EXTRACTA MATHEMATICAE Vol. **12**, Núm. 3, 291 – 299 (1997)

A BF-Regularization of a Nonstationary Two–Body Problem under the Maneff Perturbing Potential

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(Research paper presented by Jaume Llibre)

AMS *Subject Class.* (1991): 70F05, 70H15, 70M20 *Received April* 21, 1997

1. Some remarks on regularization of Keplerian systems

The process of transforming singular differential equations into regular ones is known as *regularization.* We are specially concerned with the treatment of certain systems of differential equations arising in Analytical Dynamics, in such a way that, accordingly, the regularized equations of motion will be free of singularities. On the other hand, in any case one must distinguish between singularities of a differential equation and singularities of its solutions, bearing in mind the feature that singularities of a differential equation do not necessarily entail singularities of the solutions. Thus, regularity of differential equations is more important than the regularity of their solutions. Consequently the purpose of regularization is not to obtain regular functions (in particular, solutions to differential equations) but regular differential equations. In addition to this, from the viewpoint of Numerical Analysis of differential problems, regularized equations usually allow one to employ a larger step–size during the numerical integration.

In the special case of the pure Kepler problem (that is, the two–body problem in the absence of any perturbation), the classical equations of motion derived from Newton's laws are nonlinear and unstable in the sense of Ljapunov. Unlike the case of Newtonian equations, the harmonic oscillator is stable (say, every solution of the regularized differential equations is stable.) Moreover, by application of certain transformations introducing redundant variables (see below), the spatial Kepler problem is equivalent to a linear and stable differential system corresponding to four harmonic oscillators.

Many regularization methods do not lead to linear differential equations. In our turn, we take our cue from certain classical, analytical treatments as usually applied to the stationary two–body problem, namely: on the stage of extended–phase–space Hamiltonian Mechanics (Stiefel & Scheifele 1971, §30), one often considers regularization by the method of linearization in combining (weakly) canonical transformations to sets of redundant dependent variables and reparametrizations of motion in terms of other independent variables instead of physical time (Stiefel & Scheifele 1971, §34 and §37), so as to create second–order linear differential equations of motion (with constant coefficients) for the spatial–like variables. Thus, the physical time is taken as an additional coordinate of the moving mass, and is dealt with on an equal footing as the other geometrical position coordinates.

To this end, one considers the framework of Linear and Regular Celestial Mechanics to bring the equations of motion governing Keplerian systems into linear form by giving them the form corresponding to a 4–dimensional harmonic oscillator. A well–known reference for the subject of linearization of gravitational systems is the book by Stiefel & Scheifele (1971). Recently, Deprit *et al.* (1994) have systematized and clarified the approach to the analytical treatment of linearizing transformations and put the question in a more rigorous mathematical context.

In this respect, the application of sets of redundant variables should not be contemplated as a handicap. As stated in Stiefel & Scheifele (1971, §16, p. 76),

> *"in the age of high speed automatic computation the number of differential equations is not as important as the stability and the numerical behaviour of the differential equations at hand. The attempt to reduce the number of differential equations may lead to instability, whereas redundant equations are often well behaved."*

Burdet (1969) considered the linearization of the Kepler problem in focal variables. The focal method was developed in terms of four coordinates (the direction cosines of the position vector and the inverse of the distance), the independent variable being proportional to the true anomaly. These coordinates can be interpreted as homogeneous Cartesian coordinates in a projective space, and were made canonical by Ferrándiz, in completing the coordinates with the respective conjugate momenta (Ferrándiz 1988 and 1991, and references therein; Deprit *et al.* 1994, §§4.4), so as to obtain a set of eight redundant variables of focal type, the so–called *BF variables*, in terms of which the final equations of motion are similar to those proposed by Burdet. To sum up:

the BF variables reduce the pure, stationary Kepler problem to four uncoupled and unperturbed harmonic oscillators.

Operating on the extended phase space, in the present paper we intend to investigate the analytical behaviour of an enlarged BF–type mapping when applied to a certain perturbed two–body problem with a time–dependent Keplerian parameter $\mu(t)$. To be more precise, we study the regularization of the equations of motion issued from the Hamiltonian of a perturbed Gylden system under the effect of a time–dependent Maneff–like disturbing function (see Sections 2 and 3 below). An appropriate combination of transformations, both of the dependent and independent variables, along with the introduction of first–integrals and (geometrical and dynamical) constraints into the equations, will allow us to regularize the second–order differential equations of motion obtained from the first–order canonical system generated by the said Hamiltonian.

For this purpose, we take advantage of a slight modification of the approach due to Ferrándiz & Fernández–Ferreirós (1991): our version of the BF–mapping includes time t, and we will follow their focal method canonical treatment. If Burdet–Ferrándiz focal–type variables are used, and a true–like anomaly is adopted as the new independent variable, we arrive at regularized equations of motion resembling those of the harmonic oscillator type.

2. On the Maneff perturbing potential

During the last years, several authors (Diacu 1993, 1996; Mioc & Stoica 1995) have devoted some attention to diverse aspects concerning Keplerian–like dynamical systems under the perturbation effects due to the Maneff potential (Maneff 1930, Bertrand 1921).

Within the framework of Classical Analytical Mechanics, the Maneff model of gravitational potential constitutes a nonrelativistic modification of Newton's gravitational law which can be successfully used to accurately account for the motion of the apse line (say, the secular motion of the pericentre) of some celestial bodies, at least in the Solar System (e. g. the advance of the perihelion of the inner planets, or the motion of the perigee of the Moon). Precedents in the use of this kind of gravitational model in the study of orbital motion under conservative central force laws can even be traced back to Newton's Principia, while investigating the precession of the Moon's perigee. Some considerations on the first treatments of the apsidal precession problem are found in the recent contribution by Valluri, Wilson and Harper (1997).

Newton's treatment concerning the apsidal motion of the lunar orbit and Clairaut's analysis of the same problem are also examined by Aoki (1992).

On the other hand, considered as related to a Coulombic law of force, the Maneff model can also be applied in Atomic Physics. Nevertheless, in the present paper we shall leave the three–body problem aside, and will restrict ourselves to dealing with the macroscopic two–body problem as contemplated in Classical Celestial Mechanics.

Mioc & Stoica (1995) have discussed regularization of the equations of relative motion of a stationary Kepler problem perturbed by the Maneff potential. After formulation in plane polar coordinates, in their study they have followed and extended Mangad's (1967) approach to regularization of the pure Kepler problem by means of regularizing transformations of the independent variable, and then they have analytically solved the regularized equations of the system for different choices of initial conditions. In most cases, their solutions to the said regularized equations of motion still retain the original singularity. As a final comment on this treatment, let us add that the associated energy integral plays a central role in these articles. On the contrary, no use is made of the first–integral of the angular momentum.

For the sake of definiteness, let us state that we concentrate on the so–called Gylden systems, that is, two–body problems with a time–varying Keplerian parameter $\mu(t)$ (see, e.g. Deprit 1983). On such systems, we superimpose time–dependent perturbing effects emanating from a Maneff–type nonrelativistic gravitational potential, and we study the derivation of regularized equations of motion for the resulting dynamical system formulated in homogeneous canonical formalism, although linearization cannot be assured. Our considerations and results, which are independent of the energy of the system, are uniformly valid for any type of two–body orbit.

As presented here, our treatment of the problem has some theoretical significance in itself. Possible future applications might be found in the study of the dynamics of close binary systems and other questions in Stellar Dynamics. In particular, we have in mind a simplified model for the investigation of orbital motion in the plane of the equator of the primary under certain assumptions on variability of μ.

3. Hamiltonian formulation of the model

From the outset, we coordinatize an extended, 8–dimensional phase space by the enlarged canonical set $(r, \theta, \nu, t; R, \Theta, N, T)$ of the Hill–Whittaker polar nodal variables, where r is the modulus of the two–body relative position vector, θ expresses the argument of latitude of the moving mass, and ν denotes the argument of longitude of the ascending node; the canonical momentum R represents the radial velocity of the orbiting mass, Θ designates the magnitude of the total angular momentum vector of the system, and N is the polar component of the said angular momentum. Finally, t stands for the physical time and the momentum T (canonically conjugate to t) is the negative of the total energy of the system. Notice that, in homogeneous canonical formalism, time t is introduced as an additional canonical coordinate. At a later stage (Section 5, below), extended phase–space formulation will facilitate the introduction of new independent variables other than t. Further particulars can be found in Stiefel & Scheifele (1971), Chapter VIII (specially, §30 and §34), and Chapter X, §37.

We shall deal with a perturbation formulated in these variables: the perturbing part will be proportional to the second power of the inverse of the distance r, the coefficient being some sufficiently regular function of time.

In homogeneous canonical formulation, we consider a perturbed Gylden system governed by the Hamiltonian

$$\mathcal{H}_h \equiv \mathcal{H}_h\,(r,-,-,t;R,\Theta,-,T;\varepsilon) = \mathcal{H}_0 - \frac{\varepsilon}{2\,r^2}\mu^2(t) + T, \tag{1}$$

$$\mathcal{H}_0 \equiv \mathcal{H}_0\,(r,\mu(t);R,\Theta) = \frac{1}{2}\left[R^2 + \frac{\Theta^2}{r^2}\right] - \frac{\mu(t)}{r}, \tag{2}$$

where $\mathcal{H}_0$ designates the Hamiltonian of a standard Gylden system (Deprit 1983), and $\mu(t)$ stands for the time–dependent Keplerian parameter, understanding also that $\varepsilon = 3/C^2$ is a measure of smallness, introduced to separate the small terms according to their relative size. Here C denotes the speed of light.

According to some previous knowledge on exact BF–linearization of this type of perturbation in the conservative case (Burdet 1969; Ferrándiz & Fernández–Ferreirós 1991), the equations of motion for the stationary problem can be exactly reduced to a set of second–order linear differential equations with constant coefficients corresponding to four harmonic oscillations.

In extended Cartesian variables $(X_0, X, P_0, \boldsymbol{P})$, Hamiltonian (1) is expressed as

$$\mathcal{H} \equiv \mathcal{H}\,(X_0,\boldsymbol{X},P_0,\boldsymbol{P}) = \frac{1}{2}\,\|\boldsymbol{P}\|^2 - \frac{\mu(X_0)}{r} - \frac{\varepsilon}{2\,r^2}\,\mu^2(X_0) + P_0, \tag{3}$$

where $\boldsymbol{X} = (X_1, X_2, X_3)$ denotes the position vector of the particle, $r = \|\boldsymbol{X}\|$ is the radial distance, and the vector $\boldsymbol{P} = (P_1, P_2, P_3)$ contains the respective conjugate momenta. To this set we have appended the pair of canonically conjugate variables $(X_0 = t, P_0 = T)$. As well known, the magnitude Θ of the angular momentum will be given through the relation

$$\Theta^2 = \|\boldsymbol{X} \times \boldsymbol{P}\|^2, \tag{4}$$

where $\times$ stands for the usual cross product in space $\mathbb{R}^3$.

Notice that, in the present time–dependent case, the dynamical systems under consideration still possesses the first–integral of the angular momentum.

4. A BF–TYPE TRANSFORMATION IN EXTENDED PHASE SPACE

In what follows we shall deal with a transformation of the dependent variables (that increases the dimension by two) and a regularizing change of the time parameter, and their effect on Hamiltonian (3) and the corresponding canonical equations of motion.

Inspired in Ferrándiz & Fernández–Ferreirós (1991), we perform a modified BF transformation (see also Deprit *et al.* 1994, §§4.4). The mapping is weakly canonical (Deprit *et al.*, 1994), increases the number of variables by two, and introduces the set $(x_0, \boldsymbol{x}, x_4, p_0, \boldsymbol{p}, p_4) \equiv (x_0, x_1, x_2, x_3, x_4, p_0, p_1, p_2, p_3, p_4)$ by means of the equations:

$$\begin{aligned} X_i &= \frac{x_i}{x_4}, \quad P_i = p_i\, x_4 - x_i\, \frac{x_4}{\|\boldsymbol{x}\|^2}\, \{(\boldsymbol{x} \mid \mathrm{p}) + x_4 p_4\}\ , \quad i = 1, 2, 3, \\ X_0 &= x_0\,, \quad P_0 = p_0\,, \end{aligned}$$

where it is obvious that, for convenience in writing, we have employed the abbreviations $\boldsymbol{x} \equiv (x_1, x_2, x_3)$, $\boldsymbol{p} \equiv (p_1, p_2, p_3)$, and $(\cdot \mid \cdot)$ denotes the standard, Euclidean inner product in a real vector space $\mathbb{R}^n$.

Hence, taking into account the following relations for the Euclidean norms of vectors $\boldsymbol{X}$ and $\boldsymbol{P}$,

$$\begin{aligned} \|\boldsymbol{X}\|^2 &= r^2 = \|\boldsymbol{x}\|^2 x_4^{-2}\,, \\ \|\boldsymbol{P}\|^2 &= \sum_{i=1}^{3} p_i^2 x_4^2 + x_i^2 x_4^2\, (\boldsymbol{x} \mid \boldsymbol{p})^2\, \|\boldsymbol{x}\|^{-4} + x_i^2 x_4^4 p_4^2 \|\boldsymbol{x}\|^{-4} \\ &\quad + 2\|\boldsymbol{x}\|^{-2}\left[x_i^2 x_4^3 p_4\, (\boldsymbol{x} \mid \boldsymbol{p})\, \|\boldsymbol{x}\|^{-2} - x_i p_i x_4^2\, (\boldsymbol{x} \mid \boldsymbol{p}) - x_i p_i x_4^3 p_4\right] \\ &= \|\boldsymbol{p}\|^2 x_4^2 + x_4^4 p_4^2 \|\boldsymbol{x}\|^{-2} - x_4^2\, (\boldsymbol{x} \mid \boldsymbol{p})^2\, \|\boldsymbol{x}\|^{-2}\,, \end{aligned}$$

after application of this transformation, the homogeneous Hamiltonian (3) is converted into the function

$$\widetilde{\mathcal{H}} = \frac{1}{2}\|\boldsymbol{p}\|^2 x_4^2 + \frac{1}{2}\frac{x_4^4 p_4^2}{\|\boldsymbol{x}\|^2} - \frac{1}{2}\frac{(\boldsymbol{x}\mid\boldsymbol{p})^2 x_4^2}{\|\boldsymbol{x}\|^2} - \mu(x_0)\frac{x_4}{\|\boldsymbol{x}\|} - \frac{x_4^2}{\|\boldsymbol{x}\|^2}\frac{\varepsilon}{2}\mu^2(x_0) + p_0\,, \quad (5)$$

while

$$\Theta^2 = \|\boldsymbol{x}\|^2\|\boldsymbol{p}\|^2 - (\boldsymbol{x}\mid\boldsymbol{p})^2 = c^2\,. \quad (6)$$

5. Reparametrization of motion and canonical equations

Thanks to the extended phase–space formulation, we make a change of time parameter, from t to a new fictitious time s, by reparametrizing with the help of a generalized Sundman transformation, the new independent variable (proportional to a true–like anomaly) being defined by

$$t \longrightarrow s\,: \quad dt/ds = \widetilde{f} = \|\boldsymbol{x}\|^2/x_4^2 = r^2\,. \quad (7)$$

Bearing in mind the above relation (6), the new homogeneous Hamiltonian, with the regularizing pseudo–time s as the independent variable, will adopt the form

$$\mathcal{K} = \widetilde{\mathcal{H}}\widetilde{f} = \frac{1}{2}c^2 + \frac{1}{2}p_4^2 x_4^2 - \mu(x_0)\frac{\|\boldsymbol{x}\|}{x_4} - \frac{\varepsilon}{2}\mu^2(x_0) + \frac{\|\boldsymbol{x}\|^2}{x_4^2}p_0\,. \quad (8)$$

Next, as in Ferrándiz & Fernández–Ferreirós 1991, pp. 5–6, use will be made of the first–integrals and constraints

$$\|\boldsymbol{x}\|^2 = 1\,, \quad (\boldsymbol{x}\mid\boldsymbol{p}) + x_4 p_4 = 0\,. \quad (9)$$

The canonical equations of motion derived from $\mathcal{K}$, namely

$$x'_\alpha \equiv \frac{dx_\alpha}{ds} = \frac{\partial\mathcal{K}}{\partial p_\alpha}\,, \quad p'_\alpha \equiv \frac{dp_\alpha}{ds} = -\frac{\partial\mathcal{K}}{\partial x_\alpha}\,, \quad \alpha = 0,\ldots,4\,, \quad (10)$$

can be written in the form

$$x'_0 = x_4^{-2}\,, \qquad x'_4 = p_4 x_4^2\,,$$
$$x'_i = p_i - (\boldsymbol{x}\mid\boldsymbol{p})\,x_i\,, \quad i = 1,2,3,$$
$$p'_0 = (d\mu(x_0)/dx_0)\left[x_4^{-1} + \varepsilon\mu(x_0)\right]\,, \qquad p'_4 = -p_4^2 x_4 - \mu(x_0)\,x_4^{-2} + 2p_0 x_4^{-3},$$
$$p'_i = -\|\boldsymbol{p}\|^2 x_i + (\boldsymbol{x}\mid\boldsymbol{p})\,p_i + \mu(x_0)\,x_i x_4^{-1} - 2p_0 x_i x_4^{-2}\,.$$

6. Second–order equations for spatial–like variables

Finally, by forming the s–derivatives of the first five equations for the x'_α, and taking into account the expressions for the p'_α and the above notations and constraints, we obtain a set of second–order differential equations for the x_α:

$$x''_0 = -2x'_4 x_4^{-3},$$
$$x''_i + c^2 x_i = 0\,, \quad i = 1,2,3 \text{ (three uncoupled harmonic oscillators)},$$
$$x''_4 + c^2 x_4 = \mu\,(x_0) + \varepsilon\mu^2\,(x_0)\,x_4\,.$$

Notice that these equations for the position–like variables are regular, their form resembling that of harmonic equations. In particular, for a pure Gylden system (say, $\varepsilon = 0$) the equations of motion read

$$x''_i + c^2 x_i = 0\,, \quad i = 1,2,3 \text{ (three uncoupled harmonic oscillators)},$$
$$x''_4 + c^2 x_4 = \mu\,(x_0)\;.$$

Consideration of solutions to the above equations under various special laws of variation for $\mu(t)$ and other developments will be communicated in a future paper.

Acknowledgements

L. Floría acknowledges partial financial support provided by the Grant PB 95–0807 (DGES of Spain, Ministerio de Educación y Cultura.)

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