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On Commutative FGI-Rings

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Let R be a commutative ring. An R-module M is said to satisfy property (I) if every injective endomorphism of M is an automorphism. R is called a FGI-ring if every R-module with property (I) is finitely generated. In [6] W.V. Vasconcelos proved that for a commutative ring R the following conditions are equivalent.

(a) Every finitely generated module satifies property (I)

(b) Every prime ideal of R is maximal.

The purpose of this note is to characterize commutative countable rings on which only finitely generated modules have property (I).

Proposition 1. *Let R be a commutative FGI-ring. Then every prime ideal of R is maximal. Moreover the set of all prime ideals of R is finite.*

Proof. Let P be a prime ideal of R and let B be the classical quotient field of the integral domaine R/P. It is obvious that B, considered as R/P-module, satisfies property (I). Since R/P is also a FGI-ring, then B is a finitely generated R/P-module, therefore $B = R/P$ and P is maximal. Let now L be the set of all prime ideals of R. For every $P \in L$, R/P is a simple R-module, furthermore if P, $P' \in L$ and $P \neq P'$, we have $Hom_R(R/P, R/P') = \{0\}$, hence the R-module

$$M = \bigoplus_{P \in L} R/P$$

has property (I), it follows that M is a finitely generated R-module, and this fact implies that the set L is finite. ∎

It follows from Proposition 1 that if R is a commutative FGI-ring, then the Jacobson radical $J(R)$ of R is a nilideal and that R is semilocal. We recall that a ring R is semilocal if $R/J(R)$ is semisimple.

THEOREM 1. *Let R be a countable FGI-ring. Then R is Artinian.*

Proof. By [5, Corollary 3] it suffices to show that the injective hull of each simple R-module is countable. Let E be the injective hull of a simple R-module. Since E is an indecomposable injective R-module, it has property (I), then it is finitely generated. Since R is countable so is E. ∎

In what follows C denotes an Artinian local ring with Jacobson radical $J(C) = aC$ where $a \neq 0$ and $a^2 = 0$;

$$M = \bigoplus_{i \in \mathbb{N}}^{\infty} C.e_i$$

a free C-module whith infinite countable basis $\{e_i : i \in \mathbb{N}\}$, σ the endomorphism of the C-module M, definite as follows $\sigma(e_0) = 0$, and $\sigma(e_i) = ae_{i-1}$ for $i \geq 1$; and f an injective endomorphism of the C-module M, satisfying $f\sigma = \sigma f$.

With these notations we have:

LEMMA 1. (i) $a\sigma = \sigma^2 = 0$.
(ii) *For every $i \in \mathbb{N}^*$, $\sigma[f(e_i)] = af(e_{i-1})$.*

Proof. It is obvious. ∎

LEMMA 2. *For every $i \in \mathbb{N}$,*

$$f(e_i) = \sum_{j<i} \alpha_j^i e_j + \alpha_i^i e_i + a \sum_{k>i} \alpha_k^i e_k$$

and α_i^i invertible in R.

Proof. We have $\sigma[f(e_0)] = f[\sigma(e_0)] = f(0) = 0$, and $af(e_0) = f(ae_0) \neq 0$, because f is injective. Set

$$f(e_0) = \sum_{i=0}^{m} \lambda_i e_i,$$

then from the equality $\sigma[f(e_0)] = 0$ we obtain

$$\sum_{i\geq 0}^{m-1} a\lambda_{i+1}e_i = 0,$$

and hence $a\lambda_k = 0$ for $k = 1, ..., m$. Its follows that

$\lambda_k \in J(C) = aC$ for $k = 1, ..., m$. On the other hand the relation $af(e_0) \neq 0$ implies that $a\lambda_0 \neq 0$ and that λ_0 is invertible.

Suppose now we have

$$f(e_i) = \sum_{j<i} \alpha_j^i e_j + \alpha_i^i e_i + a\sum_{k>i} \alpha_k^i e_k$$

with α_i^i invertible, and set

$$f(e_{i+1}) = \sum_{j<i+1} \alpha_j^{i+1} e_j + \alpha_{i+1}^{i+1} e_{i+1} + \sum_{k>i+1} \lambda_k^{i+1} e_k.$$

By the relation $\sigma[f(e_{i+1})] = af(e_i)$, we have

$$a\sum_{j<i+1} \alpha_j e_{j-1} + a\alpha_{i+1}^{i+1} e_i + a\sum_{k>i+1} \lambda_k^{i+1} e_{k-1} = a\sum_{j<i} \alpha_j^i e_j + a\alpha_i^i e_i.$$

So, for $k \geq i+1$, $a\lambda_k^{i+1} = 0$ which implies that $\lambda_k^{i+1} \in aC$. Since $a\alpha_{i+1}^{i+1} = a\alpha_i^i \neq 0$, then α_{i+1}^{i+1} is invertible. ∎

LEMMA 3. *For every* $i \in \mathbb{N}$, $ae_i \in \operatorname{Im} f$.

Proof. By Lemma 2, we have

$$f(e_0) = \alpha_0^0 e_0 + a\sum_{i\geq 1} \alpha_i^0 e_i$$

where α_0^0 is invertible. Hence $f(ae_0) = af(e_0) = a\alpha_0^0 e_0$, and then $ae_0 = (\alpha_0^0)^{-1} f(ae_0) = f[(\alpha_0^0)^{-1} ae_0]$. Suppose now that a $e_k \in \operatorname{Im} f$ for every $k \leq i$. By Lemma 2 we can write

$$f(e_{i+1}) = \sum_{j\leq i} \alpha_j^{i+1} e_j + \alpha_{i+1}^{i+1} e_{i+1} + \sum_{k>i+1} \lambda_k^{i+1} e_k,$$

with α_{i+1}^{i+1} invertible. So we have

$$f(ae_{i+1}) = af(e_{i+1}) = \sum_{j\leq i} a\alpha_j^{i+1} e_j + a\alpha_{i+1}^{i+1} e_{i+1}.$$

By hypothesis

$$\sum_{j\leq i} a\alpha_j^{i+1} e_j \in \operatorname{Im} f,$$

it follows then $ae_{i+1} \in \operatorname{Im} f$. ∎

LEMMA 4. *For every $i \in \mathbb{N}$, $e_i \in \operatorname{Im} f$.*

Proof. By Lemma 2, we can write

$$f(e_0) = \alpha_0 e_0 + a \sum_{k>0} \alpha_k^0 e_k$$

with α_0^0 invertible. But by Lemma 3

$$a \sum_{k>0} \alpha_k^0 e_k \in \operatorname{Im} f$$

hence $e_0 \in \operatorname{Im} f$. Assume now that $e_k \in \operatorname{Im} f$, for every $k \leq i$, and let us write

$$f(e_{i+1}) = \sum_{j\leq i} \alpha_j^{i+1} e_j + \alpha_{i+1}^{i+1} e_{i+1} + a \sum_{j>i+1} \alpha_j^{i+1} e_j$$

where α_{i+1}^{i+1} in invertible (Lemma 2). By Lemma 3 we have

$$a \sum_{j>i+1} \alpha_j^{i+1} e_j \in \operatorname{Im} f$$

and by hypothesis

$$\sum_{j\leq i} \alpha_j^{i+1} e_j \in \operatorname{Im} f,$$

hence

$$e_{i+1} = (\alpha_{i+1}^{i+1})^{-1}[f(e_{i+1}) - \sum_{j\leq i} \alpha_j^{i+1} e_j - a \sum_{j>i+1} \alpha_j^{i+1} e_j] \in \operatorname{Im} f.$$

∎

By virtue of Lemmas 1, 2, 3 and 4, we can state:

PROPOSITION 2. *Let R be a commutative Artinian ring. If R has a non principal ideal, then there exists an R-module with property (I) which is not finitely generated.*

Proof. Without loss of generality it may be assumed that R is a local ring with Jacobson radical $J(R) = aR + bR$ with the conditions $a \neq 0$, $b \neq 0$ and $a^2 = ab = b^2 = 0$. Then by [2] there exists a local Artinian principal ideal subring C of R with Jacobson radical $J(R) = aC$ such that $R = C \oplus bC$ (as C-modules). Let us consider the ring homomorphism

$$\phi : R = C \oplus bC \longrightarrow End_C M$$
$$\alpha + b\lambda \longrightarrow \alpha\, id_M + \lambda\sigma$$

where id_M denotes the identity homomorphism of the C-module

$$M = \bigoplus_{i \in \mathbb{N}} Ce_i.$$

By ϕ, M has a R-module structure whose endomorphisms are the elements f of $End_C M$ satisfiying $f\sigma = \sigma f$. It follows then from Lemmas 1, 2, 3 and 4 that the R-module M satisfies (I) and it is obvious that as R-module, M is not finitely generated. ∎

THEOREM 2. *Let R be a countable commutative ring. The following conditions are equivalent.*

(i) *R is a FGI-ring.*

(ii) *R is an Artinian principal ideal ring.*

Proof. The implication (i) $\Rightarrow$ (ii) is a consequence of Theorem 1 and Proposition 2, while the implication (ii) $\Rightarrow$ (i) result from the fact that if R is an Artinian principal ideal ring then every R-module is a direct sum of cyclic modules [3]. ∎

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