x$ is quasiinertible in $I$, then it is also quasiinertible in $A$. Conversely, let $y \in A$ such that $\frac{1}{\lambda}xy = y(\frac{1}{\lambda}x) = \frac{1}{\lambda}x + y$. But then $y = \frac{1}{\lambda}xy - \frac{1}{\lambda}x$ is already contained in $I$. The last assertion is obvious.

As a first easy application of the lemma we obtain a description of the spectrum of elements in a product of algebras:

Let $(A_i)_{i \in I}$ be a family of algebras and $x = (x_i)_{i \in I} \in A := \prod_{i \in I} A_i$. Then

$$\sigma_A(x) = \bigcup_{i \in I} \sigma_{A_i}(x_i).$$

In fact, if each $A_i$ has a unit element, then $G(A) = \prod_{i \in I} G(A_i)$, which yields the assertion. Otherwise, put $J = \{ i \in I : A_i$ does not have a unit $\}$; then $A$ is a proper ideal without unit element in $\prod_{i \in J} (A_i)_e \times \prod_{i \in I \setminus J} A_i$ and the lemma yields the assertion.

Next we are going to study the spectrum of elements in algebras $C(T, A)$ of all continuous functions $f : T \to A$ where $T$ is a completely regular Hausdorff space and $A$ a locally convex algebra (provided with pointwise operations). In [2] the set $\sigma(C(T, A))$ of characters on $C(T, A)$ was described, in the case that $A$ is metrizable and realcompact (as a topological space): The characters on $C(T, A)$ are exactly of the form $\chi \circ \delta_x$, where $\chi \in \sigma(A)$ and $\delta_x$ is the evaluation (of the continuous extension) in a point $x$ in the realcompactification $\nu X$ of $X$. We will now describe the spectrum of an element in such an algebra.

**Proposition 1.** Let $T$ be a completely regular Hausdorff space, $A$ a locally convex algebra with continuous quasiinversion $q$ and let $f \in C(T, A)$. Then

$$\sigma_{C(T, A)}(f) = \bigcup_{t \in T} \sigma_A(f(t)).$$

**Proof.** We may assume that $A$ contains a unit element $e$. In fact, if $A$ does not, $C(T, A)$ is a proper ideal without unit in $C(T, A_e)$, which by the lemma gives $\sigma_{C(T, A)}(f) = \sigma_{C(T, A_e)}(f)$ and clearly, $\sigma_{A_e}(f(t)) = \sigma_A(f(t))$ for all $t \in T$. Moreover, quasiinversion and hence inversion are continuous in $A_e$.

We will show that for any $g \in G(C(T, A))$,

$$g \in G(C(T, A)) \iff g(t) \in G(A), \forall t \in T,$$

(which yields the assertion). In fact, if $g(t) \in G(A)$ for all $t \in T$, then $h : T \to A, t \mapsto g(t)^{-1}$ is continuous, hence inverse to $g$ in $C(T, A)$. The converse implication is trivial. 


Remarks. 1) The hypothesis about continuity of quasi-inversion in $A$ is essential. In fact, there exist even complete metrizable locally convex algebras with unit and discontinuous quasi-inversion hence quasi-inversion, e.g. the Arens-algebra $\bigcap_{p>1} L^p([0,1])$ (see [1]). $T := G(A)$ provided the relative topology is metrizable hence completely regular and Hausdorff, and the inclusion $j: T \rightarrow A$ is not invertible in $C(T,A)$, but of course $j(t) \in G(A)$ for all $t \in T$. Thus $0 \in \sigma_{G(T,A)}(j) \cup \sigma_{G(A)}(j(t))$. On the other hand, every locally $m$-convex algebra has continuous quasi-inversion.

2) Let $T$ be a completely regular Hausdorff space and $A$ a locally convex algebra with continuous quasi-inversion such that

$$\{ \chi(x): \chi \in \sigma(A) \} \subset \sigma_A(x) \subset \{ \chi(x): \chi \in \sigma(A) \} \cup \{0\}$$

for all $x \in A$. (All commutative Banach algebras have this last property). Then $C(T,A)$ has the same properties. In fact, we must only show that

$$\sigma_{C(T,A)}(f) \subset \{ \chi(f): \chi \in \sigma(C(T,A)) \} \cup \{0\}.$$ 

Let $\lambda \in \sigma_{C(T,A)}(f) \setminus \{0\}$. Then, by proposition 1, there is $t \in T$ such that $\lambda \in \sigma_A(f(t))$. By hypothesis, there is $\psi \in \sigma(A)$ such that $\lambda = \psi(f(t))$.

Certainly $\chi: C(T,A) \rightarrow \mathbb{C}$, $g \mapsto \psi(g(t))$ is a character on $C(T,A)$.

Next we will study the spectrum of elements in algebras of vector-valued sequences.

Let $(\lambda, \|\cdot\|)$ be a normal Banach sequence space, i.e. $(\lambda, \|\cdot\|)$ is a Banach space such that $\bigoplus_{\mathbb{N}} \mathbb{C} \subset \lambda \subset \mathbb{C}^\mathbb{N}$ such that for all $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in \lambda$ and all $\beta = (\beta_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$,

$$||\beta_n| \leq |\alpha_n|, \forall n \in \mathbb{N} \Rightarrow (\beta \in \lambda \text{ and } ||\beta|| \leq ||\alpha||).$$

For every $n \in \mathbb{N}$ the number $\rho_n := ||(\beta_{k,n})_{k \in \mathbb{N}}||$ is positive. Provided with the multiplication $\lambda \times \lambda \rightarrow \lambda$, $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}} \mapsto (\rho_n \alpha_n \beta_n)_{n \in \mathbb{N}}$, the Banach space $(\lambda, ||\cdot||)$ is a Banach algebra. If $A$ is a locally convex algebra and $cs(A)$ the set of continuous seminorms on $A$, then

$$\lambda(A) := \{ (a_n)_{n \in \mathbb{N}} \in A^\mathbb{N}: (p(a_n))_{n \in \mathbb{N}} \in \lambda, \forall p \in cs(A) \}$$

provided with the locally convex topology generated by the seminorms

$$\hat{p}: \lambda(A) \rightarrow [0, \infty), (a_n)_{n \in \mathbb{N}} \mapsto \{(p(a_n))_{n \in \mathbb{N}}\}$$

is a locally convex algebra with respect to the multiplication $(a_n)_{n \in \mathbb{N}}(b_n)_{n \in \mathbb{N}} := (\rho_n a_n b_n)_{n \in \mathbb{N}}$. For the case that $(\lambda, ||\cdot||)$ has sectional convergence (i.e., $||(0)_{k \leq n,}$


\((\alpha_k)_{k \geq 0}|| \stackrel{n \to \infty}{\longrightarrow} 0\) for all \((\alpha_n)_{n \in \mathbb{N}} \in \lambda\), the set of characters on \(\lambda(A)\) was characterized in [2], namely

\[
\sigma(\lambda(A)) = \{\chi \circ \text{pr}_n : n \in \mathbb{N}, \chi \in \sigma(A)\}.
\]

We will now investigate the spectrum of elements in such algebras.

**Proposition 2.** Let \((\lambda, ||.||)\) be a normal Banach sequence space with sectional convergence, let \(A\) be a locally convex algebra with sequentially continuous quasi-inversion and let \(x = (x_n)_{n \in \mathbb{N}} \in \lambda(A)\). Then

\[
\sigma_{\lambda(A)}(x) = \bigcup_{n \in \mathbb{N}} \sigma_A(\rho_n x_n) \cup \{0\}
\]

(where \(\rho_n := ||(\delta_{kn})_{k \in \mathbb{N}}||, \ (n \in \mathbb{N})\)).

**Proof.** The map \(\lambda(A) \longrightarrow c_0(A), (a_n)_{n \in \mathbb{N}} \longmapsto (\rho_n a_n)_{n \in \mathbb{N}}\) is injective, linear, multiplicative and its range is an ideal in \(c_0(A)\) without unit element. Then by the lemma, we may assume that \((\lambda, ||.||) = (c_0, ||.||_{\infty})\), and we must prove that for each \(y = (y_n)_{n \in \mathbb{N}} \in c_0(A)\) one has \(\sigma_{c_0(A)}(y) = \bigcup_{n \in \mathbb{N}} \sigma_A(y_n) \cup \{0\}\).

As, clearly, \(0 \in \sigma_{c_0(A)}(y)\), we have to show that \(z \in Q(c_0(A))\) if and only if \(z_n \in Q(A)(n \in \mathbb{N})\), whenever \(z = (z_n)_{n \in \mathbb{N}} \in c_0(A)\). The only part is obvious. So let \(z = (z_n)_{n \in \mathbb{N}} \in c_0(A)\) be given such that \(z_n \in Q(A)\) for all \(n \in \mathbb{N}\). We will be done, if we show that \(y := (q(z_n))_{n \in \mathbb{N}}\) belongs to \(c_0(A)\) \((q\) denoting quasi-inversion in \(A\), as before). But this is clear, as \((z_n)_{n \in \mathbb{N}}\) tends to \(0\) in \(Q(A) \subset A\) and \(q\) is sequentially continuous on \(Q(A)\).

**Remarks.**

1) Again the assumption about sequential continuity of the quasi-inversion in \(A\) cannot be omitted. In fact, let \(A\) again denote the Arens-algebra (see the Remark above). Then by the metrizability of \(A\) there is a sequence \((a_n)_{n \in \mathbb{N}} \subset Q(A)\) tending to an element \(a\) in \(Q(A)\) such that \((q(a_n))_{n \in \mathbb{N}}\) does not converge to \(q(a)\). Then \(x_n := (x_n)_{n \in \mathbb{N}} := (a_n + q(a_n - x_n a_n)_{n \in \mathbb{N}}\) belongs to \(c_0(A)\), \(x_n \in Q(A)\) and \(q(x_n) = a + q(a_n) - a q(a_n)\). But \((q(x_n))_{n \in \mathbb{N}}\) does not belong to \(c_0(A),\) because otherwise \((q(a_n))_{n \in \mathbb{N}}\) would converge to \(-e - a^{-1} a = -(e - q(a))a = q(a)\). Thus \(1 \in \sigma_{c_0(A)}(x) \setminus \bigcup_{n \in \mathbb{N}} \sigma_A(x_n)\).

2) Let \((\lambda, ||.||)\) be a normal Banach sequence space with sectional convergence and \(A\) a locally convex algebra with sequentially continuous quasi-inversion such that \(\{\chi(x) : \chi \in \sigma(A)\}\) \(\subset \sigma_A(x) \subset \{\chi(x) : \chi \in \sigma(A)\}\) \(\cup \{0\}\) for all \(x \in A\). Then \(\lambda(A)\) has the same properties, as is immediately clear from the description of \(\sigma_{\lambda(A)}((x_n)_{n \in \mathbb{N}})\).
3) Let \((\lambda, \|\cdot\|)\) be a normal Banach sequence space, \(\rho_n := \|(\delta_{kn})_{k \in \mathbb{N}}\| (n \in \mathbb{N})\); then for every locally convex algebra \(A\) the map

\[
\lambda(A) \rightarrow l^\infty(A), (a_n)_{n \in \mathbb{N}} \mapsto (\rho_n a_n)_{n \in \mathbb{N}}
\]
is injective, linear, multiplicative and its range is an ideal in \(l^\infty(A)\). Thus if one could describe the characters on \(l^\infty(A)\) or the spectrum \(\sigma_{l^\infty(A)}((a_n)_{n \in \mathbb{N}})\), this would allow such a description for all \(\lambda(A)\). Unfortunately, the case \(l^\infty(A)\) is not easy to handle (unless the bounded sets in \(A\) are relatively compact, which leads to \(C(\beta\mathbb{N}, A)\)) even if \(A\) is a Banach algebra.

We owe the following two observations to L. Frerick and J. Wengenroth (oral communication):

\(\alpha\) Let \(A\) be a commutative \(C^*\)-algebra with unit element \(e\). Then

\[
\sigma_{l^\infty(A)}((x_n)_{n \in \mathbb{N}}) = \bigcup_{n \in \mathbb{N}} \sigma_A(x_n)
\]
for all \((x_n)_{n \in \mathbb{N}} \in l^\infty(A)\). In fact, it suffices to prove "\(\subset\)". Let

\[
\lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{N}} \sigma_A(x_n).
\]

Then \(\lambda e - x_n \in G(A)\) for all \(n \in \mathbb{N}\). Assume that the sequence \(\|(\lambda e - x_n)^{-1}\|\) is unbounded. Let \(\varepsilon > 0\) be given. Then there is \(n \in \mathbb{N}\) such that \(||(\lambda e - x_n)^{-1}|| > \frac{1}{\varepsilon} \) and therefore (\(A\) being a \(C^*\)-algebra) there is \(\mu \in \sigma_A((e - x_n)^{-1})\) such that \(|\mu| > \frac{1}{\varepsilon}\). Consequently

\[
\mu e - (\lambda e - x_n)^{-1} = (\mu(e - x_n) - e)(\lambda e - x_n)^{-1}
\]

\[
= \mu((\lambda - \frac{1}{\mu})e - x)(\lambda e - x_n)^{-1} \not\in G(A)
\]

and thus \(\lambda - \frac{1}{\mu} \in \sigma_A(x_n) \subset \bigcup_{m \in \mathbb{N}} \sigma_A(x_m)\). As \(\frac{1}{|\mu|} < \varepsilon\), we obtain that

\[
\lambda \in \bigcup_{m \in \mathbb{N} \sigma_A(x_m),}
\]

Thus \(\|(\lambda e - x_n)^{-1}\|_{n \in \mathbb{N}} \in l^\infty(A)\), hence \(\lambda e_n \in G(l^\infty(A))\) and \(\lambda \not\in \sigma_{l^\infty(A)}((x_n)_{n \in \mathbb{N}})\).

\(\beta\) The description in \(\alpha\) for the spectrum of elements in \(l^\infty(A)\) is no longer true for non-commutative \(C^*\)-algebra \(A\). In fact, let \(A\) be the \(C^*\)-algebra of linear bounded operators in \(l^2\). For each \(n \in \mathbb{N}\) let \(x_n \in A\) be defined by

\[
x_n((\delta_{kl})_{l \in \mathbb{N}}) := \begin{cases} (\delta_{k+1,l})_{l \in \mathbb{N}} & \text{if } k \leq n + 1 \\ (0)_{l \in \mathbb{N}} & \text{if } k > n + 1 \end{cases}
\]
Then \( \|x_n\|_{op} = 1 \) for all \( n \in \mathbb{N} \), hence \((x_n)_{n \in \mathbb{N}} \in l^\infty(A)\). Since each \( x_n \) is nilpotent, \( \sigma_A(x_n) = \{0\} \) for all \( n \in \mathbb{N} \). On the other hand, \( \sqrt{\sup_{n \in \mathbb{N}} \|x_n^k\|} = 1 \) for each \( k \in \mathbb{N} \), as \( \|x_n^k\| = 1 \). Therefore there is \( \lambda \in \sigma_{l^\infty(A)}((x_n)_{n \in \mathbb{N}}) \) such that \( |\lambda| = 1 \).

**References**
