$R$–Schauder Decompositions. Some Applications

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1. INTRODUCTION AND NOTATION

In recent years several authors have been interested in describing the bidual of some subspaces of $\mathcal{P}(X)$ (the space of continuous polynomials on a Banach space $X$) as subspaces of $\mathcal{P}(X^{**})$. See, for instance, [2], [5], [6] and [10]. The aim of this work is to extend these results to holomorphic functions. Related to this, Prieto obtains some interesting results in [9]. According to her one has the following situation: $(\mathcal{P}(mX))_m$ and $(\mathcal{P}_{wu}(mX))_m$ are Schauder decompositions of $\mathcal{H}_b(X)$ and $\mathcal{H}_{wu}(X)$ respectively; hence, topological isomorphisms between $\mathcal{P}(mX)$ and $\mathcal{P}_{wu}(mX)$ for all $n \in \mathbb{N}$ apparently yield to a topological isomorphism between $\mathcal{H}_b(X)$ and $\mathcal{H}_{wu}(X)$ (Theorem 12 of [9]). However, $\mathcal{H}(\mathbb{C})$ and $\mathcal{H}(\Delta)$, where $\Delta$ is the open unit ball of $\mathbb{C}$, have the same Schauder decomposition, $(\mathcal{P}(m\mathbb{C}))_m$, but they are not topologically isomorphic (see the remark after Corollary 10.6.12 of [7] or Theorem 2.3). This example shows that to obtain a topological isomorphism between Fréchet spaces it is not enough that they have the same Schauder decomposition. In order to clarify this situation we introduce a new class of Schauder decompositions: the $R$–Schauder decompositions. Some applications to the study of the bidual space of some closed subspaces of $\mathcal{H}_b(U)$ are given in Section 3.

In the sequel we use the notation $E^*$ for the strong dual of an arbitrary Fréchet space $E$, $X$ for an arbitrary complex Banach space and $B$, $B^*$ and $B^{**}$ for the open unit ball of $X$, $X^*$ and $X^{**}$ respectively. For a balanced open subset $U$ of $X$ let $\mathcal{H}_b(U)$ be the space of all holomorphic functions of bounded type on $U$, that is, the space of all holomorphic functions on $U$ which are bounded on all $U$–bounded sets. We recall that the $U$–bounded sets are, in

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the case $U = X$, the bounded subsets of $X$, whereas, in the case of an arbitrary open set $U$, they are the bounded subsets of $U$ whose distance to the boundary of $U$ is greater than zero. If $A$ is a $U$-bounded set, we set $\|f\|_A = \sup_{x \in A} |f(x)|$, for $f \in \mathcal{H}_b(U)$. \(\mathcal{H}_b(U)\) will be endowed with the topology $\tau_b$ defined by the seminorms $\|\cdot\|_A$. It is well known that $(\mathcal{H}_b(U), \tau_b)$ is a Fréchet space. Let \(\mathcal{H}_{wu}(U)\) denote the closed subspace of $\mathcal{H}_b(U)$ of all holomorphic functions on $U$ which are weakly uniformly continuous on all $U$-bounded sets. If $G$ is a balanced open subset of $X^*$, \(\mathcal{H}_{wu}(G)\) is the closed subspace of $\mathcal{H}_b(G)$ of all holomorphic functions on $G$ which are weak*-uniformly continuous on all $G$-bounded sets. Let $\mathcal{P}(^mX)$ be the space of all continuous $m$-homogeneous polynomials on $X$. Let $\mathcal{P}_{wu}(^mX) = \mathcal{P}(^mX) \cap \mathcal{H}_{wu}(X)$ and $\mathcal{P}_{wu}(^mX^*) = \mathcal{P}(^mX^*) \cap \mathcal{H}_{wu}(X^*)$.

2. Main results

**Definition 2.1.** Let $E$ be a Fréchet space and let $(E_n, \|\cdot\|_n)_n$ be a sequence of Banach spaces that is a Schauder decomposition of $E$. The sequence $(E_n)_n$ is said to be an $R$-Schauder decomposition of $E$, $0 < R \leq \infty$, if whenever $x_n \in E_n$, the series $\sum_{n=0}^{\infty} x_n$ converges if and only if $\limsup_n \|x_n\|_n^R \leq \frac{1}{R}$.

**Example 2.2.** By using Cauchy inequalities we obtain that the family $(\mathcal{P}(^mX), \|\cdot\|_m)_m$ is, at the same time, an $\infty$-Schauder decomposition of $\mathcal{H}_b(X)$ and an $R$-Schauder decomposition of $\mathcal{H}_b(RB)$. Moreover, given a bounded balanced open set $U \subset X$, the sequence $(\mathcal{P}(^mX), \|\cdot\|_U)_m$ is a 1-Schauder decomposition of $\mathcal{H}_b(U)$. Analogously for $\mathcal{P}_{wu}(^mX)$, $\mathcal{P}_{wu}(^mX^*)$ and their corresponding spaces of holomorphic functions.

Every $R$-Schauder decomposition is $S$-absolute (Definition 3.7 of [3]). However, the converse is not true. There exist essentially two types of $R$-Schauder decompositions: the $\infty$-Schauder decompositions and the 1-Schauder decompositions. Indeed, if $(E_n, \|\cdot\|_n)_n$ is an $R$-Schauder decomposition of $E$, $0 < R < \infty$, then $(E_n, R^n \|\cdot\|_n)_n$ is a 1-Schauder decomposition of $E$. Then a natural question arises: is it possible to establish a topological isomorphism between two Fréchet spaces, one having a 1-Schauder decomposition and the other one an $\infty$-Schauder decomposition? Or, better: is it possible to find a Banach space $X$ such that $\mathcal{H}_b(X)$ is topologically isomorphic to $\mathcal{H}_b(B)$? The answer to both questions is negative and has been told us by José Bonet in a personal communication, which we gratefully acknowledge, where he pointed out the following power
series approach to R-Schauder decompositions: Every Fréchet space $E$ with an $R$-Schauder decomposition can be identified with the power series space $\lambda^1(A_R; (E_n)_n)$ (where $A_R = \{(r^n)_n : 0 < r < R\}$) defined by $\lambda^1(A_R; (E_n)_n) := \{x = (x_n) \in \prod_n E_n : p_r(x) := \sum_{n=0}^{\infty} \|x_n\|_n r^n < \infty, \forall r : 0 < r < R\}$, endowed with the locally convex topology given by the family of seminorms $\{p_r : 0 < r < R\}$. He obtains the following theorem.

**Theorem 2.3.** If $E$ and $F$ are Fréchet spaces having a $R$-Schauder, $0 < R < \infty$, and an $\infty$-Schauder decomposition respectively, then there exists no topological isomorphism between $E$ and $F$.

Therefore, given a Banach space $X$, the space $\mathcal{H}_b(X)$ (resp. $\mathcal{H}_{wu}(X)$, $\mathcal{H}_{w^*}(X^*)$) is not topologically isomorphic to $\mathcal{H}_b(B)$ (resp. $\mathcal{H}_{wu}(B)$, $\mathcal{H}_{w^*}(B^*)$).

Our main theorem characterizes when a topological isomorphism occurs between spaces $E$ and $F$ having $R$-Schauder decompositions of the same type:

**Theorem 2.4.** Let $(E_n, \|\cdot\|_n)_n$ and $(F_n, \|\cdot\|_n)_n$ be $R$-Schauder decompositions of the Fréchet spaces $E$ and $F$ respectively $(0 < R \leq \infty)$. Assume that there exist algebraic isomorphisms $T_m : E_m \rightarrow F_m$ for all $m \in \mathbb{N}$ so that:

(i) (Condition I) In case $0 < R < \infty$, for each $t > 1$ there exist $a_t, b_t > 0$ such that, for every $m \in \mathbb{N}$ and every $x_m \in E_m$,

$$\|T_m(x_m)\|_m \leq a_t t^m \|x_m\|_m \quad \text{and} \quad \|x_m\|_m \leq b_t t^{-m} \|T_m(x_m)\|_m.$$  

(ii) (Condition II) In case $R = \infty$, there exist $t, t' > 0$ and $a_t, b_t' > 0$ such that, for every $m \in \mathbb{N}$ and every $x_m \in E_m$,

$$\|T_m(x_m)\|_m \leq a_t t^m \|x_m\|_m \quad \text{and} \quad \|x_m\|_m \leq b_t (t')^{-m} \|T_m(x_m)\|_m.$$  

Then the map $T : x = \sum_{m=0}^{\infty} x_m \in E \rightarrow T(x) := \sum_{m=0}^{\infty} T_m(x_m) \in F$ is a topological isomorphism.

Conversely, if there exists a topological isomorphism $T : E \rightarrow F$ so that $T(E_m) \subset F_m$, for all $m \in \mathbb{N}$, then $T(E_m) = F_m$ and $T_m := T|_{E_m}$ are topological isomorphisms satisfying Condition I in case $0 < R < \infty$ and Condition II in case $R = \infty$.

**Corollary 2.5.** Let $(E_n, \|\cdot\|_n)_n$ and $(F_n, \|\cdot\|_n)_n$ be $R$-Schauder decompositions of $E$ and $F$ respectively $(0 < R \leq \infty)$. If $E_n$ is isometrically isomorphic to $F_n$ for every $n \in \mathbb{N}$, then $E$ and $F$ are topologically isomorphic.
3. Applications

We now state some applications of these results to the study of biduals of spaces of holomorphic functions.

**Corollary 3.1.** Let $G \subset X^*$ be either $B^*$ or $X^*$. If $\mathcal{P}_{w^*}(^{m}X^*)$ contains no copy of $l^1$, for every $m \in \mathbb{N}$, then $\mathcal{H}_{w^*}(G)^{**}$ is topologically isomorphic to $\mathcal{H}_{w^*}(G)^{**}$, the closure of $\mathcal{H}_{w^*}(G)$ in $(\mathcal{H}_b(G), \tau_0)$. In particular, the isomorphism holds whenever $X$ is an Asplund space.

**Corollary 3.2.** Let $X$ be a Banach space such that $X^*$ has the approximation property. Let $G \subset X^*$ be either $B^*$ or $X^*$. If $\mathcal{P}_{w^*}(^{m}X^*)$ contains no copy of $l^1$ for all $m \in \mathbb{N}$, then $\mathcal{H}_{w^*}(G)^{**}$ is topologically isomorphic to $\mathcal{H}_b(G)$. In particular, the isomorphism holds whenever $X$ is an Asplund space such that $X^*$ has the approximation property.

Corollaries 3.1 and 3.2 have been obtained by Valdivia in [11] for entire functions under the assumption that $l^1$ is not contained in the space of entire functions. J.C. Diaz pointed out to us that this assumption is equivalent to the non-containment of $l^1$ in $\mathcal{P}_{w^*}(^{m}X^*)$ for all $m \in \mathbb{N}$ (Corollary 1.25 of [8]).

**Corollary 3.3.** Let $U \subset X$ be either a bounded convex balanced open subset of $X$ or $U = X$. Assume that $\mathcal{P}_{wu}(^{m}X)$ contains no copy of $l^1$ for all $m \in \mathbb{N}$ (for example when $X^*$ is an Asplund space). Then

(a) $\mathcal{H}_{wu}(U)^{**}$ is topologically isomorphic to $\mathcal{H}_{w^*}(U^{**})^{**}$, where $U^{**}$ is the interior on $X^{**}$ for the norm topology of the closure of $U$ for the weak$^*$-topology on $X^{**}$. In particular, $\mathcal{H}_{wu}(B)^{**}$ is topologically isomorphic to $\mathcal{H}_{w^*}(B^{**})^{**}$ and $\mathcal{H}_{wu}(X)^{**}$ is topologically isomorphic to $\mathcal{H}_{w^*}(X^{**})^{**}$.

(b) Moreover, if $X^{**}$ has the approximation property then $\mathcal{H}_{wu}(U)^{**}$ is topologically isomorphic to $\mathcal{H}_b(U^{**})$.

**Corollary 3.4.** Let $X$ be a Banach space and let $U \subset X$ be either $B$ or $X$. If for every $m \in \mathbb{N}$ $\mathcal{P}_{wu}(^{m}X)^{**}$ is isometrically isomorphic to $\mathcal{P}(^{m}X)$, then $\mathcal{H}_{wu}(U)^{**}$ is isomorphic to $\mathcal{H}_b(U)$.

Corollary 3.4 and Corollary 3.5 below clarify Theorem 12 of [9].

Let us now consider the map $\tilde{\delta}_m : z \in X^{**} \rightarrow \tilde{\delta}_{m,z} \in \mathcal{P}(^{m}X)^*$ given by $\tilde{\delta}_{m,z}(P) = \tilde{P}(z)$, where $\tilde{P}$ denotes the Aron-Berner extension [1] of $P$ to $X^{**}$. González in [5] has defined, extending an earlier definition of Aron and Dineen [2], a Banach space $X$ to be $Q$-reflexive if the adjoint map $\tilde{\delta}_m^* : \mathcal{P}(^{m}X)^{**} \rightarrow$
\( \mathcal{P}(^mX^{**}) \) of \( \tilde{\delta}_n \) is bijective and hence, a topological isomorphism for every \( m \in \mathbb{N} \). Since \( \| \tilde{\delta}_n^* \| \leq 1 \), in order to satisfy the remaining inequalities in the hypothesis of Theorem 2.4 one has to assume that the maps \( \tilde{\delta}_n^* \) have some additional properties, for example to be isometries (in this case let us call \( X \) to be isometrically \( Q \)-reflexive).

**Corollary 3.5.** Let \( X \) be an isometrically \( Q \)-reflexive Banach space and let \( U \subseteq X \) be either \( B \) or \( X \). Then the space \( \mathcal{H}_b(U)^{**} \) is topologically isomorphic to \( \mathcal{H}_b(U^{**}) \).

Compare with Proposition 16 of [2].

**Theorem 3.6.** Let \( X \) be a Banach space and let \( U \subseteq X \) be either the open unit ball of \( X \) or \( U = X \). Then the space \( \mathcal{H}_b(U)^{**} \) is topologically isomorphic to \( \mathcal{H}_b(U^{**}) \) if, and only if, \( X \) is \( Q \)-reflexive and the sequence \( (\tilde{\delta}_n) \) satisfies either Condition I if \( U \neq X \) or Condition II if \( U = X \).

The proofs of these results are detailed in [4].

**REFERENCES**


