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Arbitrary Exponential Decay of Energy for a Class of Bilinear Control Problems

M. Aassila

IRMA, 7 rue René Descartes, 67084 *Strasbourg, France.*
e-mail: aassila@math.u-strasbg.fr

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1. Statement of results

This work considers the question of feedback stabilizability for the bilinear system

$$\begin{cases} u'(t) = Au(t) + v(t)Bu(t), \\ u(0) = u_0. \end{cases} \tag{P1}$$

Here A is the infinitesimal generator of a linear C_0–semigroup of contractions e^{At} on a real Hilbert space H with inner product $(\cdot,\cdot)$, so that A is dissipative, i.e. $(A\Psi,\Psi) \le 0$ for all $\Psi \in D(A)$. B is a (possibly nonlinear) operator from H into H and $v(t)$ is a real valued control.

The mains novelty of this paper is the statement that there exists a feedback control $v(u)$ which gives a uniform decay rate of the solution to the closed-loop problem (P1) with an arbitrarily decay rate.

Let ω be an arbitrarily large positive number, and choose

$$v(t) = \frac{-\omega\|u\|^2}{(Bu,u)} - (Bu,u) \quad \text{in} \quad (\text{P1}),$$

then (P1) may be written in the first order form

$$\begin{cases} u' = Au + F(u), \\ u(0) = u_0, \end{cases}$$

where

$$F(u) = \left(\frac{-\omega \|u\|^2}{(Bu, u)} - (Bu, u) \right) Bu.$$

Under the hypotheses

(H1) A is the infinitesimal generator of a linear C_0-semigroup of contractions e^{At} on a real Hilbert space H;

(H2) there exists a positive constant α such that $(B\Psi, \Psi) \geq \alpha \|\Psi\|^2$ for all $\Psi \in D(B)$, $B(0) = 0$ and $B : H \to H$ is locally Lipschitz,

it can be shown that problem

$$\begin{cases} u' = Au + \left(\dfrac{-\omega \|u\|^2}{(Bu, u)} - (Bu, u) \right) Bu, \\ u(0) = u_0, \end{cases} \tag{P2}$$

has a unique weak solution $u(t)$ on $\mathbb{R}_+$. Let us note that the existence of such control follows from a theorem of Ball [1].

MAIN RESULT. Fix an arbitrarily large positive number ω, and let $u(t)$ denotes the unique weak global solution of (P2), then we have

$$\|u(t)\| \leq \|u_0\| e^{-\omega t} \quad \text{for all} \quad t \geq 0.$$

Proof. We have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (e^{2\omega t} \|u(t)\|^2) &= e^{2\omega t} (\omega \|u(t)\|^2 + (Au, u) + (F(u), u)) \\ &\leq e^{2\omega t} (\omega \|u(t)\|^2 - \omega \|u(t)\|^2 - (Bu, u)^2) \leq 0. \end{aligned}$$

∎

2. APPLICATION

Let Ω be a bounded, open, connected set in $\mathbb{R}^n$ $(n \geq 1)$ having a boundary Γ of class C^2. Let $\nu_1, \nu_2, \dots, \nu_m$ be m real numbers strictly positive, and $f_i : \mathbb{R}^m \to \mathbb{R}$, m functions of class C^1 in $\mathbb{R}^m$.

Let us consider the following system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \nu_i \Delta u_i + v f_i(u_1, \cdots, u_m) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \quad i = 1, 2, \dots, m, \\ u_i = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \quad i = 1, 2, \dots, m, \\ u(x, 0) = u_0(x). \end{cases} \tag{P3}$$

If we set

$$U = (u_1, u_2, \dots, u_m);$$
$$F(u_1, u_2, \dots, u_m) = (f_1(u_1, \dots, u_m), \dots, f_m(u_1, \dots, u_m));$$
$$H = (L^2(\Omega))^m \quad \text{and} \quad F(U) := BU \quad \text{for all} \quad U \in H,$$

then (P3) may be written in the form

$$\begin{cases} U' = AU + F(U), \\ U(0) = U_0. \end{cases}$$

Assume that

(i) $f_i(0, 0, \dots, 0) = 0, \quad i = 1, 2, \dots, m;$

(ii) there exists $\alpha > 0$ such that $f_i(u_1, u_2, \dots, u_m) \geq \alpha u_i$;

(iii) there exists $M > 0$ such that

$$\left| \frac{\partial f_i}{\partial u_i}(u_1, u_2, \dots, u_m) \right| \leq M \quad \text{for all} \quad (u_1, \dots, u_m) \in B_S(0),$$

where $B_S(0)$ denotes the ball of center 0 and radius S,

then (H1)–(H2) are satisfied.

Putting

$$v = \frac{-\omega \sum_{i=1}^m \int_\Omega u_i^2 \, dx}{\sum_{i=1}^m \int_\Omega u_i f_i(u_1, u_2, \dots, u_m) \, dx} - \sum_{i=1}^m \int_\Omega u_i f_i(u_1, u_2, \dots, u_m) \, dx,$$

then the solution of (P3) satisfies

$$\|u(t)\|_H \leq \|u_0\|_H e^{-\omega t} \quad \text{for all} \quad t \geq 0.$$

A special case of (P3) is when $f_i(u_1, u_2, \dots, u_m) = u_i$. Then, the feedback $v = -\omega - \sum_{i=1}^m \int_\Omega u_i^2 \, dx$ gives an arbitrarily exponential decay rate.

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