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Lectures on Maximal Monotone Operators *

R.R. Phelps

Dept. Math. Box 354350, Univ. of Washington, Seattle WA 98195.
e-mail: phelps@math.washington.edu

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Introduction

These lectures will focus on those properties of maximal monotone operators which are valid in arbitrary real Banach spaces. Most applications (to nonlinear partial differential equations, optimization, calculus of variations, etc.) take place in reflexive spaces, in part because several key properties have only been shown to hold in such spaces. (See, for instance, [6], [17] and [28].) We will generally isolate the reflexivity hypothesis, hoping that by doing so, it will eventually be possible to decide their validity without that hypothesis. In Section 1 we define maximal monotone operators and prove some of their main elementary properties. Section 2 is devoted to the prototypical class of subdifferentials of convex functions. Gossez's subclass of monotone operators of type (D) is examined in Section 3 and Section 4 gives a brief treatment of another subclass, the locally maximal monotone operators. We have attempted to keep the exposition self-contained, using only standard tools of elementary functional analysis. One exception is the application of the Brouwer fixed-point theorem in the proof of the Debrunner–Flor theorem (Lemma 1.7).

1. Monotone Operators

Definition 1.1. A set-valued map T from a Banach space E into the subsets of its dual E^* is said to be a monotone operator provided

$$\langle x^* - y^*, x - y\rangle \geq 0 \quad \forall x, y \in E \text{ and } x^* \in T(x),\ y^* \in T(y).$$

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We do not require that $T(x)$ be nonempty. The domain (or effective domain) of T is the set $D(T) = \{x \in E\colon\ T(x) \neq \emptyset\}$.

EXAMPLSE 1.2. (a) The simplest examples of such operators are linear and single–valued. For instance, if H is a real Hilbert space and $T\colon H \to H^* \equiv H$ is a linear map, then T is monotone if and only if it is, in the usual sense, a positive operator: $\langle T(x), x\rangle \geq 0$ for all x.

(b) Let D be a nonempty subset of the real numbers $\mathbb{R}$. A function $\varphi\colon D \to \mathbb{R}^* \equiv \mathbb{R}$ defines a monotone operator if and only if φ is monotone nondecreasing in the usual sense: That is,

$$[\varphi(t_2) - \varphi(t_1)]\cdot(t_2 - t_1) \geq 0 \quad \forall t_1, t_2 \in D \quad \text{iff} \quad \varphi(t_1) \leq \varphi(t_2) \text{ whenever } t_1 < t_2.$$

(c) Examples of set–valued monotone functions from $\mathbb{R}$ to subsets of $\mathbb{R}$ are easy to exhibit: For instance, let $\varphi(x) = 0$ if $x < 0$, $\varphi(x) = 1$ if $x > 0$ and let $\varphi(0)$ be any subset of $[0, 1]$.

(d) Here is an important single–valued but nonlinear example: Let f be a continuous real–valued function on E which is Gateaux differentiable (that is, for each $x \in E$ the limit

$$df(x)(y) = \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}, \qquad y \in E$$

exists and is a bounded linear functional of y). Such a function f is convex if and only if the mapping $x \to df(x)$ is monotone. Indeed, suppose that f is convex; then for $0 < t < 1$, convexity implies that

$$\frac{f(x + t(y-x)) - f(x)}{t} \leq \frac{(1-t)f(x) + tf(y) - f(x)}{t} = f(y) - f(x).$$

It follows that $df(x)(y - x) \leq f(y) - f(x)$, for any $x, y \in E$. Thus, if $x, y \in E$ and $x^* = df(x)$, $y^* = df(y)$, then

$$\langle x^*, y - x\rangle \leq f(y) - f(x) \text{ and } -\langle y^*, y - x\rangle = \langle y^*, x - y\rangle \leq f(x) - f(y);$$

now add these two inequalities. A proof of the converse may be found in [18, p.17] .

(e) The next example arises in fixed-point theory. Let C be a bounded closed convex nonempty subset of Hilbert space H and let U be a (generally nonlinear) nonexpansive map of C into itself: $\|U(x) - U(y)\| \leq \|x - y\|$ for all

x, $y \in C$. Let I denote the identity map in H; then $T = I - U$ is monotone, with $D(T) = C$. Indeed, for all x, $y \in C$,

$$\begin{aligned}\langle T(x) - T(y), x - y\rangle &= \langle x - y - (U(x) - U(y)), x - y\rangle \\ &= \|x - y\|^2 - \langle U(x) - U(y), x - y\rangle \\ &\geq \|x - y\|^2 - \|U(x) - U(y)\| \cdot \|x - y\| \geq 0.\end{aligned}$$

Note that 0 is in the range of T if and only if U has a fixed point in C; this hints at the importance for applications of studying the ranges of monotone operators.

(f) Again, in Hilbert space, let C be a nonempty closed convex set and let P be the metric projection of H onto C; that is, $P(x)$ is the unique element of C which satisfies $\|x - P(x)\| = \inf\{\|x - y\| : y \in C\}$. We first prove the fundamental fact that the mapping P satisfies (in fact, it is characterized by) the following variational inequality: For all $x \in H$,

$$\langle x - P(x), z - P(x)\rangle \leq 0 \quad \text{for all } z \in C. \tag{1.1}$$

Indeed, if $z \in C$ and $0 < t < 1$, then $z_t \equiv tz + (1 - t)P(x) \in C$ and hence $\|x - P(x)\| \leq \|x - z_t\| = \|(x - P(x)) - t(z - P(x))\|$. Squaring both sides of this inequality, expanding and then cancelling $\|x - P(x)\|^2$ on both sides yields

$$0 \leq -2t\langle x - P(x), z - P(x)\rangle + t^2\|z - P(x)\|^2.$$

If we then divide by t and take the limit as $t \to 0$ we obtain (1.1). Moreover, if $y \in H$ and we write down (1.1) again, using y in place of x, then take $z = P(y)$ in the first equation, $z = P(x)$ in the second one and add the two, we obtain

$$\langle x - y, P(x) - P(y)\rangle \geq \|P(x) - P(y)\|^2 \quad \text{for all } x, y \in H, \tag{1.2}$$

which shows that P is in monotone in a very strong sense. Note that P is an example of a nonexpansive mapping in the sense of the previous example: We always have the inequality $\langle x - y, P(x) - P(y)\rangle \leq \|x - y\| \cdot \|P(x) - P(y)\|$, so combined with the monotonicity inequality above, we have $\|P(x) - P(y)\| \leq \|x - y\|$ for all $x, y \in H$.

(g) Here is a fundamental example of a set–valued monotone mapping, the duality mapping from E into 2^{E^*}. For any $x \in E$ define

$$J(x) = \{x^* \in E^* : \langle x^*, x\rangle = \|x^*\| \cdot \|x\| \text{ and } \|x^*\| = \|x\|\}.$$

By the Hahn–Banach theorem, $J(x)$ is nonempty for each x, so $D(J) = E$. Suppose that $x^* \in J(x)$ and $y^* \in J(y)$. Then

$$\begin{aligned}\langle x^* - y^*, x - y\rangle &= \|x^*\|^2 - \langle x^*, y\rangle - \langle y^*, x\rangle + \|y^*\|^2 \\ &\geq \|x^*\|^2 - \|x^*\| \cdot \|y\| - \|y^*\| \cdot \|x\| + \|y^*\|^2 \\ &= \|x^*\|^2 - 2\|x^*\| \cdot \|y^*\| + \|y^*\|^2 = (\|x^*\| - \|y^*\|)^2,\end{aligned}$$

so this is also monotone in a rather strong sense.

In order to define maximal monotone operators we must consider their graphs.

DEFINITION 1.3. A subset G of $E \times E^*$ is said to be monotone provided $\langle x^* - y^*, x - y\rangle \geq 0$ whenever (x, x^*), $(y, y^*) \in G$. A set–valued mapping $T\colon E \to 2^{E^*}$ is a monotone operator if and only if its graph

$$G(T) = \{(x, x^*) \in E \times E^*\colon x^* \in T(x)\}$$

is a monotone set. A monotone set is said to be maximal monotone if it is maximal in the family of monotone subsets of $E \times E^*$, ordered by inclusion. An element $(x, x^*) \in E \times E^*$ is said to be monotonically related to the subset G provided

$$\langle x^* - y^*, x - y\rangle \geq 0 \quad \text{for all } (y, y^*) \in G.$$

We say that a monotone operator T is maximal monotone provided its graph is a maximal monotone set.

The most frequently used form of the definition of maximality of T is the following condition: Whenever $(x, x^*) \in E \times E^*$ is monotonically related to $G(T)$, then $x \in D(T)$ and $x^* \in T(x)$.

There is an obvious one-to-one correspondence between monotone sets and monotone operators. An easy application of Zorn's lemma shows that every monotone operator T can be extended to a maximal monotone operator $\overline{T}$, in the sense that $G(T) \subset G(\overline{T})$.

DEFINITION 1.4. If $T\colon E \to 2^{E^*}$ is a monotone operator, its inverse T^{-1} is the set valued mapping from E^* to 2^E defined by $T^{-1}(x^*) = \{x \in E\colon\ x^* \in T(x)\}$. Obviously,

$$G(T^{-1}) = \{(x^*, x) \in E^* \times E\colon\ x^* \in T(x)\},$$

which is (within a permutation) the same as the monotone set $G(T)$. In particular,then, T^{-1} is maximal monotone if and only if T is maximal monotone.

EXAMPLSE 1.5. (a) The monotone mapping φ defined in Example 1.2 (c) is maximal if and only if $\varphi(0) = [0,1]$. More generally, it is easily seen that a monotone nondecreasing function φ on $\mathbb{R}$ is maximal monotone if and only if $\varphi(x) = [\varphi(x^-), \varphi(x^+)]$ for each $x \in \mathbb{R}$ (where, for instance, $\varphi(x^-) \equiv \lim_{t\to x^-} \varphi(t)$).

(b) Any positive linear operator T on Hilbert space is maximal monotone. Indeed, suppose $(x, x^*) \in H \times H$ is monotonically related to $G(T)$. Then, for any $z \in H$ and $\lambda > 0$ we have

$$\begin{aligned} 0 \leq \langle T(x \pm \lambda z) - x^*, (x \pm \lambda z) - x\rangle &= \pm\lambda\langle T(x) \pm \lambda T(z) - x^*, z\rangle \\ &= \pm\lambda\langle T(x) - x^*, z\rangle + \lambda^2\langle T(z), z\rangle. \end{aligned}$$

Dividing by λ and then letting $\lambda \to 0$ shows that $\langle T(x) - x^*, z\rangle = 0$ for all $z \in H$, hence that $x^* = T(x)$.

EXERCISE 1.6. Prove that if T is maximal monotone, then $T(x)$ is a convex set, for every $x \in E$.

A major goal of these lectures is to examine the consequences of maximal monotonicity with regards to questions of convexity of $D(T)$ and $R(T)$ (or convexity of their interiors or closures). Other basic questions involve conditions under which $D(T) = E$ or $R(T) = E^*$, or whether these sets might be dense in E or E^*, respectively.

Perhaps the most fundamental result concerning monotone operators is the extension theorem of Debrunner–Flor [5]. An easy consequence of the following version of their theorem states that if T is maximal monotone and if its range is contained in a weak* compact convex set C, then $D(T) = E$. (Given $x_0 \in E$, let ϕ (below) be the constant mapping $\phi(x^*) = x_0$ for all $x^* \in C$, by the lemma, there exists $x_0^* \in C$ such that $\{(x_0, x_0^*)\} \cup G(T)$ is monotone; by maximality, $x_0 \in D(T)$.)

LEMMA 1.7. (Debrunner–Flor) *Suppose that C is a weak* compact convex subset of E^*, that $\phi\colon C \to E$ is weak* to norm continuous and that $M \subset E \times C$ is a monotone set. Then there exists $x_0^* \in C$ such that $\{(\phi(x_0^*), x_0^*)\} \cup M$ is a monotone set.*

Proof. For each element $(y, y^*) \in M$ let

$$U(y, y^*) = \{x^* \in C\colon \langle x^* - y^*, \phi(x^*) - y\rangle < 0\}.$$

Since $x^* \to \langle x^* - y^*, \phi(x^*) - y \rangle$ is weak* continuous on the bounded set C, each of these sets is relatively weak* open. If the conclusion of the lemma fails, then $C = \bigcup\{U(y,y^*)\colon\ (y,y^*) \in M\}$. By compactness, there must exist (y_1,y_1^*), (y_2,y_2^*), $\ldots$, (y_n,y_n^*) in M such that $C = \bigcup_{i=1}^n\{U(y_i,y_i^*)\}$. Let $\beta_1,\beta_2,\ldots,\beta_n$ be a partition of unity subordinate to this cover of C; that is, each β_i is weak* continuous on C, $0 \le \beta_i \le 1$, $\Sigma\beta_i = 1$ and $\{x^* \in C\colon\ \beta_i(x^*) > 0\} \subset U(y_i,y_i^*)$ for each i. Let $K = \text{co}\{y_i^*\} \subset C$ and define the weak* continuous map p of K into itself by

$$p(x^*) = \Sigma\beta_i(x^*)y_i^*, \quad x^* \in K.$$

Note that K is a finite dimensional compact convex set which (since the weak* topology is the same as the norm topology in finite dimensional spaces) is homeomorphic to a finite dimensional ball. Thus, the Brouwer fixed–point theorem is applicable. (See [11] for several proofs of the latter.) It follows that there exists $z^* \in K$ such that $p(z^*) = z^*$. We therefore have

$$\begin{aligned}0 &= \langle p(z^*) - z^*, \Sigma\beta_j(z^*)(y_j - \phi(z^*))\rangle\\ &= \langle \Sigma\beta_i(z^*)(y_i^* - z^*), \Sigma\beta_j(z^*)(y_j - \phi(z^*))\rangle\\ &= \Sigma_{i,j}\beta_i(z^*)\beta_j(z^*)\langle y_i^* - z^*, y_j - \phi(z^*)\rangle.\end{aligned}$$

Define $\alpha_{ij} = \langle y_i^* - z^*, y_j - \phi(z^*)\rangle$. It is straightforward to verify that

$$\alpha_{ij} + \alpha_{ji} = \alpha_{ii} + \alpha_{jj} + \langle y_i^* - y_j^*, y_j - y_i\rangle \le \alpha_{ii} + \alpha_{jj},$$

the inequality following from the monotonicity of M. Next, to simplify notation, let $\beta_i(z^*) = \beta_i$. Note that for all i,j,

$$\beta_i\beta_j\alpha_{ij} + \beta_j\beta_i\alpha_{ji} = \beta_i\beta_j(\frac{\alpha_{ij}+\alpha_{ji}}{2}) + \beta_j\beta_i(\frac{\alpha_{ij}+\alpha_{ji}}{2}).$$

It follows that

$$0 = \Sigma\beta_i\beta_j\alpha_{ij} = \Sigma\beta_i\beta_j(\frac{\alpha_{ij}+\alpha_{ji}}{2}) \le \Sigma\beta_i\beta_j(\frac{\alpha_{ii}+\alpha_{jj}}{2}).$$

We claim that this inequality implies that $\beta_i\beta_j = 0$ for all i,j. Indeed, for every pair i,j such that $\beta_i\beta_j > 0$ we must have $z^* \in U(y_i,y_i^*)\cap U(y_j,y_j^*)$, hence both $\alpha_{ii} < 0$ and $\alpha_{jj} < 0$ so that $\Sigma\beta_i\beta_j(\frac{\alpha_{ii}+\alpha_{jj}}{2}) < 0$, a contradiction. We conclude that $\beta_i \equiv \beta_i(z^*) = 0$ for all i, an impossibility, since $\Sigma\beta_i(z^*) = 1$. ∎

Definition 1.8. A set–valued mapping $T\colon E \to 2^{E^*}$ is said to be locally bounded at the point $x \in E$ provided there exists a neighborhood U of x such that $T(U)$ is a bounded set.

Note that this does not require that the point x actually be in $D(T)$. Thus, it is true (but not interesting) that T is locally bounded at each point of $E \setminus \overline{D(T)}$.

There are at least four proofs (all based on the Baire category theorem) that a maximal monotone operator T is locally bounded at the interior points of $D(T)$. (See the discussion in [18, Sec. 2].) The first one was by Rockafellar [22], who showed that much of it can be carried out in locally convex spaces. His proof is the longest one, but it has the great advantage of simultaneously proving that the interior of $D(T)$ is convex. By specializing his proof to Banach spaces (below), it is possible to shorten it considerably. We first make some simple general observations about an arbitrary subset $D \subset E$, its convex hull $\operatorname{co} D$ and its interior $\operatorname{int} D$.

(i) Always, $D \subset \operatorname{co} D$, so $\operatorname{int} D \subset \operatorname{int}(\operatorname{co} D)$. Suppose it is shown that $\operatorname{int}(\operatorname{co} D) \subset D$. Since $\operatorname{int}(\operatorname{co} D)$ is open, it is therefore a subset of int D and hence $\operatorname{int}(\operatorname{co} D) = \operatorname{int} D$, showing that int D is convex. (None of these assertions assume that $\operatorname{int} D$ is nonempty!)

(ii) However, if $\operatorname{int}(\operatorname{co} D)$ is nonempty and contained in D (so that $\operatorname{int}(\operatorname{co} D) = \operatorname{int} D$), then $\overline{D} = \overline{\operatorname{int}(\operatorname{co} D)}$, hence is convex. [This follows from the fact that for any convex set C with interior we have $C \subset \overline{\operatorname{int} C}$, hence $D \subset \operatorname{co} D \subset \overline{\operatorname{int}(\operatorname{co} D)}$ therefore $\overline{D} \subset \overline{\operatorname{int}(\operatorname{co} D)} = \overline{\operatorname{int} D} \subset \overline{D}$.]

(iii) Another useful elementary fact is the following: If $\{C_n\}$ is an increasing sequence of closed convex sets having nonempty interior, then $\operatorname{int} \bigcup C_n \subset \bigcup \operatorname{int} C_n$ (and, in fact, equality holds). [Here is a proof, the reader might have a simpler one: If $x \in \operatorname{int} \bigcup C_n$, then it is in some C_n, hence in the closure of $\bigcup \operatorname{int} C_n$. The latter is an open convex set, so if it doesn't contain x, there must exist a closed half-space H supporting its closure at x. Thus, $\operatorname{int} C_n \subset H$ for each n, therefore $C_n = \overline{\operatorname{int} C_n} \subset H$. This leads to the contradiction that $x \in \operatorname{int} \bigcup C_n \subset \operatorname{int} H$.]

NOTATION. We will denote the closed unit ball in E [resp. in E^*] by B [resp. B^*]. Thus, if $r > 0$, say, then

$$rB^* = \{x^* \in E^* \colon \|x^*\| \le r\}.$$

THEOREM 1.9. (Rockafellar) *Suppose that T is maximal monotone and that $\operatorname{int} \operatorname{co} D(T)$ is nonempty. Then $\operatorname{int} D(T) = \operatorname{int} \operatorname{co} D(T)$ (so $\operatorname{int} D(T)$ is convex) and T is locally bounded at each point of $\operatorname{int} D(T)$. Moreover, $\overline{D(T)} = \overline{\operatorname{int} D(T)}$, hence it is also convex.*

Proof. Let $C = \operatorname{int} \operatorname{co} D(T)$. For each $n \geq 1$ let

$$S_n = \{x \in nB \colon\ T(x) \cap nB^* \neq \emptyset\}.$$

Then $S_n \subset S_{n+1}$ and $D(T) = \bigcup S_n \subset \bigcup \operatorname{co} S_n$. Since the S_n's are increasing, this last union is convex, so it contains $\operatorname{co} D(T)$ and therefore contains C. As an open subset of a Banach space, C has the Baire property and therefore there exists an integer n_0 such that the closure (relative to C) of $C \cap \operatorname{co} S_n$ has nonempty interior for all $n \geq n_0$. In particular, the larger set $\operatorname{int}(\overline{\operatorname{co}}\, S_n)$ is nonempty for each such n. We have

$$\operatorname{int} \operatorname{co} D(T) \subset \operatorname{int} \bigcup_{n \geq n_0} \overline{\operatorname{co}}\, S_n \subset \bigcup_{n \geq n_0} \operatorname{int}(\overline{\operatorname{co}}\, S_n),$$

the last inclusion being a special case of the elementary fact (iii) described above. We will show two things; first, that T is locally bounded at any point in each set $\operatorname{int}(\overline{\operatorname{co}}\, S_n)$ $(n \geq n_0)$ and second, that each such point is in $D(T)$ (which, as noted in (i) above, will imply that $\operatorname{int} D(T)$ is convex). For the first step, then, suppose that $x_0 \in \operatorname{int}(\overline{\operatorname{co}}\, S_n)$ (for a fixed $n \geq n_0$). Assume without loss of generality that n is sufficiently large that $R(T) \cap nB^* \neq \emptyset$ and for each $m \geq n$ let

$$M_m = \{(u, u^*) \in E \times E^* \colon\ u \in D(T) \text{ and } u^* \in T(u) \cap mB^*\};$$

this is a nonempty monotone subset of $E \times mB^*$. For each $m \geq n$ and $x \in E$ define

$$A_m(x) = \{x^* \in E^* \colon\ \langle x^* - u^*, x - u\rangle \geq 0 \quad \forall u \in D(T) \text{ and } u^* \in T(u) \cap mB^*\}.$$

Since $A_m(x)$ is the set of all x^* such that (x, x^*) is monotonically related to M_m, the Debrunner–Flor Lemma 1.7 guarantees that it is nonempty. For each such m and every $x \in E$ we have $A_{m+1}(x) \subset A_m(x)$ and $T(x) \subset A_m(x)$. Moreover, as the intersection of weak*–closed half-spaces, each $A_m(x)$ is weak* closed. Suppose that $u^* \in nB^*$;then $S_n \subset \{x \in E \colon\ |\langle u^*, x\rangle| \leq n^2\}$ and therefore $\overline{\operatorname{co}}\, S_n$ is contained in the same set. By hypothesis, there exists $\epsilon > 0$ such that $x_0 + 2\epsilon B \subset \overline{\operatorname{co}}\, S_n$. Now, choose any $x \in x_0 + \epsilon B$ and $x^* \in A_n(x)$. For all $u \in S_n$ and $u^* \in T(u) \cap nB^*$ we must have

$$\langle x^*, u - x\rangle \leq \langle u^*, u - x\rangle \leq 2n^2,$$

which shows that $S_n \subset \{u \in E \colon\ \langle x^*, u - x\rangle \leq 2n^2\}$ hence $\overline{\operatorname{co}}\, S_n$ is contained in this same (closed and convex) set. Thus,

$$x_0 + 2\epsilon B \subset \overline{\operatorname{co}} S_n \subset \{u \in E \colon\ \langle x^*, u - x\rangle \leq 2n^2\}.$$

Suppose that $\|v\| \le \epsilon$, so that $x + v \in x_0 + 2\epsilon B \subset \overline{\mathrm{co}}\, S_n$. We then have $\langle x^*, v\rangle = \langle x^*, (x+v) - x\rangle \le 2n^2$. Thus, $\epsilon\|x^*\| = \sup\{\langle x^*, v\rangle \colon \|v\| \le \epsilon\} \le 2n^2$, which implies that $\|x^*\| \le 2n^2/\epsilon$. We have shown, then, that if $x \in x_0 + \epsilon B$ and $x^* \in A_n(x)$, then $x^* \in \frac{2n^2}{\epsilon} B^*$. Since

$$T(x_0+\epsilon B) = \bigcup\{T(x)\colon\ x \in x_0+\epsilon B\} \subset \bigcup\{A_n(x)\colon\ x \in x_0+\epsilon B\} \subset (2n^2/\epsilon)B^*,$$

we see that T is locally bounded at x_0. (Note that this part of the proof did not require that T be maximal.)

To show that $x_0 \in D(T)$, note that $A_n(x_0) \subset (2n^2/\epsilon)B^*$ and is weak* closed, hence is weak* compact. For $m \ge n$ we have $A_m(x_0) \subset A_n(x_0)$ and therefore the sequence $\{A_m(x_0)\}$ is a decreasing family of nonempty weak* compact sets. Let $x_0^* \in \bigcap_{m\ge n} A_m(x_0)$. If $u \in D(T)$ and $u^* \in T(u)$, then $\|u^*\| \le m$ for some $m \ge n$. By definition of $A_m(x_0)$ we have $\langle x_0^* - u^*, x_0 - u\rangle \ge 0$. By maximality of T, this implies that $x_0^* \in T(x_0)$ and therefore $x_0 \in D(T)$. ∎

COROLLARY 1.10. *Suppose that E is reflexive and that $T\colon E \to 2^{E^*}$ is maximal monotone. Then $\operatorname{int} R(T)$ is convex. If $\operatorname{int} R(T)$ is nonempty, then $\overline{R(T)}$ is convex.*

Proof. Since T^{-1} is a maximal monotone operator from E^* to 2^E, by the previous theorem, $\operatorname{int} D(T^{-1}) \equiv \operatorname{int} R(T)$ is convex.

That the first part of this corollary can fail in a nonreflexive space will be shown by Example 2.21 (below).

DEFINITION 1.11. A subset $A \subset E$ (not necessarily convex) which contains the origin is said to be absorbing if $E = \cup\{\lambda A : \lambda > 0\}$. Equivalently, A is absorbing if for each $x \in E$ there exists $t > 0$ such that $tx \in A$. A point $x \in A$ is called an absorbing point of A if the translate $A - x$ is absorbing.

It is obvious that any interior point of a set is an absorbing point. If A_1 is the union of the unit sphere and $\{0\}$, then A_1 is absorbing, even though it has empty interior. A proof of the following theorem of J. Borwein and S. Fitzpatrick [1] may be found in [18].

THEOREM 1.12. *Suppose that $T\colon E \to 2^{E^*}$ is monotone and that $x \in D(T)$. If x is an absorbing point of $D(T)$ (in particular, if $x \in \operatorname{int} D(T)$), then T is locally bounded at x.*

Note that the foregoing result does not require that $D(T)$ be convex nor that T be maximal. There are trivial examples which show that 0 can be an absorbing point of $D(T)$ but not an interior point (for instance, let T be the restriction of the duality mapping J to the set A_1 defined above). Even if $D(T)$ is convex and T is maximal monotone, $D(T)$ can have empty interior, as shown by the following example. (In this example, T is an unbounded linear operator, hence it is not locally bounded at any point and therefore $D(T)$ has no absorbing points.)

EXAMPLE 1.13. In the Hilbert space ℓ^2 let $D = \{x = (x_n) \in \ell^2\colon\ (2^n x_n) \in \ell^2\}$ and define $Tx = (2^n x_n)$, $x \in D$. Then $D(T) = D$ is a proper dense linear subspace of ℓ^2 and T is a positive operator, hence – by Example 1.5 (b) – it is maximal monotone.

It is conceivable that for a maximal monotone T, any absorbing point of $D(T)$ is actually an interior point. That this is true if $\overline{D(T)}$ is assumed to be convex is shown by combining Theorem 1.12 with the following result.

THEOREM 1.14. (Libor Veselý) *Suppose that T is maximal monotone and that $\overline{D(T)}$ is convex. If $x \in \overline{D(T)}$ and T is locally bounded at x, then $x \in \operatorname{int} D(T)$.*

Proof. The first step doesn't use the convexity hypothesis: Suppose that T is a maximal monotone operator which is locally bounded at the point $x \in \overline{D(T)}$; then $x \in D(T)$. Indeed, by hypothesis, there exists a neighborhood U of x such that $T(U)$ is a bounded set. Choose a sequence $\{x_n\} \subset D(T) \cap U$ such that $x_n \to x$ and choose $x_n^* \in T(x_n)$. By weak* compactness of bounded subsets of E^* there exists a subnet of $\{(x_n, x_n^*)\}$ – call it $\{(x_\alpha, x_\alpha^*)\}$ – and $x^* \in E^*$ such that $x_\alpha^* \to x^*$ (weak*). It follows that for all $(y, y^*) \in G(T)$,

$$\langle x^* - y^*, x - y\rangle = \lim_\alpha \langle x_\alpha^* - y^*, x_\alpha - y\rangle \geq 0;$$

by maximal monotonicity, $x^* \in T(x)$, so $x \in D(T)$. Next, if x is in the boundary of the closed convex set $\overline{D(T)}$, then T is not locally bounded at x: Suppose there were a neighborhood U of x such that $T(U)$ were bounded. By the Bishop–Phelps theorem there would exist a point $z \in U \cap \overline{D(T)}$ and a nonzero element $w^* \in E^*$ which supported $\overline{D(T)}$ at z; that is, $\langle w^*, z\rangle = \sup\langle z^*, D(T)\rangle$. Now, T would also be locally bounded at $z \in U$, so by the first step, $z \in D(T)$ and we could choose $z^* \in T(z)$. For any $(y, y^*) \in G(T)$

and any $\lambda \geq 0$ we would have

$$\langle z^* + \lambda w^* - y^*, z - y\rangle = \langle z^* - y^*, z - y\rangle + \lambda\langle w^*, z - y\rangle \geq 0.$$

By the maximality of T this would imply that $z^* + \lambda w^* \in T(z)$ for each $\lambda \geq 0$, which shows that $T(z)$ is not bounded, a contradiction. Since $x \notin \operatorname{bdry} \overline{D(T)}$, it must be in $\operatorname{int} \overline{D(T)}$. By local boundedness, we can choose an open set U such that $x \in U \subset \operatorname{int} \overline{D(T)}$ and $T(U)$ is bounded. Thus, T is locally bounded at every point of U, which – by the first step proved above – implies that $U \subset D(T)$ and therefore $x \in \operatorname{int} D(T)$. ∎

Note that this result, combined with Theorem 1.9, implies that for any maximal monotone T, if $\operatorname{int} D(T)$ is nonempty, then it is precisely the set of points in $\overline{D(T)}$ where T is locally bounded. It remains open as to what happens if $\operatorname{int} D(T)$ is empty and $\overline{D(T)}$ is not convex.

EXERCISE 1.15. Prove that if T is maximal monotone, then for all $x \in \operatorname{int} D(T)$ the set $T(x)$ is weak* compact and convex.

DEFINITION 1.16. Let X and Y be Hausdorff spaces and suppose that $T\colon X \to 2^Y$ is a set-valued mapping. We say that T is upper semicontinuous at the point $x \in X$ if the following holds: For every open set $U \subset Y$ such that $T(x) \subset U$ there exists an open subset V of X such that $x \in V$ and $T(V) \subset U$. Upper semicontinuity on a set is defined in the obvious way.

EXERCISE 1.17. Prove that if $T\colon E \to 2^{E^*}$ is maximal monotone, then it is norm–to–weak* upper semicontinuous on $\operatorname{int} D(T)$.

The following "fixed–point" result, which will be useful to us in Section 3, illustrates both the utility of the upper semicontinuity property as well as the strength of the Debrunner–Flor Lemma 1.7. It is a special case of Theorem 4 of [4].

LEMMA 1.18. *Suppose that E is a reflexive Banach space and that K is a nonempty compact convex subset of E. Let $R\colon K \to 2^K$ be an upper semicontinuous mapping such that $R(u)$ is nonempty, closed and convex, for each $u \in K$. Then there exists $u_0 \in K$ such that $u_0 \in R(u_0)$.*

Proof. Suppose there were no such point; then $0 \notin u - R(u)$ for each $u \in K$. By the separation theorem applied to the compact convex set $u - R(u)$ and 0, for each u there would exist $x^* \in E^*$, $\|x^*\| = 1$, and $\delta > 0$ such that $\langle x^*, v\rangle > \delta$ for each $v \in u - R(u)$. For each $x^* \in E^*$, define

$$W(x^*) = \{u \in K\colon \ \langle x^*, v\rangle > 0 \quad \forall v \in u - R(u)\}.$$

For each $\|x^*\| = 1$ let $U(x^*) = \{v \in E\colon \ \langle x^*, v\rangle > 0\}$, so $u \in W(x^*)$ if and only if $u \in K$ and $u - R(u) \subset U(x^*)$. Now, if $u \in K$, then our supposition implies that there exists $\|x^*\| = 1$ and $\delta > 0$ such that $u - R(u) + \delta B \subset U(x^*)$. Upper semicontinuity of R (hence of $-R$) at u implies that for some $0 < \epsilon < \delta/2$ we will have $-R(y) \subset -R(u) + \frac{\delta}{2}B$ whenever $y \in (u + \epsilon B) \cap K$. It follows that for all such y, we have

$$y - R(y) \subset u + \frac{\delta}{2}B - R(u) + \frac{\delta}{2}B \subset U(x^*),$$

that is, $(u + \epsilon B) \cap K \subset W(x^*)$. This shows that every point $u \in K$ is in the interior of some $W(x^*)$, so that the sets $\{\operatorname{int} W(x^*)\}$ form an open cover of K. As in the proof of Lemma 1.7, there is a finite subcover $\{\operatorname{int} W(x_j^*)\}_{j=1}^n$ of K and a continuous partition of unity $\{\beta_1, \beta_2, \dots, \beta_n\}$ subordinate to this cover. Define

$$r(x) = \sum \beta_j(x) x_j^*, \quad x \in K.$$

This is a continuous map from K into E^* and for all $u \in K$ and $v \in u - R(u)$ we have $\langle r(u), v\rangle = \sum \beta_j(u)\langle x_j^*, v\rangle > 0$ (since $\beta_j(u) > 0$ implies that $u \in W(x_j^*)$ hence that $\langle x_j^*, v\rangle > 0$). In Lemma 1.7 let $\phi = -r$, reverse the roles of E and E^* (reflexivity permits this) and let M be the monotone set $K \times \{0\}$ to obtain $u_0 \in K$ such that $-\langle r(u_0), u_0 - u\rangle \geq 0$ for all $u \in K$. In particular, this is true if $u = v_0$ where v_0 is any element of $R(u_0)$; that is, the element $v \equiv u_0 - v_0 \in u_0 - R(u_0)$ satisfies $\langle r(u_0), v\rangle \leq 0$, in contradiction to the fact that $\langle r(u_0), v\rangle > 0$, completing the proof. ∎

2. Subdifferentials of Convex Functions

This section introduces what is perhaps the most basic class of maximal monotone operators. We assume that the reader has some familiarity with real-valued convex functions. (A good reference for their elementary properties is [20].) Their use in optimization and convex analysis is most simply handled by introducing the seeming complication of admitting extended real-valued functions, that is, functions with values in $\mathbb{R} \cup \{\infty\}$.

Definition 2.1. Let X be a Hausdorff space and let $f\colon X \to \mathbb{R} \cup \{\infty\}$. The effective domain of f is the set $\operatorname{dom}(f) = \{x \in X\colon \ f(x) < \infty\}$. Recall that f is lower semicontinuous provided $\{x \in X\colon\ f(x) \leq r\}$ is closed in X for every $r \in \mathbb{R}$. This is equivalent to saying that the epigraph of f

$$\text{epi}(f) = \{(x, r) \in X \times \mathbb{R}\colon\ r \geq f(x)\}$$

is closed in $X \times \mathbb{R}$. Equivalently, f is lower semicontinuous provided

$$f(x) \leq \liminf f(x_\alpha)$$

whenever $x \in X$ and (x_α) is a net in X converging to x. We say that f is proper if $\text{dom}(f) \neq \emptyset$.

Note that if f is defined on a Banach space E and is convex, then so is $\text{dom}(f)$. Also, a function f is convex if and only if $\text{epi}(f)$ is convex. This last fact is important; it implies that certain properties of lower semicontinuous convex functions can be deduced from properties of these (rather special) closed convex subsets of $E \times \mathbb{R}$. One can view this as saying that the study of lower semicontinuous convex functions is a special case of the study of closed convex sets.

EXAMPLSE 2.2. (a) Let C be a nonempty convex subset of E; then the indicator function δ_C, defined by

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{otherwise,} \end{cases}$$

is a proper convex function which is lower semicontinuous if and only if C is closed.

This example is one reason for introducing extended real-valued functions, since it makes it possible to deduce certain properties of a closed convex set from properties of its lower semicontinuous convex indicator function. Thus, one can cite this example to support the view that the study of closed convex sets is a special case of the study of lower semicontinuous convex functions. It's all a matter of which approach is more convenient, the geometrical or the analytical. It is useful to be able to switch easily from one to the other.

(b) Let A be any nonempty subset of E^* such that the weak* closed convex hull of A is not all of E^* (or, more simply, let A be a weak* closed convex proper subset of E^*) and define the support function σ_A of A by

$$\sigma_A(x) = \sup\{\langle x^*, x\rangle\colon\ x^* \in A\}, \quad x \in E.$$

This is easily seen to be a proper lower semicontinuous convex function.

(c) If f is a continuous convex function defined on a nonempty closed convex set C, extend f to be ∞ at the points of $E \setminus C$; the resulting function is a proper lower semicontinuous convex function.

The next proposition uses completeness of E to describe a set where a lower semicontinuous convex function is necessarily continuous.

PROPOSITION 2.3. *Suppose that f is a proper lower semicontinuous convex function on a Banach space E and that $D = \operatorname{int}\operatorname{dom}(f)$ is nonempty; then f is continuous on D.*

Proof. We need only show that f is locally bounded in D, since this implies that it is locally Lipschitzian in D (see [18, Prop. 1.6]). First, note that if f is bounded above (by M, say) in $B(x;\delta) \subset D$ for some $\delta > 0$, then it is bounded below in $B(x;\delta)$. Indeed, if y is in $B(x;\delta)$, then so is $2x - y$ and

$$f(x) \le \frac{1}{2}[f(y) + f(2x-y)] \le \frac{1}{2}[f(y) + M],$$

so $f(y) \ge 2f(x) - M$ for all $y \in B(x;\delta)$. Thus, to show that f is locally bounded in D, it suffices to show that it is locally bounded above in D. For each $n \ge 1$, let $D_n = \{x \in D\colon\ f(x) \le n\}$. The sets D_n are closed and $D = \cup D_n$; since D is a Baire space, for some n we must have $U \equiv \operatorname{int} D_n$ nonempty. We know that f is bounded above by n in U; without loss of generality, we can assume that $B(0;\delta) \subset U$ for some $\delta > 0$. If y is in D, with $y \ne 0$, then there exists $\mu > 1$ such that $z = \mu y \in D$ and hence (letting $0 < \lambda = \mu^{-1} < 1$), the set

$$V = \lambda z + (1-\lambda)B(0;\delta) = y + (1-\lambda)B(0;\delta)$$

is a neighborhood of y in D. For any point $v = (1-\lambda)x + \lambda z \in V$ (where $x \in B(0;\delta)$) we have

$$f(v) \le (1-\lambda)n + \lambda f(z),$$

so f is bounded above in V and the proof is complete. ∎

EXAMPLSE 2.4. (a) The function f defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0,\infty), \\ \infty & \text{if } x \in (-\infty, 0] \end{cases}$$

shows that f can be continuous at a boundary point x of $\mathrm{dom}(f)$ where $f(x) = \infty$. (Recall that the neighborhoods of ∞ in $(-\infty, \infty]$ are all the sets $(a, \infty]$, $a \in \mathbb{R}$.)

(b) Suppose that C is nonempty closed and convex; then the lower semicontinuous convex indicator function δ_C is continuous at $x \in C$ if and only if $x \in \operatorname{int} C$. Thus, if $\operatorname{int} C = \emptyset$, then δ_C is not continuous at any point of $C = \mathrm{dom}(\delta_C)$.

DEFINITION 2.5. Recall that if E is a Banach space, then so is $E \times \mathbb{R}$, under any norm which restricts to give the original topology on the subspace E, for instance, $\|(x, r)\| = \|x\| + |r|$. Recall, also, that $(E \times \mathbb{R})^*$ can be identified with $E^* \times \mathbb{R}$, using the pairing

$$\langle (x^*, r^*), (x, r) \rangle = \langle x^*, x \rangle + r^* \cdot r.$$

Remark 2.6. If a proper lower semicontinuous convex function f is continuous at some point $x_0 \in \mathrm{dom}(f)$, then $\mathrm{dom}(f)$ has nonempty interior and $\mathrm{epi}(f)$ has nonempty interior in $E \times \mathbb{R}$. (Indeed, $f(x) = \infty$ outside of $\mathrm{dom}(f)$, so x_0 cannot be a boundary point of the latter. Moreover, there exists an open neighborhood U of x_0 in $\mathrm{dom}(f)$ in which $f(x) < f(x_0) + 1$, so the open product set $U \times \{r\colon\ r > f(x_0) + 1\}$ is contained in $\mathrm{epi}(f)$.)

DEFINITION 2.7. If $x \in \mathrm{dom}(f)$, define the subdifferential mapping ∂f by

$$\begin{aligned} \partial f(x) &= \{x^* \in E^*\colon\ \langle x^*, y - x \rangle \le f(y) - f(x) \text{ for all } y \in E\} \\ &= \{x^* \in E^*\colon\ \langle x^*, y \rangle \le f(x + y) - f(x) \text{ for all } y \in E\}, \end{aligned}$$

while $\partial f(x) = \emptyset$ if $x \in E \setminus \mathrm{dom}(f)$. It may also be empty at points of $\mathrm{dom}(f)$, as shown in the first example below.

It is easy to see that ∂f is a monotone operator: If $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$, then

$$\langle x^*, y - x \rangle \le f(y) - f(x) \quad \text{and} \quad -\langle y^*, y - x \rangle = \langle y^*, x - y \rangle \le f(x) - f(y);$$

now add these two inequalities.

It is not obvious that ∂f is maximal monotone; in fact, it is not even obvious that it is nontrivial (i.e., that $D(\partial f) \neq \emptyset$.) Much of the rest of this section is devoted to establishing these properties.

DEFINITION 2.8. If $x \in \mathrm{dom}(f)$ we define its right–hand directional derivative $d^+f(x)$ by

$$d^+f(x)(y) = \lim_{t \to 0^+} t^{-1}[f(x+ty) - f(x)], \quad y \in E.$$

It follows from the convexity of f that this limit always exists (see [18]). Note that $d^+f(x)(y) = \infty$ if $x + ty \in E \setminus \mathrm{dom}(f)$ for all $t > 0$. (It is also possible to have $d^+f(x)(y) = -\infty$; consider, for instance, $d^+f(0)(1)$ when $f(x) = -x^{1/2}$ for $x \geq 0$, $= \infty$ elsewhere.) We have the following important relationship: For any point $x \in \mathrm{dom}(f)$,

$$x^* \in \partial f(x) \text{ if and only if } \langle x^*, y\rangle \leq d^+f(x)(y) \text{ for all } y \in E.$$

It follows from this that for the example given above ($f(x) = -\sqrt{x}$ for $x \geq 0$), it must be true that $\partial f(0) = \emptyset$. In the first example below, one sees that it is possible to have $\partial f(x) = \emptyset$ for a dense set of points $x \in \mathrm{dom}(f)$.

EXAMPLSE 2.9. (a) Let C be the closed (in fact, compact) convex subset of ℓ^2 defined by

$$C = \{x \in \ell^2 \colon\ |x_n| \leq 2^{-n},\ n = 1,\, 2,\, 3,\, \ldots\}$$

and define f on C by $f(x) = \Sigma[-(2^{-n} + x_n)^{1/2}]$. Since each of the functions $x \to -(2^{-n} + x_n)^{1/2}$ is continuous, convex and bounded in absolute value by $2^{(-n+1)/2}$, the series converges uniformly, so f is continuous and convex. We claim that $\partial f(x) = \emptyset$ for any $x \in C$ such that $x_n > -2^{-n}$ for infinitely many n. Indeed, let e_n denote the n-th unit vector in ℓ^2. If $x^* \in \partial f(x)$ (so that, as noted above, $x^* \leq d^+f(x)$), then for all n such that $x_n > -2^{-n}$, we have

$$-\|x^*\| \leq \langle x^*, e_n\rangle \leq d^+f(x)(e_n) = -(1/2)(2^{-n} + x_n)^{-1/2},$$

an impossibility which implies that $\partial f(x) = \emptyset$. Note that if we make the usual extension (setting $f(x) = \infty$ for $x \in \ell^2 \setminus C$), then f is lower semicontinuous, but not continuous at any point of C ($= \mathrm{bdry}\, C$).

(b) Let C be a nonempty closed convex subset of E; then for any $x \in C$, the subdifferential $\partial\delta_C(x)$ of the indicator function δ_C is the cone with vertex 0 of all $x^* \in E^*$ which "support" C at x, that is, which satisfy

$$\langle x^*, x\rangle = \sup\{\langle x^*, y\rangle \colon\ y \in C\} \equiv \sigma_C(x^*).$$

(Indeed, $x^* \in \partial\delta_C(x)$ if and only if $\langle x^*, y - x\rangle \leq \delta_C(y) - \delta_C(x)$, while x^* attains its supremum on C at x if and only if the left hand side of this latter inequality is at most 0, while the right side is always greater or equal to 0.)

The following notion of an approximate subdifferential is useful in many parts of convex analysis.

DEFINITION 2.10. Let f be a proper convex lower semicontinuous function and suppose $x \in \mathrm{dom}(f)$. For any $\epsilon > 0$ define the ϵ–subdifferential $\partial_\epsilon f(x)$ by

$$\partial_\epsilon f(x) = \{x^* : \langle x^*, y\rangle \le f(x+y) - f(x) + \epsilon \text{ for all } y \in E\}.$$

It follows easily from the definition that $\partial_\epsilon f(x)$ convex and weak* closed. That fact that it is nonempty for every $x \in \mathrm{dom}(f)$ follows from the convexity of $\mathrm{epi}(f)$ and the separation theorem (in $E \times \mathbb{R}$) (see [18, Prop. 3.14]).

The basic maximality technique which was used in proving the Bishop–Phelps theorem [18] was applied in $E \times \mathbb{R}$ by Brøndsted and Rockafellar to prove the following fundamental lemma, which shows, among other things, that ∂f is nontrivial.

LEMMA 2.11. *Suppose that f is a convex proper lower semicontinuous function on the Banach space E. Then given any point $x_0 \in \mathrm{dom}(f)$, $\epsilon > 0$, $\lambda > 0$ and any functional $x_0^* \in \partial_\epsilon f(x_0)$, there exist $x \in \mathrm{dom}(f)$ and $x^* \in E^*$ such that*

$$x^* \in \partial f(x), \quad \|x - x_0\| \le \frac{\epsilon}{\lambda} \quad \text{and} \quad \|x^* - x_0^*\| \le \lambda.$$

In particular, the domain of ∂f is dense in $\mathrm{dom}(f)$, *so* $\overline{D(\partial f)} = \overline{\mathrm{dom}(f)}$ *is convex.*

The next result can be looked at in two ways. On the one hand, it is simply a verification that (under a certain hypothesis), the subdifferential operation is additive. (We use the usual vector sum of sets.) On the other hand, it will be seen later as a verification in a special case that the sum of maximal monotone operators is again maximal monotone.

THEOREM 2.12. *Suppose that f and g are convex proper lower semicontinuous functions on the Banach space E and that there is a point in* $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ *where one of them, say f, is continuous. Then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \quad x \in D(\partial f) \cap D(\partial g).$$

Remark. It is immediate from the definitions that for $x \in \mathrm{dom}(f+g)$ (which is identical to $\mathrm{dom}(f) \cap \mathrm{dom}(g)$), one must have

$$\partial f(x) + \partial g(x) \subset \partial(f+g)(x).$$

This inclusion can be proper. To see this, let $E = \mathbb{R}^2$, let f denote the indicator function δ_C and let $g = \delta_L$, where C is the epigraph of the quadratic function $y = x^2$ and L is the x-axis. Obviously, C and L intersect only at the origin 0 and it is easily verified that $\partial f(0) = \mathbb{R}^- e$, where e is the vector $(0,1)$, and $\partial g(0) = \mathbb{R}e$, while

$$\partial(f+g)(0) = \mathbb{R}^2 \neq \partial f(0) + \partial g(0).$$

Proof. Suppose that $x_0^* \in \partial(f+g)(x_0)$. In order to simplify the argument, we can replace f and g by the functions

$$f_1(x) = f(x+x_0) - f(x_0) - \langle x_0^*, x\rangle \quad \text{and} \quad g_1(x) = g(x+x_0) - g(x_0), \quad x \in E;$$

it is readily verified from the definitions that if $x_0^* \in \partial(f+g)(x_0)$, then $0 \in \partial(f_1+g_1)(0)$ and if $0 \in \partial f_1(0) + \partial g_1(0)$, then $x_0^* \in \partial f(x_0) + \partial g(x_0)$. Without loss of generality, then, we assume that $x_0 = 0$, $x_0^* = 0$, $f(0) = 0$ and $g(0) = 0$. We want to conclude that 0 is in the sum $\partial f(0) + \partial g(0)$, under the hypothesis that $0 \in \partial(f+g)(0)$. This last means that

$$(f+g)(x) \geq (f+g)(0) = 0 \quad \text{for all} \quad x \in E. \tag{2.1}$$

We now apply the separation theorem in $E \times \mathbb{R}$ to the two closed convex sets $C_1 = \operatorname{epi}(f)$ and $C_2 = \{(x,r)\colon r \leq -g(x)\}$; this is possible because f has a point of continuity in $\operatorname{dom}(f) \cap \operatorname{dom}(g)$ and hence – recall Remark 2.6 – C_1 has nonempty interior. Moreover, it follows from (2.1) that C_2 misses the interior of $C_1 = \{(x,r)\colon r > f(x)\}$. Since $(0,0)$ is common to both sets, it is contained in any separating hyperplane. Thus, there exists a functional $(x^*, r^*) \in E^* \times \mathbb{R}$, $(0,0) \neq (x^*, r^*)$, such that

$$\langle x^*, x\rangle + r^* \cdot r \geq 0 \quad \text{if} \quad r \geq f(x) \quad \text{and} \quad \langle x^*, x\rangle + r^* \cdot r \leq 0 \quad \text{if} \quad r \leq -g(x).$$

Since $1 > f(0) = 0$ we see immediately that $r^* \geq 0$. To see that $r^* \neq 0$, (that is, that the separating hyperplane is not "vertical"), we argue by contradiction: If $r^* = 0$, then we must have $x^* \neq 0$; also $\langle x^*, x\rangle \geq 0$ for all $x \in \operatorname{dom}(f)$ and $\langle x^*, x\rangle \leq 0$ for all $x \in \operatorname{dom}(g)$. This says that x^* separates these two sets. This is impossible; by the continuity hypothesis, their intersection contains an interior point of $\operatorname{dom}(f)$. Without loss of generality, then, we can assume that $r^* = 1$ and hence, for any $x \in E$,

$$\langle -x^*, x-0\rangle \leq f(x) - f(0) \quad \text{and} \quad \langle x^*, x-0\rangle \leq g(x) - g(0),$$

that is, $0 = -x^* + x^* \in \partial f(0) + \partial g(0)$, which completes the proof. ∎

The next two lemmas lead easily to S. Simons' recent proof [25], [27] of Rockafellar's maximal monotonicity theorem for subdifferentials.

LEMMA 2.13. *Suppose that f is a lower semicontinuous proper convex function on E. If $\alpha, \beta > 0$, $x_0 \in E$ and $f(x_0) < \inf_E f + \alpha\beta$, then there exist $x \in E$ and $x^* \in \partial f(x)$ such that $\|x - x_0\| < \beta$ and $\|x^*\| < \alpha$.*

Proof. Choose $\epsilon > 0$ such that $f(x_0) - \inf_E f < \epsilon < \alpha\beta$ and then choose λ such that $\epsilon/\beta < \lambda < \alpha$. It follows that $0 \in \partial_\epsilon f(x_0)$ so by the Brøndsted–Rockafellar Lemma 2.11, there exist $x \in \operatorname{dom}(f)$ and $x^* \in \partial f(x)$ such that $\|x^*\| \le \lambda < \alpha$ and $\|x - x_0\| \le \epsilon/\lambda < \beta$. ∎

LEMMA 2.14. *With f as in the previous lemma, suppose that $x \in E$ (not necessarily in $\operatorname{dom}(f)$) and that $\inf_E f < f(x)$. Then there exist $z \in \operatorname{dom}(f)$ and $z^* \in \partial f(z)$ such that*

$$f(z) < f(x) \quad \text{and} \quad \langle z^*, x - z\rangle > 0.$$

Proof. Fix $\lambda \in \mathbb{R}$ such that $\inf_E f < \lambda < f(x)$ and let

$$K = \sup_{y\in E,\ y\neq x} \frac{\lambda - f(y)}{\|y - x\|}.$$

We first show that $0 < K < \infty$. To that end, let $F = \{y \in E\colon\ f(y) \le \lambda\}$, so F is closed, nonempty and $x \notin F$. Since $\operatorname{dom}(f) \neq \emptyset$, one can apply the separation theorem in $E \times \mathbb{R}$ to find $u^* \in E^*$ and $r \in \mathbb{R}$ such that $f \ge u^* + r$. Suppose that $y \in E$ and that $y \neq x$. If $y \in F$, then

$$\lambda - f(y) \le \lambda - \langle u^*, y\rangle - r \le |\lambda - \langle u^*, x\rangle - r| + \langle u^*, x - y\rangle,$$

hence

$$\frac{\lambda - f(y)}{\|y - x\|} \le \frac{|\lambda - \langle u^*, x\rangle - r|}{\operatorname{dist}(x, F)} + \|u^*\|.$$

If $y \notin F$, then $\frac{\lambda - f(y)}{\|y-x\|} < 0$. In either case, there is an upper bound for $\frac{\lambda - f(y)}{\|y-x\|}$, so $K < \infty$. To see that $K > 0$, pick any $y \in E$ such that $f(y) < \lambda$. Since $\lambda < f(x)$, we have $y \neq x$ and $K \ge \frac{\lambda - f(y)}{\|y-x\|} > 0$.

Suppose, now, that $0 < \epsilon < 1$, so that $(1 - \epsilon)K < K$ and hence, by definition of K, there exists $x_0 \in E$ such that $x_0 \neq x$ and

$$\frac{\lambda - f(x_0)}{\|x_0 - x\|} > (1 - \epsilon)K.$$

For $z \in E$, let $N(z) = K\|z-x\|$; we have shown that $(1-\epsilon)N(x_0)+f(x_0) < \lambda$, that is, $(N+f)(x_0) < \lambda + \epsilon N(y)$. We claim that $\lambda \leq \inf_E(N+f)$. Indeed, if $z = x$, then we have $\lambda < f(x) = (N+f)(z)$, while if $z \neq x$, then $\frac{\lambda - f(z)}{\|z-x\|} \leq K$, from which it follows that $\lambda \leq (N+f)(z)$. Thus, we have shown that there is a point $x_0 \in E$, $x_0 \neq x$, such that

$$(N+f)(x_0) < \inf_E(N+f) + \epsilon K\|x_0 - x\|.$$

We now apply Lemma 2.13 to $N+f$, with $\beta = \|x_0 - x\|$ and $\alpha = \epsilon K$. Thus, there exists $z \in \operatorname{dom}(N+f) \equiv \operatorname{dom}(f)$ and $w^* \in \partial(N+f)(z)$ such that $\|z - x_0\| < \|x - x_0\|$ and $\|w^*\| < \epsilon K$. It follows that $\|z - x\| > 0$. From the sum formula (Theorem 2.12),

$$\partial(N+f)(z) = \partial N(z) + \partial f(z),$$

so there exist $y^* \in \partial N(z)$ and $z^* \in \partial f(z)$ such that $w^* = y^* + z^*$. Since $y^* \in \partial N(z)$, we must have $\langle y^*, z - x\rangle \geq N(z) - N(x) = K\|z - x\|$. Thus

$$\begin{aligned}\langle z^*, x - z\rangle &= \langle y^*, z - x\rangle + \langle w^*, x - z\rangle \\ &\geq K\|z - x\| - \|w^*\| \cdot \|x - z\| \\ &> (1-\epsilon)K\|z - x\| > 0.\end{aligned}$$

Since $z^* \in \partial f(z)$, we have $f(x) \geq f(z) + \langle z^*, x - z\rangle > f(z)$, which completes the proof. ∎

THEOREM 2.15. (Rockafellar) *If f is a proper lower semicontinuous convex function on a Banach space E, then its subdifferential ∂f is a maximal monotone operator.*

Proof. Suppose that $x \in E$, that $x^* \in E^*$ and that $x^* \notin \partial f(x)$. Thus, $0 \notin \partial(f - x^*)(x)$, which implies that $\inf_E(f - x^*) < (f - x^*)(x)$. By Lemma 2.14 there exists $z \in \operatorname{dom}(f - x^*) \equiv \operatorname{dom}(f)$ and $z^* \in \partial(f - x^*)(z)$ such that $\langle z^*, z - x\rangle < 0$. Thus, there exists $y^* \in \partial f(z)$ such that $z^* = y^* - x^*$, so that $\langle y^* - x^*, z - x\rangle < 0$. ∎

EXAMPLE 2.16. In a Banach space E define $j(x) = (1/2)\|x\|^2$; this is clearly continuous and convex. The monotone duality mapping J is actually the subdifferential of j, and hence is maximal monotone.

Proof. It is readily computed that $d^+j(x)(y) = \|x\| \cdot d^+\|x\|(y)$. If $x = 0$, then $d^+j(0)(y) = 0$ for all y, hence is linear and therefore $\partial j(0) = \{0\}$. Suppose, then, that $x \neq 0$. We know (from the remark following Definition 2.8) that $x^* \in \partial j(x)$ if and only if $x^* \leq d^+j(x)$, that is, if and only if $\|x\|^{-1}x^* \leq d^+\|x\|$, which is equivalent to $y^* \equiv \|x\|^{-1}x^* \in \partial\|x\|$, that is, if and only if $\langle y^*, y - x\rangle \leq \|y\| - \|x\|$ for all $y \in E$. If, in this last inequality, we take $y = x + z$, $\|z\| \leq 1$ and apply the triangle inequality, we conclude that $\|y^*\| \leq 1$. If we take $y = 0$, we conclude that $\|x\| \leq \langle y^*, x\rangle \leq \|y^*\| \cdot \|x\|$, so $\|y^*\| = 1$ and $\langle y^*, x\rangle = \|x\|$, which is equivalent to what we want to prove. The converse is easy: If $\|y^*\| = 1$ and $\langle y^*, x\rangle = \|x\|$, then for all y in E, we necessarily have $\langle y^*, y - x\rangle \leq \|y\| - \|x\|$, so $y^* \in \partial\|x\|$. ∎

There are other interesting and useful properties of the duality mapping J. For instance, it is immediate from the original definition that it satisfies $J(-x) = -J(x)$ and $J(\lambda x) = \lambda J(x)$ for $\lambda > 0$. Since it is the subdifferential of the function $\frac{1}{2}\|x\|^2$, it is not surprising that it reflects properties of the norm.

PROPOSITION 2.17. (a) *The norm in E is Gateaux differentiable (at nonzero points) if and only if J is single valued.*

(b) *The mapping J is "one–to–one" (that is, $J(x) \cap J(y) = \emptyset$ whenever $x \neq y$) if and only if the norm in E is strictly convex; that is, $\|x + y\| < 2$ whenever $\|x\| = 1$, $\|y\| = 1$ and $x \neq y$.*

(c) *Surjectivity of J is equivalent to reflexivity of E.*

Proof. Parts (a) and (b) are straightforward exercises. Part (c) is an easy consequence of R. C. James' deep result that a Banach space E is reflexive if (and only if) each functional in E^* attains its supremum on the unit ball of E at some point. Indeed, if J is surjective and $x^* \in E^*$, then there exists $x \in E$ such that $x^* \in J(x)$, hence $\langle x^*, \frac{x}{\|x\|}\rangle = \|x^*\| \equiv \sup\{\langle x^*, y\rangle \colon \|y\| \leq 1\}$, showing x^* attains its supremum on the unit ball at $\frac{x}{\|x\|}$. ∎

DEFINITION 2.18. If S and T are monotone operators and $x \in D(S) \cap D(T)$, we define

$$(S + T)(x) = S(x) + T(x) \equiv \{x^* + y^* \colon\ x^* \in S(x),\ y^* \in T(x)\},$$

while $(S + T)(x) = \emptyset$ otherwise.

It is immediate that $S + T$ is also a monotone operator and that – by definition – $D(S + T) = D(S) \cap D(T)$.

The following theorem of Rockafellar [21] is basic to many applications of maximal monotone operators.

THEOREM 2.19. (Rockafellar) *Suppose that E is reflexive, that S and T are maximal monotone operators on E and that $D(T) \cap \operatorname{int} D(S) \neq \emptyset$. Then $S+T$ is maximal.*

We refer to [21] for the proof of this theorem (see, also, [3]), which relies partly on the fact that any reflexive space can be renormed so that both it and its dual norm are Gateaux differentiable (at nonzero points) [7]. Since maximal monotonicity does not depend on which norm defines the topology of E, this guarantees that the duality mapping J can be assumed to be single–valued, one–to–one and onto, a useful step in the proof.

The situation is wide open in arbitrary Banach spaces.

PROBLEM 2.20. Suppose that E is a nonreflexive Banach space and that S and T are maximal monotone operators such that $D(T) \cap \operatorname{int} D(S) \neq \emptyset$; is $S+T$ necessarily maximal? What about the special case when S is the subdifferential of the indicator function δ_C of a closed convex set C for which $\operatorname{int} C \cap D(T) \neq \emptyset$?

Remark. The answer to the first question is affirmative when $D(S) = E = D(T)$; this was pointed out to us by Martin Heisler. One can use the local boundedness of S and T and weak* compactness to prove that the graph of $G(S+T)$ is closed in the norm $\times$ weak* topology in $E \times E^*$; Lemma 2.2 of [9] then applies to show that $S+T$ is maximal.

Rockafellar [21, Theorem 3] has shown that the answer to the second question is affirmative for certain single–valued monotone operators T.

The answer to the first question is also affirmative whenever $S = \partial f$ and $T = \partial g$, where f and g are proper lower semicontinuous convex functions; this is a consequence of Theorem 2.12 (that the subdifferential of the sum of two convex functions is the sum of their subdifferentials). Indeed, if $\operatorname{int} D(T) \neq \emptyset$, then [since $\operatorname{int}\operatorname{dom}(\partial g) \subset \operatorname{int}\operatorname{dom}(g)$ and g is continuous on $\operatorname{int}\operatorname{dom}(g)$] we conclude that g is continous at some point of $D(S) \cap D(T) \subset \operatorname{dom}(f) \cap \operatorname{dom}(g)$, so Theorem 2.12 implies that $S+T = \partial(f+g)$ and Theorem 2.15 implies that the latter is maximal monotone. The remark following the statement of Theorem 2.12 shows that, even in a two dimensional Banach space, maximality of a sum can fail if $D(S) \cap \operatorname{int} D(T)$ is empty; in that example, the graph of

$\partial f + \partial g$ is a proper subset of the graph of the maximal monotone operator $\partial(f+g)$.

The foregoing discussion is an example of how subdifferentials fulfill their role as prototypes when considering general questions about maximal monotone operators. Whenever a property is valid for subdifferentials in arbitrary Banach spaces there is some hope that it also holds for all maximal monotone operators. On the other hand, of course, if it fails for subdifferentials on nonreflexive spaces, the situation is obviously hopeless. The following example illustrates this with respect to the first assertion in Corollary 1.10 (that int $R(T)$ is convex when E is reflexive).

EXERCISE 2.21. (Simon Fitzpatrick) There exists a continuous convex function f on the Banach space c_0 such that the interior of $R(\partial f)$ is not convex.

Proof. With the usual supremum norm on c_0, define $g(x) = \|x\|$ and $h(x) = \|x - e_1\|$, where $e_1 = (1, 0, 0, \dots)$, and let $f = g + h$. Since g and h are continous and convex, we have $\partial f(x) = \partial g(x) + \partial h(x)$ for each $x \in c_0$. It is straightforward to compute that $\partial g(x)$ is either B^* if $x = 0$, or is contained in the set F of all finitely nonzero sequences in ℓ_1 otherwise. It is also easy to see that $\partial h(x) = \partial g(x - e_1)$ for each x. It follows (letting e_1^* denote the corresponding element of ℓ_1) that $\partial f(0) = -e_1^* + B^*$ and $\partial f(e_1) = e_1^* + B^*$, while $\partial f(x)$ is contained in F if $x \neq 0, e_1$. Since $\operatorname{int} R(\partial f) \supset \operatorname{int} B^* \pm e_1^*$, if it were convex, it would contain $0 = \frac{1}{2}(e_1^* - e_1^*)$ and hence a neighborhood U of 0. But, for any $\lambda > 0$, the element $(0, \frac{\lambda}{2^2}, \frac{\lambda}{2^3}, \frac{\lambda}{2^4}, \dots)$ is not in F, has distance from $\pm e_1^*$ equal to $1 + \frac{\lambda}{2} > 1$ and (for sufficiently small λ) is in U. ∎

Another place one looks for prototypical properties is in Hilbert space (see, for instance, [2]), where things are made much easier by the fact that the duality mapping J is replaced by the identity mapping.

We conclude this section with a look at an additional property which characterizes subdifferentials within the class of maximal monotone operators. Details may be found, for instance, in [18].

DEFINITION 2.22. A set-valued map $T\colon E \to 2^{E^*}$ is said to be n-cyclically monotone provided

$$\sum_{1 \le k \le n} \langle x_k^*, x_k - x_{k-1} \rangle \ge 0$$

whenever $n \geq 2$ and $x_0, x_1, x_2, \dots x_n \in E$, $x_n = x_0$, and $x_k^* \in T(x_k)$, $k = 1, 2, 3, \dots, n$. We say that T is cyclically monotone if it is n-cyclically monotone for every n. Clearly, a 2-cyclically monotone operator is monotone.

EXAMPLSE 2.23. (a) The linear map in $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, -x_1)$ is positive, hence maximal monotone, but it is not 3-cyclically monotone: Look at the points $(1,1)$, $(0,1)$ and $(1,0)$.

(b) Let f be a proper lower semicontinuous convex function; then ∂f is cyclically monotone.

The final theorem of this section shows that this is the only such example [23].

THEOREM 2.24. (Rockafellar) *If $T\colon E \to 2^{E^*}$ is maximal monotone and cyclically monotone, with $D(T) \neq \emptyset$, then there exists a proper convex lower semicontinuous function f on E such that $T = \partial f$.*

3. GOSSEZ'S MONOTONE OPERATORS OF TYPE (D)

In 1971, J.–P. Gossez [12] introduced the class of monotone operators of "dense type" in order to extend to nonreflexive spaces some of the basic known results about maximal monotone operators on reflexive spaces. He subsequently modified his definition [14], [16] to the one given below. One first identifies a Banach space E with its canonically embedded image $\widehat{E}$ in E^{**}. Having done this, it is natural to consider the graph $G(T)$ of a monotone operator T to be a subset of $E^{**} \times E^*$.

DEFINITION 3.1. A monotone operator $T\colon E \to 2^{E^*}$ is said to of type (D) provided it satisfies the following property: If $(x^{**}, x^*) \in E^{**} \times E^*$ is monotonically related to $G(T)$, then there exists a net $(x_\alpha, x_\alpha^*) \in G(T)$ such that $x_\alpha \to x^{**}$ in the $\sigma(x^{**}, x^*)$ topology, (x_α) is bounded and $x_\alpha^* \to x^*$ in norm.

In using this definition it is convenient to extend T to a mapping $\overline{T}\colon E^{**} \to 2^{E^*}$ as follows: We let $\overline{T}$ be the map whose graph $G(\overline{T}) \subset E^{**} \times E^*$ consists of all elements $(x^{**}, x^*) \in E^{**} \times E^*$ which are monotonically related to $G(T)$.

The map $\overline{T}$ need not be monotone [15]. However, T is monotone of type (D), then $\overline{T}$ is maximal monotone. Indeed, suppose that (x^{**}, x^*) is monotonically related to $G(\overline{T})$; it is then obviously monotonically related to $G(T)$, hence is in $G(\overline{T})$. It only remains to show that $\overline{T}$ is monotone. Suppose that

(x^{**}, x^*), $(y^{**}, y^*) \in G(\overline{T})$. By hypothesis, there exists a net $(x_\alpha, x_\alpha^*) \in G(T)$ as above, hence

$$0 \le \langle y^{**} - \hat{x}_\alpha, y^* - x_\alpha^* \rangle = \langle y^{**} - x^{**}, y^* - x^* \rangle + \langle x^{**} - \hat{x}_\alpha, y^* - x^* \rangle + \langle y^{**} - \hat{x}_\alpha, x^* - x_\alpha^* \rangle.$$

Taking limits, we see that $\langle y^{**} - x^{**}, y^* - x^* \rangle \ge 0$.

A word about terminology: When T is monotone of type (D), then any element of $G(\overline{T})$ is the limit of a certain net in $G(T)$, so we can consider the latter as being dense in the former, hence Gossez's earlier "dense type" and the later use of "type (D)".

EXAMPLSE 3.2. (a) Define $T\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by $T(x) = 0$ for all $x \ne 0$. It is easily verified that $G(\overline{T}) = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$, hence is maximal monotone, while T is monotone of type (D) (but not maximal).

(b) Suppose that E is reflexive. If T is maximal monotone, then $\overline{T} = T$, hence T is trivially maximal monotone of type (D), that is, in reflexive spaces, the maximal monotone operators coincide with the maximal monotone operators of type (D).

(c) If f is a proper lower semicontinuous convex function on E, then ∂f is maximal monotone of type (D) [12]. (This is not obvious. In essence, it uses the first step of Rockafellar's original proof [23] of the maximal monotonicity of ∂f.)

The fundamental fact about maximal monotone operators of type (D) is contained in Theorem 3.6 (below). In order to formulate it, we need to recall a few facts about convex functions and their Fenchel duals.

DEFINITION 3.3. If f is a proper convex lower semicontinuous function on E, then the function f^* defined on E^* by

$$f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x)\}, \quad x^* \in E^*$$

is proper, convex and weak* lower semicontinuous on E^*. One defines $f^{**} \equiv (f^*)^*$ on E^{**} analogously.

EXERCISES 3.4. (a) If we let $j(x) = \frac{1}{2}\|x\|^2$, then $j^*(x^*) = \frac{1}{2}\|x^*\|^2$ and $j^{**}(x^{**}) = \frac{1}{2}\|x^{**}\|^2$.

(b) If f is convex proper and lower semicontinuous, and $\epsilon \ge 0$, then $\partial_\epsilon f$ can be characterized as follows: $x^* \in \partial_\epsilon f(x)$ if and only if $f(x) + f^*(x^*) \le \langle x^*, x \rangle + \epsilon$. (Here, we take $\partial_0 f \equiv \partial f$.)

The following theorem, which is a very special case of a result of F. Browder [4, Theorem 10], provides a crucial step in proving Theorem 3.6 (below).

THEOREM 3.5. *Suppose that F is a finite dimensional Banach space with unit ball B and duality map J. Fix $r > 0$ and let A denote the restriction of $-J$ to rB. If $T\colon\ rB \to 2^{F^*}$ is monotone, then there exists $(x, x^*) \in G(A)$ such that $G(T) \cup \{(x, x^*)\}$ is monotone, that is, there exists $x \in rB$ and $x^* \in -J(x) \subset rB^*$ such that $\langle x^* - y^*, x - y\rangle \geq 0$ for all $(y, y^*) \in G(T)$.*

Proof. For each $\epsilon > 0$ and finite subset $G \subset G(T)$, let

$$H_{\epsilon,G} = \{(x, x^*) \in G(A)\colon\ \langle x^* - y^*, x - y\rangle \geq -\epsilon \quad \forall\, (y, y^*) \in G\}$$

and

$$H = \{(x, x^*) \in G(A)\colon\ (x, x^*) \text{ is monotonically related to } G(T)\}.$$

We will have proved the theorem if we show that the set H is nonempty. It is straightforward to verify that $H = \bigcap\{H_{\epsilon,G}\colon\ \epsilon > 0,\ G \text{ finite},\ G \subset G(T)\}$. Moreover, from the compactness of $rB \times rB^*$ and the definition of J, it follows that each $H_{\epsilon,G}$ is compact. The intersection of any finite number of these sets contains a set of the same form, so to show that the family has the finite intersection property it suffices to show that each of them is nonempty; it will then follow by compactness that H is nonempty. Fix ϵ and G. To show that $H_{\epsilon,G}$ is nonempty, we define, for each $x \in rB$,

$$S(x) = \{x^* \in rB^*\colon\ \langle x^* - y^*, x - y\rangle > -\epsilon \quad \forall\, (y, y^*) \in G\}$$

and for $x^* \in rB^*$,

$$S^{-1}(x^*) = \{x \in rB\colon\ \langle x^* - y^*, x - y\rangle > -\epsilon \quad \forall\, (y, y^*) \in G\}.$$

Each set $S(x)$ is a finite intersection of relatively open subsets of rB^*, hence is relatively open and each set $S^{-1}(x^*)$ is convex. Moreover, each of the latter sets is nonempty: This is an application of Lemma 1.7, using the monotonicity of G, reversing the roles played there by E and E^* and letting $\phi\colon\ rB \to rB^*$ be the constant map with value x^*. It follows that the sets $\{S(x)\}$ form an open cover of the compact set rB^*, hence there exist $\{x_1, x_2, \ldots, x_n\} \subset rB$ such that $rB^* = \bigcup_{j=1}^n S(x_j)$. As in the proof of Lemma 1.7, there exists a partition of unity $\{\beta_1, \beta_2, \ldots, \beta_n\}$ subordinate to this covering. Define the continuous mapping $p\colon\ rB^* \to rB$ by $p(x^*) = \sum_{j=1}^n \beta_j(x^*)x_j$. We claim that

$x^* \in S(p(x^*))$ for all $x^* \in rB^*$. Indeed, for every j such that $\beta_j(x^*) > 0$ we have $x^* \in S(x_j)$, that is, $x_j \in S^{-1}(x^*)$. Since the latter is convex and since $p(x^*)$ is a convex combination of x_j's, it follows that $p(x^*) \in S^{-1}(x^*)$, which is equivalent to $x^* \in S(p(x^*))$. Next, define the set-valued mapping R: $rB^* \to 2^{rB^*}$ by $R(x^*) = -J(p(x^*))$. Since p is continuous and J is upper semicontinuous (Exercise 1.17 and Example 2.16), R is upper semicontinuous. Let $K = rB^*$; by Lemma 1.18 there exists $x_0^* \in rB^*$ such that $x_0^* \in R(x_0^*)$, that is, $x_0^* \in -J(p(x_0^*))$ and, of course, $x_0^* \in S(p(x_0^*))$. Letting $x_0 = p(x_0^*)$, this means that $x_0^* \in (-J)(x_0) \cap S(x_0) \subset H_{\epsilon,G}$, which completes the proof. ∎

Note that since ∂j^*: $E^* \to 2^{E^{**}}$, its inverse $(\partial j^*)^{-1}$ is a mapping from E^{**} to 2^{E^*}.

THEOREM 3.6. (Gossez) *If T is maximal monotone of type (D), then for all $\lambda > 0$, $R(\overline{T} + \lambda(\partial j^*)^{-1}) = E^*$.*

Proof. Since T is of type (D) if and only if the same is true of $\lambda^{-1}T - x^*$ for each $\lambda > 0$ and $x^* \in E^*$, we need only show that $0 \in R(\overline{T} + (\partial j^*)^{-1})$. Let $\mathcal{F}$ denote the directed family of all finite dimensional subspaces $F \subset E$ such that $D(T) \cap F \neq \emptyset$, partially ordered by inclusion. For each such F, let i_F: $F \to E$ denote the natural injection, with adjoint i_F^*: $E^* \to F^*$ (the restriction mapping to F). Suppose, now that $F \in \mathcal{F}$ and that $r > 0$. Apply Theorem 3.5 to F, F^*, $i_F^* T i_F$ and $K_r = \{x \in F\colon \|x\| \leq r\}$, as follows: Let G be the graph in $K_r \times F^*$ of the restriction to K_r of the monotone operator $i_F^* T i_F$. (We assume that r is sufficiently large that $K_r \cap D(T) \neq \emptyset$.) Let A: $K_r \to F^*$ be $-i_F^* J i_F$. Thus, there exists an element of $G(A)$ – call it $(x_{F,r}, -x_{F,r}^*)$ – which is montonely related to G; that is, $\|x_{F,r}\| \leq r$, $x_{F,r}^* \in i_F^* J(x_{F,r})$ (this uses $Ji_F = J$ in F) and $\langle -x_{F,r}^* - y^*, x_{F,r} - y\rangle \geq 0$ whenever $\|y\| \leq r$ and $y^* \in i_F^* T i_F(y)$.

Suppose, now, that both r_0 and F_0 are sufficiently large so that there exists $y_0 \in F_0$ with $\|y_0\| \leq r_0$ and $y_0^* \in i_{F_0}^* T i_{F_0}(y_0)$. Fix $F \supset F_0$; for each $r \geq r_0$ we will show that the following set H_r is bounded: Define H_r to be the set of all $(x_{F,r}, x_{F,r}^*) \in F \times F^*$ such that

$$x_{F,r}^* \in i_F^* J(x_{F,r}) \quad \text{and} \quad \langle -x_{F,r}^* - y^*, x_{F,r} - y\rangle \geq 0$$

whenever

$$\|y\| \leq r \quad \text{and} \quad y^* \in i_F^* T i_F(y).$$

For any such $(x_{F,r}, x^*_{F,r})$ we have

$$\frac{1}{2}\|x_{F,r}\|^2 + \frac{1}{2}\|x^*_{F,r}\|^2 = \langle x^*_{F,r}, x_{F,r}\rangle \le \langle x^*_{F,r}, y_0\rangle - \langle y_0^*, x_{F,r}\rangle + \langle y_0^*, y_0\rangle$$
$$\le \|x^*_{F,r}\| \cdot \|y_0\| + \|y_0^*\| \cdot \|x_{F,r}\| + \langle y_0^*, y_0\rangle.$$

The subset of the plane where a positive quadratic function is dominated by a linear function is necessarily bounded, so there exists an upper bound on each of the sets $\{\|x_{F,r}\|\}$ and $\{\|x^*_{F,r}\|\}$. That is, each of the sets H_r is bounded. Moreover, each of them is closed (in the product of the norm topologies), since $i_F^* J i_F$ is easily seen to have closed graph and the function $(x_{F,r}, x^*_{F,r}) \to \langle -x^*_{F,r} - y^*, x_{F,r} - y\rangle$ is continuous, for each $(y, y^*) \in F \times F^*$. Clearly, $H_r \supset H_{r'}$ whenever $r' > r > 0$, so for increasing r, the H_r's form a decreasing family of nonempty compact sets and they therefore have nonempty intersection. This shows that there exists $x_F \in F$ and $x_F^* \in i_F^* J x_F$ such that

$$\langle -x_F^* - y^*, x_F - y\rangle \ge 0 \quad \text{whenever} \quad y \in F, \quad \text{and} \quad y^* \in i_F^* T i_F(y). \tag{1}$$

Note that any Hahn-Banach extension of x_F^* from F to all of E is in $J(x_F)$, so we can assume that $x_F^* \in J(x_F) \subset E^*$. Since the nets (x_F) and (x_F^*) are bounded, and since we can regard the x_F's as elements of E^{**}, we see that there exists a subnet (call it (x_F, x_F^*)) in $E^{**} \times E^*$ converging to an element $(x^{**}, x^*) \in E^{**} \times E^*$ in the $\sigma(E^{**}, E^*) \times \sigma(E^*, E)$ topology. We want to show that $(x^{**}, -x^*)$ is monotonically related to $G(T) \subset E^{**} \times E^*$, that is,

$$\langle x^{**} - \hat{y}, -x^* - y^*\rangle \ge 0 \quad \text{whenever} \quad (y, y^*) \in G(T). \tag{2}$$

To see this, note that (using the weak* lower semicontinuity of both j^* and j^{**})

$$\langle x^{**}, x^*\rangle \le j^{**}(x^{**}) + j^*(x^*) \le \liminf[j(x_F) + j^*(x_F^*)] = \liminf\langle x_F^*, x_F\rangle, \tag{3}$$

while (1) implies that, for all $(y, y^*) \in G(T)$,

$$\limsup\langle x_F^*, x_F\rangle \le \langle y^*, y\rangle + \langle x^*, y\rangle - \langle x^{**}, y^*\rangle; \tag{4}$$

together, these yield (2). Since T is assumed to be of type (D), there exists a net $(y_\alpha, -y_\alpha^*)$ in $G(T)$ such that (y_α) is bounded, converges to x^{**} in the $\sigma(E^{**}, E^*)$ topology and $\|y_\alpha^* - x^*\| \to 0$. This fact, applied to (4), shows that

$$\limsup\langle x_F^*, x_F\rangle \le \langle x^{**}, x^*\rangle + \langle x^{**}, x^*\rangle - \langle x^{**}, x^*\rangle = \langle x^{**}, x^*\rangle.$$

Now, from (3),

$$\langle x^{**}, x^* \rangle \leq j^{**}(x^{**}) + j^*(x^*) \leq \liminf \langle x^*_F, x_F \rangle \leq \limsup \langle x^*_F, x_F \rangle \leq \langle x^{**}, x^* \rangle,$$

which shows that $x^{**} \in \partial j^*(x^*)$. Thus, $-x^* \in \overline{T}(x^{**})$ and $x^* \in \partial(j^*)^{-1}(x^{**})$, which completes the proof. ∎

COROLLARY 3.7. *If E is reflexive and $T\colon E \to 2^{E^*}$ is maximal monotone, then $R(T + \lambda J) = E^*$ for every $\lambda > 0$.*

Proof. It follows directly from the definitions that for a reflexive space E, one has $\partial(j^*)^{-1} = J$ and, as has been noted earlier, $\overline{T} = T$, so the corollary is immediate. ∎

THEOREM 3.8. *If T is of maximal monotone of type (D), then $\overline{R(T)}$ is convex.*

Before proving this, we need the fact that if T is maximal monotone of type (D) and $x^* \in \operatorname{co} R(T)$, then there exists $x \in E$ such that

$$\sup_{(y^{**}, y^*) \in G(\overline{T})} \langle y^{**} - \hat{x}, x^* - y^* \rangle < \infty.$$

This follows in a straightforward way from the following lemma and the definition of type (D). Note that if T is of type (D), then $R(\overline{T}) \subset \overline{R(T)}$.

LEMMA 3.9. *Suppose that E and F are linear spaces in duality and that $T\colon E \to 2^F$ is monotone. If $x^* \in \operatorname{co} R(T)$, then there exists $x \in \operatorname{co} D(T)$ such that*

$$\sup_{(y, y^*) \in G(T)} \langle y^* - x^*, x - y \rangle < \infty.$$

Proof. Suppose that $x^* = \sum t_i x_i^*$ where $t_i \geq 0$, $\sum t_i = 1$ and $x_i^* \in R(T)$, so there exist $x_i \in E$ such that $x_i^* \in T(x_i)$. Take $x = \sum t_i x_i$; then for any $(y, y^*) \in G(T)$,

$$\begin{aligned}\langle y^* - x^*, x - y \rangle = \langle y^* - \textstyle\sum t_i x_i^*, \sum t_j x_j - y \rangle &= \sum_{i,j} t_i t_j \langle y^* - x_i^*, x_j - y \rangle \\ &= \sum_{i,j} t_i t_j \langle y^* - x_j^*, x_j - y \rangle + \sum_{i,j} t_i t_j \langle x_j^* - x_i^*, x_j - y \rangle \\ &\leq \sum_{i,j} t_i t_j \langle x_j^* - x_i^*, x_j - y \rangle = \sum_{i<j} t_i t_j \langle x_j^* - x_i^*, x_j - x_i \rangle,\end{aligned}$$

which proves the lemma, since the last term does not depend on y or y^*. ∎

Proof of Theorem 3.8. Note that it suffices to show that $\operatorname{co} R(T) \subset \overline{R(T)}$, since this implies that $\overline{\operatorname{co}}\, R(T) \subset \overline{R(T)} \subset \overline{\operatorname{co}}\, R(T)$. Suppose, then, that $x^* \in \operatorname{co} R(T)$. By Theorem 3.6, for each $\lambda > 0$ there exists $y_\lambda^* \in E^*$, $x_\lambda^{**} \in j^*(y_\lambda^*)$ and $z_\lambda^* \in \overline{T}(x_\lambda^{**})$ such that $x^* = \lambda y_\lambda^* + z_\lambda^*$. By the foregoing remark, there exists $x \in E$ such that

$$\langle x_\lambda^{**} - \hat{x}, x^* - z_\lambda^* \rangle \equiv \lambda \langle x_\lambda^{**} - \hat{x}, y_\lambda^* \rangle$$

is bounded above for all $\lambda > 0$. It follows that for some $M > 0$ (and all $\lambda > 0$),

$$\lambda \|y_\alpha^*\|^2 \le \lambda \|x_\lambda^{**}\|^2 + \lambda \|y_\lambda^*\|^2 = 2\lambda \langle x_\lambda^{**}, y_\lambda^* \rangle \le M + 2\lambda \langle y_\lambda^*, x \rangle \le M + 2\lambda \|y_\lambda^*\| \cdot \|x\|.$$

From this we see that $\lambda y_\lambda^* \to 0$ as $\lambda \to 0$; indeed, if there were a sequence $\lambda_n \to 0$ such that $\|\lambda_n y_{\lambda_n}^*\|$ were bounded away from 0, then we would necessarily have $\|y_{\lambda_n}^*\| \to \infty$ and dividing both sides of the inequality above by $\lambda_n \|y_{\lambda_n}^*\|$ would lead to a contradiction. Thus, $x^* - z_\lambda^* = \lambda y_\lambda^* \to 0$; since $z_\lambda^* \in R(\overline{T})$, this shows that $x^* \in \overline{R(\overline{T})} \subset \overline{R(T)}$. (Since $R(T) \subset R(\overline{T})$, their closures are in fact equal.) ∎

COROLLARY 3.10. *If E is reflexive and $T\colon E \to 2^{E^*}$ is maximal monotone, then both $\overline{R(T)}$ and $\overline{D(T)}$ are convex.*

Proof. By Example 3.2(b), if T is maximal monotone, then it is maximal monotone of type (D), so it follows from Theorem 3.8 that $\overline{R(T)}$ is convex. By applying this result to the maximal monotone operator T^{-1} we obtain convexity of $\overline{R(T^{-1})} \equiv \overline{D(T)}$. ∎

DEFINITION 3.11. An operator $T\colon E \to 2^{E^*}$ is said to be coercive provided $D(T)$ is bounded or there exists a function $c\colon \mathbb{R}^+ \to \mathbb{R}$ such that $c(r) \to \infty$ when $r \to \infty$ and $\langle x^*, x \rangle \ge c(\|x\|) \cdot \|x\|$ for each $(x, x^*) \in G(T)$.

Remark. It is easily verified that if $D(T)$ is unbounded, then T is coercive if and only if for every $M > 0$ there exists $r > 0$ such that

$$\frac{\langle x^*, x \rangle}{\|x\|} \ge M \quad \text{whenever} \quad \|x\| \ge r \quad \text{and} \quad x^* \in T(x).$$

Indeed, if this holds, take $c(r) = \inf\{\frac{\langle x^*, x \rangle}{\|x\|}\colon \|x\| \ge r \text{ and } x^* \in T(x)\}$.

EXAMPLSE 3.12. (a) The duality mapping J is an obvious example of a coercive operator, since $\langle x^*, x\rangle = \|x\|^2$ whenever $x^* \in J(x)$.

(b) If T is a positive linear operator and $\lambda > 0$, then $T + \lambda J$ is coercive: If $x \in E$ and $x^* \in (T + \lambda J)(x)$, then $x^* = T(x) + \lambda z^*$ for some $z^* \in J(x)$ and hence $\langle x^*, x\rangle = \langle T(x), x\rangle + \lambda\langle z^*, x\rangle \geq \lambda\|x\|^2$. From the Remark following Problem 2.20, it follows that $T + \lambda J$ is also maximal monotone.

(c) Recall that a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a monotone operator if and only if it is nondecreasing. It is easily seen that φ is coercive if and only if $\varphi(t) \to \pm\infty$ as $t \to \pm\infty$.

EXERCISE 3.13. Show that if T is coercive, then so is $\overline{T}\colon E^{**} \to 2^{E^*}$.

THEOREM 3.14. (Gossez) *Suppose that T is a coercive maximal monotone operator of type (D). Then $R(\overline{T}) = E^*$ and hence $\overline{R(T)} = E^*$.*

Proof. It is clear from the definition of type (D) that one always has $R(\overline{T}) = \overline{R(T)}$. Suppose, then, that $x^* \in E^*$. By Theorem 3.6, for each $\lambda > 0$ there exist $y^*_\lambda \in E^*$, $x^{**}_\lambda \in \partial j^*(y^*_\lambda)$ and $z^*_\lambda \in \overline{T}(x^{**}_\lambda)$ such that $x^* = \lambda y^*_\lambda + z^*_\lambda$. We have

$$\langle x^{**}_\lambda, x^*\rangle = \lambda\langle x^{**}_\lambda, y^*_\lambda\rangle + \langle x^{**}_\lambda, z^*_\lambda\rangle = \lambda\|x^{**}_\lambda\|^2 + \langle x^{**}_\lambda, z^*_\lambda\rangle.$$

Since (Exercise 3.13) $\overline{T}$ is coercive, if $\{\|x^{**}_\lambda\|\}$ were unbounded as $\lambda \to 0$, the right side of

$$\|x^*\| \geq \frac{\langle x^{**}_\lambda, x^*\rangle}{\|x^{**}_\lambda\|} = \lambda\|x^{**}_\lambda\| + \frac{\langle x^{**}_\lambda, z^*_\lambda\rangle}{\|x^{**}_\lambda\|}$$

would be unbounded, an impossibility. Thus, the bounded net $\{x^{**}_\lambda\}$ has a subnet (call it $\{x^{**}_\lambda\}$) converging in the $\sigma(E^{**}, E^*)$ topology to an element $x^{**} \in E^{**}$. We will show that $x^* \in \overline{T}(x^{**})$ by showing that (x^{**}, x^*) is monotonically related to $G(T)$. Suppose, then, that $(u, u^*) \in G(T)$. Since $x^*_\lambda = x^* - \lambda y^*_\lambda$, we have

$$0 \leq \langle x^{**}_\lambda - \hat{u}, x^*_\lambda - u^*\rangle = \langle x^{**}_\lambda - \hat{u}, x^* - u^*\rangle - \lambda\langle x^{**}_\lambda - \hat{u}, y^*_\lambda\rangle.$$

Recall that $x^{**}_\lambda \in \partial j^*(y^*_\lambda)$ implies boundedness of $\|y^*_\lambda\| = \|x^{**}_\lambda\|$, so the second term on the right converges to 0 as $\lambda \to 0$, yielding $0 \leq \langle x^{**} - \hat{u}, x^* - u^*\rangle$. ∎

COROLLARY 3.15. *If E is reflexive and T is a coercive maximal monotone operator on E, then $R(T) = E^*$.*

Proof. Simply use the fact that reflexivity implies that $T = \overline{T}$. ∎

The fact (Proposition 2.17(c)) that E is reflexive if $R(J) = E^*$ shows that one cannot omit reflexivity from this result.

In order that $\overline{R(T)}$ be convex it is not necessary for T to be maximal monotone of type (D). This was shown by Gossez [13], [14] with the help of the following example.

EXAMPLE 3.16. Let A be defined on the nonreflexive space ℓ_1 as follows: For each $x = (x_k) \in \ell_1$, let $\{A(x)_n\}$ be the ℓ_∞ sequence defined by

$$(Ax)_n = -\sum_{k<n} x_k + \sum_{k>n} x_k.$$

It is not hard to verify that A is bounded, linear and antisymmetric (that is, $\langle Ax, y\rangle = -\langle Ay, x\rangle$ for all $x, y \in \ell_1$) hence is monotone, satisfying $\langle Ax, x\rangle = 0$ for all x. This latter means that, in particular, A is a positive operator, hence it is maximal monotone. The range $R(A)$ of A is a linear subspace (hence is convex and has convex closure) which is properly contained in the proper closed subspace c of ℓ_∞ consisting of all convergent sequences. (Indeed, $\lim_{n\to\infty}(Ax)_n = -\sum_{k=1}^{\infty} x_k$.)

Gossez [13] uses the operator A by showing that there exists $\lambda > 0$ such that $R(A + \lambda J)$ is not dense in ℓ_∞. This shows that $A + \lambda J$ is not of type (D), in view of Theorem 3.14 and the fact that $A + \lambda J$ is maximal monotone and coercive (Example 3.12(b)). Thus not all maximal monotone operators (not even the coercive ones) are of type (D). Subsequently, he showed [14] that the fact that $R(A + \lambda J)$ is not dense in ℓ_∞ implies that its closure is not convex, that is, there exists a coercive maximal monotone operator T on ℓ_1 such that $\overline{R(T)}$ is not convex.

4. Locally maximal monotone operators

As we have seen, some of the nice properties of maximal monotone operators on reflexive spaces fail to hold in general, but are valid for the subclass of maximal monotone operators of type (D). In this section we introduce another subclass which shares some of the same properties.

DEFINITION 4.1. A set-valued mapping $T\colon E \to 2^{E^*}$ is said to be locally maximal monotone if, for each norm-open convex subset $U \subset E^*$ which intersects $R(T)$, the restriction of the inverse operator T^{-1} to U is maximal monotone in U. The latter means that the graph $G((T^{-1})|_U) \subset U \times E$ is a maximal monotone subset of $U \times E$.

The "working definition" of this property is the following: If U is an open convex subset of E^* which intersects $R(T)$ and if $(x, x^*) \in E \times U$ is montonically related to each $(y, y^*) \in G(T) \cap (E \times U)$, then $(x, x^*) \in G(T)$. It is clear (take $U = E^*$) that every locally maximal monotone operator is maximal monotone.

The locally maximal monotone operators were introduced in [9] because they are the precise class for which a certain approximation scheme is valid. While their exact position within the class of all maximal monotone operators is still unclear, some important properties are known.

PROPOSITION 4.2. (i) *If T is locally maximal monotone, then $\overline{R(T)}$ is convex.*

(ii) *If f is a proper lower semicontinuous convex function on E, then ∂f is locally maximal monotone.*

The proof for (i) may be found in [9]. Property (ii), which is a nontrivial extension of Rockafellar's maximality theorem (Theorem 2.15), was proved by S. Simons [26]; see, also, [27]. In order to see that maximal monotone operators in reflexive spaces are locally maximal monotone, we first reformulate the definition.

PROPOSITION 4.3. *A monotone operator T on E is locally maximal monotone if and only if it satisfies the following condition: For any weak* closed convex and bounded subset C of E^* such that $R(T) \cap \operatorname{int} C \neq \emptyset$ and for each $x \in E$ and $x^* \in \operatorname{int} C$ with $x^* \notin T(x)$, there exists $z \in E$ and $z^* \in T(z) \cap C$ such that $\langle x^* - z^*, x - z \rangle < 0$.*

Proof. In one direction, if T is locally maximal monotone and C is given, let $U = \operatorname{int} C$. In the other direction, if U is open and convex in E^*, if $u \in E$ and $x \in E$ with $u^* \in T(u) \cap U$ and $x^* \in U$ but $x^* \notin T(x)$, then there exists $\epsilon > 0$ such that $u^* + \epsilon B^* \subset U$ and $x^* + \epsilon B^* \subset U$. By convexity, $C \equiv [u^*, x^*] + \epsilon B^*$ is a weak* closed, convex and bounded subset of U which can be used to verify that U has the required property. ∎

The only use of reflexivity in the next proposition is an application of Theorem 2.19 (on the sum of two maximal monotone operators in a reflexive space).

PROPOSITION 4.4. *If E is reflexive and T is maximal monotone on E, then it is locally maximal monotone.*

Proof. Suppose that C is weak* closed and convex and that $\operatorname{int} C \cap R(T) \neq \emptyset$. Suppose also that $x^* \in \operatorname{int} C$ but $x^* \notin T(x)$. Let T_1 denote the inverse T^{-1} of T and let $T_2 = \partial\delta_C$. Since $\operatorname{int} D(T_2) = \operatorname{int} C$, these are maximal monotone operators from E^* into E for which $D(T_1) \cap \operatorname{int} D(T_2) \neq \emptyset$; by Theorem 2.19, their sum $T_1 + T_2$ is maximal monotone. Now $x^* \notin T(x)$ implies that $x \notin T_1(x^*)$, and since $T_2(x^*) = \{0\}$, we see that $x \notin T_1(x^*) + T_2(x^*)$. By maximality of $T_1 + T_2$, there exists $z^* \in D(T_1) \cap D(T_2) \equiv R(T) \cap C$ and $z \in (T_1 + T_2)(z^*)$ such that $\langle x^* - z^*, x - z\rangle < 0$. We can write $z = u + v$, where $u \in T_1(z^*)$ (that is, $z^* \in T(u)$) and $v \in T_2(z^*)$. The latter means that $\langle z^* - w^*, v\rangle \geq 0$ for all $w^* \in C$. We have thus produced $z^* \in T(u) \cap C$ such that

$$0 > \langle x^* - z^*, x - u\rangle - \langle x^* - z^*, v\rangle \geq \langle x^* - z^*, x - u\rangle,$$

showing that T satisfies the condition in Proposition 4.3 and is therefore locally maximal monotone. ∎

Recall Gossez's Example 3.15 of the linear maximal monotone operator $A\colon \ell_1 \to \ell_\infty$. Its interest in this context is the fact that, even though $\overline{R(A)}$ is linear (hence convex), A is not locally maximal monotone, so not every maximal monotone operator is locally maximal monotone.

EXAMPLE 4.5. The operator A is not locally maximal monotone.

Proof. Let $e = (1, 0, 0, \dots)$, considered as an element of either ℓ_1 or ℓ_∞, and let

$$z = \left(-\frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \dots\right) \in \ell_1.$$

Some computations using the definition show that $(Az)_1 = \frac{1}{4}$ while for $n \geq 2$, $(Az)_n = \frac{1}{4} + \frac{1}{2^n} + \frac{1}{2^{n+1}}$. Moreover, $e - Ae = (1, 1, 1, \dots)$ and $\|e - Az\|_\infty = \frac{3}{4}$, so if U is the open unit ball in ℓ_∞, then $x^* \equiv e - Az \in U$. Let $x = e - z$. If $u \in \ell_1$ and $Au \in U$, then $\lim_{n\to\infty} |(Au)_n| = |\sum_{k=1}^\infty u_k| \leq 1$ and hence

$$\langle x^* - Ax, u\rangle = \langle e - Ae, u\rangle = \sum_{k=1}^\infty u_k \leq 1,$$

while

$$\langle x^*, x\rangle = \langle e, e\rangle - \langle Az, e\rangle - \langle e, z\rangle = 1 - (Az)_1 - z_1 = 1 - \frac{1}{4} + \frac{1}{2} > 1.$$

Thus, $x^* \neq Ax$ even though

$$\langle x^* - Au, x - u\rangle = \langle x^*, x\rangle - \langle Au, x\rangle - \langle x^*, u\rangle = \langle x^*, x\rangle + \langle Ax, u\rangle - \langle x^*, u\rangle \geq 0$$

whenever $Au \in U$, contradicting the definition of locally maximal monotone. ∎

We still do not know whether the class of maximal monotone operators of type (D) is actually different from the class of locally maximal monotone operators. To see that for coercive operators, the former class is contained in the latter, we need two preliminary results. The proof of the following identity consists of an elementary but tedious computation.

PROPOSITION 4.6. *If* $u, v, x \in E$, $u^*, v^*, x^* \in E^*$ *and* $\lambda \in [0,1]$, *then*

$$\begin{aligned}\langle \lambda u^* + (1-\lambda)v^* - x^*, \lambda u + (1-\lambda)v - x\rangle \\ = \lambda\langle u^* - x^*, u - x\rangle + (1-\lambda)\langle v^* - x^*, v - x\rangle \\ - \lambda(1-\lambda)\langle u^* - v^*, u - v\rangle\end{aligned}$$

LEMMA 4.7. *Suppose that T is a maximal monotone operator, that U is an open subset of E^* and that $z^* \in U \setminus T(z)$ is such that $\langle x^* - z^*, x - z\rangle \geq 0$ for all $x^* \in T(x) \cap U$. Then there exist $b \in E$, $b^* \in U$ and $r > 0$ such that for all $x^* \in T(x) \cap U$,*

$$\langle x^* - b^*, x - b\rangle \geq r.$$

Proof. Since T is maximal monotone there exists $y^* \in T(y)$ such that $\langle y^* - z^*, y - z\rangle < 0$. Let $1 > \lambda > 0$ be such that $b^* := \lambda z^* + (1-\lambda)y^* \in U$ and let $b = \lambda z + (1-\lambda)y$. Then, using the identity (4.1), for all $x^* \in T(x) \cap U$ we have

$$\begin{aligned}\langle x^* - b^*, x - b\rangle &= \lambda\langle x^* - z^*, x - z\rangle + (1-\lambda)\langle x^* - y^*, x - y\rangle \\ &\quad - \lambda(1-\lambda)\langle z^* - y^*, z - y\rangle \\ &\geq -\lambda(1-\lambda)\langle z^* - y^*, z - y\rangle > 0\end{aligned}$$

so we may set $r = -\lambda(1-\lambda)\langle z^* - y^*, z - y\rangle > 0$. ∎

THEOREM 4.8. *Suppose that T is a maximal monotone operator such that either* (i) $R(T) = E^*$ *or* (ii) $\overline{R(T)} = E^*$ *and T is coercive. Then T is locally maximal monotone.*

Proof. Suppose, first, that $R(T) = E^*$, that $U \subset E^*$ is open and convex and that $z \in E$, $z^* \in U$ are such that $\langle z^* - x^*, z - x\rangle \geq 0$ for all $x \in E$ such that $x^* \in T(x) \cap U$. If $z^* \notin T(z)$, then there would exist $b \in E$, $b^* \in U$ and $r > 0$ as in Lemma 4.7. Since $b^* \in R(T)$ by hypothesis, there exists $x \in E$ such that $b^* \in T(x) \cap U$ and hence by Lemma 4.7, $\langle b^* - b^*, x - b\rangle \geq r > 0$, a contradiction.

Suppose, next, that $R(T)$ is dense in E^* and that T is coercive. If T were not locally maximal monotone, we could find an open convex subset $U \subset E^*$ with $U \cap R(T) \neq \emptyset$ and elements $z \in E$ and $z^* \in U \backslash T(z)$ such that $\langle z^* - x^*, z - x\rangle \geq 0$ whenever $x \in E$ and $x^* \in T(x) \cap U$. Choose b, b^* and r as in Lemma 4.7. Since $R(T)$ is dense in E^*, we can find $x_n \in E$ and $x_n^* \in T(x_n)$ such that $\|b^* - x_n^*\| \to 0$. But then for all sufficiently large n, we would have $x_n^* \in U$ and hence

$$r \leq \langle x_n^* - b^*, x_n - b\rangle \leq \|x_n^* - b^*\| \|x_n - b\|,$$

which would imply that $\|x_n - b\| \to \infty$. Coercivity would then imply that $\|x_n^*\| \to \infty$, again a contradiction. ∎

COROLLARY 4.9. *If T is maximal monotone, coercive and of type (D), then it is locally maximal monotone.*

Proof. Recall that by Theorem 3.14, the fact that T is coercive and maximal monotone of type (D) implies that $\overline{R(T)} = E^*$. ∎

Recall Problem 2.20: If E is a nonreflexive Banach space and S and T are maximal monotone operators such that $D(T) \cap \operatorname{int} D(S) \neq \emptyset$; is $S + T$ necessarily maximal? What about the special case when S is the subdifferential of the indicator function δ_C of a closed convex set C for which $\operatorname{int} C \cap D(T) \neq \emptyset$?

It is not unreasonable to ask whether these questions have affirmative answers when the maximal monotone operators are of type (D), or are locally maximal monotone.

ADDED IN PROOF: For the most recent results on Problem 2.20 (as well as a detailed treatment of many aspects of monotone operators on general Banach spaces) see the forthcoming monograph "Minimax and Monotonicity" by S. Simons (in preparation).

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