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# Quotients of $L_1$ by Reflexive Subspaces *

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Here we present an example and some results suggesting that there is no infinite-dimensional reflexive subspace $Z$ of $L_1 \equiv L_1[0,1]$ such that the quotient $L_1/Z$ is isomorphic to a subspace of $L_1$.

Observe that such a subspace $Z$ cannot be complemented, because $L_1$ has the Dunford-Pettis property [12, III.D.33]. Moreover, $Z$ is isomorphic to a subspace of $L_p[0,1]$ for some $p \in (1,2]$ [12, III.H.13]; in particular, it is superreflexive. On the other hand, there are many examples of reflexive subspaces of $L_1$. For instance, the closed space generated by the Rademacher functions on $[0,1]$, which are given by $r_n(t) = \operatorname{sgn} \sin 2^n \pi t$ for $n \in \mathbb{N}$, is isomorphic to $\ell_2$ [7, Theorem 2.b.3]. Also, it is known [8, Theorem 2.f.5] that for every $r \in (1,2]$ there exists a subspace of $L_1$ isomorphic to $L_r$.

## 1. The example

An operator $T \in \mathcal{B}(X,Y)$ between Banach spaces $X$ and $Y$ is 1-summing if it takes weakly unconditionally Cauchy series into absolutely convergent series. A Banach space $X$ has the Gordon-Lewis property if every 1-summing operator $T \in \mathcal{B}(X,Y)$ factors through a $L_1(\mu)$-space.

The subspaces of $L_1$ have the Gordon-Lewis property. Indeed, every 1-summing operator factors through a $L_\infty(\mu)$-space. Therefore, by the extension property of the $L_\infty(\mu)$-spaces, every 1-summing operator defined on a subspace $Z$ of $L_1$ can be extended to the whole space.

Proposition. [9] *There exists a subspace $Z_0$ of $L_1$ isomorphic to $\ell_2$ such that $L_1/Z_0$ fails the Gordon-Lewis property. In particular, $L_1/Z_0$ is not isomorphic to a subspace of $L_1$.*

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This subspace $Z_0$ is obtained using an ultraproduct argument applied to the Kasin decompositions of finite dimensional spaces $\ell_2^n$. We do not know, for example, whether the quotient of $L_1$ by the subspace generated by the Rademacher functions is isomorphic to a subspace of $L_1$.

## 2. The results

Let $X$ and $Y$ be Banach spaces. An operator $T \in \mathcal{B}(X,Y)$ is upper semi-Fredholm if its kernel $N(T)$ is finite dimensional and its range $R(T)$ is closed. It is tauberian [5] if $T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y$. We denote by $\mathcal{F}_+(X,Y)$ and $\mathcal{T}_+(X,Y)$ the classes of upper semi-Fredholm operators and tauberian operators from $X$ into $Y$, respectively. It follows from Theorem 1 (and is not difficult to see) that $\mathcal{F}_+ \subset \mathcal{T}_+$.

*Remark.* If $Z$ is an infinite dimensional reflexive subspace of $L_1$, then the quotient map $Q : L_1 \longrightarrow L_1/Z$ belongs to $\mathcal{T}_+ \setminus \mathcal{F}_+$ [5]. However, it is not known whether $\mathcal{F}_+(L_1, L_1)$ coincides with $\mathcal{T}_+(L_1, L_1)$. In the remaining of the paper we describe some results suggesting that these two classes coincide. This would imply that $L_1/Z$ is not isomorphic to a subspace of $L_1$ when $Z$ is an infinite dimensional reflexive subspace of $L_1$.

In the following result we give perturbative characterizations of the classes $\mathcal{F}_+$ and $\mathcal{T}_+$, showing that there are some formal similarities between these two classes.

THEOREM 1. [4] *An operator $T \in \mathcal{B}(X,Y)$ is upper semi-Fredholm (tauberian) if and only if for every compact operator $K \in \mathcal{B}(X,Y)$ the kernel $N(T+K)$ is finite-dimensional (reflexive).*

In the case of operators from $L_1$ into a Banach space, more specific characterizations are avaible.

THEOREM 2. [3] *For $T \in \mathcal{B}(L_1, Y)$, the following statements are equivalent:*

(1) *$T$ is tauberian;*

(2) $\liminf_n \|Tf_n\| > 0$ *for every normalized disjoint sequence $(f_n)$ in $L_1$;*

(3) *there exists $r > 0$ so that* $\liminf_n \|Tf_n\| > r$ *for every normalized disjoint sequence $(f_n)$ in $L_1$;*

(4) *there exists* $s > 0$ *so that for every* $f \in L_1$ *with* $m(\{t : f(t) \neq 0\}) < s$ *and* $\|f\| = 1$ *we have* $\|Tf\| > s$.

This theorem has some interesting consequences: the class $\mathcal{T}_+(L_1, Y)$ is norm open, every $T \in \mathcal{T}(L_1, Y)$ can be seen a "superposition" of a finite number of isomorphisms, and every quotient of $L_1$ by a reflexive subspace contains a copy of $L_1$. Observe that, in general, $\mathcal{T}_+(X, Y)$ is not open [1].

COROLLARY. (1) *For every* $T \in \mathcal{T}_+(L_1, Y)$ *there exists* $\delta_T > 0$ *so that if* $A \in \mathcal{B}(L_1, Y)$ *and* $\|A\| < \delta_T$ *then* $T + A \in \mathcal{T}_+(L_1, Y)$.

(2) *For every* $T \in \mathcal{T}(L_1, Y)$ *we can find a partition* $\{I_1, \cdots, I_n\}$ *of* $[0, 1]$ *in subintervals so that the restrictions* $T|_{L_1(I_i)}$ *are isomorphisms (into). In particular, if* $\mathcal{T}_+(L_1, Y)$ *is non-empty then the space* $Y$ *contains a subspace isomorphic to* $L_1$.

(3) *For every reflexive subspace* $Z$ *of* $L_1$*, the quotient* $L_1/Z$ *contains a subspace isomorphic to* $L_1$.

With respect to the last part of Corollary, observe that it is not known whether $L_1/Z$ contains a copy of $L_1$ when $Z$ is isomorphic to a dual space (see [11, page 10]). The answer is positive for $Z$ isomorphic to $\ell_1$ [11, Proposition 1.2]. Moreover, the following result of Talagrand shows that the containment of copies of $L_1$ by $L_1/Z$ is quite unstable.

THEOREM 3. [11, Theorem 1.1] *There exist two subspaces* $Y$ *and* $Z$ *of* $L_1$*, both of them isomorphic to* $\ell_1$*-sums of spaces (not uniformly) isomorphic to* $\ell_1$*, such that*

(1) *the spaces* $L_1/Y$ *and* $L_1/Z$ *contain no copies of* $L_1$*, but*

(2) *the canonical map from* $L_1$ *into* $L_1/Y \times L_1/Z$ *is an isomorphism into.*

Given $T \in \mathcal{B}(X, Y)$, we consider the operator $\widetilde{T} : X^{**}/X \longrightarrow Y^{**}/Y$ defined by

$$\widetilde{T}(x^{**} + X) := T^{**}(x^{**}) + Y \text{ for every } x^{**} \in X^{**}.$$

Note that an operator $T$ is tauberian if and only if $\widetilde{T}$ is injective. Rosenthal [10] has recently introduced the strongly tauberian operators as those operators $T \in \mathcal{B}(X, Y)$ for which $\widetilde{T}$ is an isomorphism into. Obviously, if $T$ is strongly tauberian then $T$ is tauberian. Moreover, Rosenthal proves that if $T$

has a tauberian ultrapower, then it is strongly tauberian. We refer to [2] for the properties of operators with tauberian ultrapowers. In our case, we have the following result.

PROPOSITION. *An operator $T \in \mathcal{B}(L_1, Y)$ is tauberian if and only if the induced operator $\widetilde{T} : L_1^{**}/L_1 \longrightarrow Y^{**}/Y$ is an isomorphism into. In this case, the second conjugate $T^{**}$ of $T$ is also tauberian.*

*Remarks.* (a) An example of an operator $T \in \mathcal{T}_+(X, Y)$ such that $T^{**}$ is not tauberian is given in [1].

(b) It follows from the results in [2] that the ultrapowers of an operator $T \in \mathcal{T}_+(L_1, Y)$ are tauberian.

(c) The proof given in [10] of $T \in \mathcal{B}(X, Y)$ strongly tauberian implies $T^{**}$ strongly tauberian is essentially as follows:

If $T \in \mathcal{B}(X, Y)$ is strongly tauberian, then $\widetilde{T} : X^{**}/X \longrightarrow Y^{**}/Y$ is an isomorphism. Moreover, we can identify canonically $(X^{**}/X)^{**}$ with $X^{****}/X^{**}$, and

$$\widetilde{T}^{**}(X^{**}/X)^{**} \longrightarrow (Y^{**}/Y)^{**} \text{ with } \widetilde{T^{**}} : X^{****}/X^{**} \longrightarrow Y^{****}/Y^{**}.$$

Then $\widetilde{T^{**}}$ is an isomorphism, hence $T^{**}$ is strongly tauberian.

For a Banach space $\mathcal{A}$ and a subset $\mathcal{S} \subset A$, Lebow and Schechter [6] define the perturbation class $P(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{A}$ in the following way.

$$P(\mathcal{S}) := \{a \in \mathcal{A} : a + s \in \mathcal{S} \text{ for all } s \in \mathcal{S}\}.$$

We say that $\mathcal{C} \subset A$ is an admissible class for $\mathcal{S}$ if $\mathcal{C} \subset P(\mathcal{S})$.

Recall that $T \in \mathcal{B}(X, Y)$ is strictly singular if no restriction of $T$ to an infinite dimensional subspace is an isomorphism, and $T$ is weakly precompact if $(Tx_n)$ contains a weakly Cauchy subsequence for eveiy bounded sequence $(x_n) \subset X$. The class $\mathcal{SS}$ of strictly singular operators is admissible for $\mathcal{F}_+$, and it is a well-known open problem whether $P(\mathcal{F}_+) = \mathcal{SS}$ [6]. Moreover, the weakly compact operators form an admissible for $\mathcal{T}_+$. The perturbation class $P(\mathcal{T}_+(X, Y))$ is not well-known in general, but in the case $X = L_1$ it coincides with the class of weakly precompact operators.

PROPOSITION. *Let $Y$ be a Banach space such that $\mathcal{T}_+(L_1, Y) \neq \emptyset$. An operator $K \in \mathcal{B}(L_1, Y)$ is weakly precompact if and only if for every operator $T \in \mathcal{T}_+(L_1, Y)$ we have that $T + K$ is also tauberian.*

Since the space $L_1$ is weakly sequentially complete, weakly precompact operators in $\mathcal{B}(L_1, L_1)$ are weakly compact, and the Dunford-Pettis property of $L_1$ implies that they coincide with the strictly singular operators.

COROLLARY. $P(\mathcal{T}_+)(L_1, L_1) = P(\mathcal{F}_+)(L_1, L_1) = \mathcal{SS}(L_1, L_1)$.

*Remark.* In some cases the class of weakly precompact operators is not admissible for $\mathcal{T}_+(X, Y)$. For instance, it is not difficult to see that the inclusion of James' quasireflexive space $J$ into $c_0$ is weakly precompact (but not weakly compact). However, the null operator $0 \in \mathcal{B}(J, c_0)$ is not tauberian.

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