EXTRACTA MATHEMATICAE Vol. **12**, Núm. 2, 129 – 134 (1997)

c_0, ℓ_1 and ℓ_∞ in Function Spaces *

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AMS *Subject Class.* (1991): 46E40, 46B20

Since the birth of Banach space theory, it has been an important goal to know how are the subspaces of a given Banach space. An interesting part of that study has been focused in the search of criteria for a Banach space to have any of the classical sequence spaces as a subspace. Several deep results have revealed how the presence (or the absence) of such subspaces provides a very good insight in the internal structure of the Banach spaces involved. Think, for instance, in the fundamental structural results on the spaces having an unconditional basis, due to James (1950); or in the criterion for spaces containing c_0 given by Bessaga-Pelczynski (1958); or in the remarkable Rosenthal's result (1974) characterizing spaces containing ℓ_1.

A particular aspect of this research emerged in the mid-seventies when Hoffmann-Jørgensen (1974), in one of the pioneering works on probability on Banach space, left open the problem of characterizing when $L_1(\mu, X)$ contains a copy of c_0. Since then, many authors have investigated in which conditions different vector-valued function spaces contain copies of classical sequence spaces, especially copies of c_0, ℓ_1 and ℓ_∞. Most of their work centers on the spaces $L_p(\mu, X)$ and $C(K, X)$, which are the most important examples of vector-valued function spaces. Thanks to them we know today an (almost) complete answer to the following question:

When do the spaces $C(K, X)$ and $L_p(\mu, X)$ contain a copy or a complemented copy of c_0, ℓ_1 or ℓ_∞?

Notice that actually this is not one question, but a collection of questions.

J. Mendoza and myself are writing a monograph which contains a detailed exposition of all the answers, and I have been asked to give some idea of our work. In [8] one can find a condensed preliminary version of it, which summarizes almost all known results and contains the adequate references. Therefore,

*Partialy supported by the D.G.I.C.Y.T. grant PB94–0243.

our aim here will be first to sketch briefly some aspects of its content, with especial mention to some curious examples and counterexamples; and secondly, to comment two concrete points which does not appear in [8]: the behavior of c_0-sequences in $L_p(\mu, X)$ and the last solution obtained to the posed question.

Let us begin fixing the notation. X will be a Banach space, K a compact Hausdorff space and (Ω, Σ, μ) a finite measure space. For $1 \leq p \leq \infty$, $L_p(\mu, X)$ denotes the Banach space of all X-valued, p-Bochner μ-integrable (μ-essentially bounded, when $p = \infty$) functions; and $C(K, X)$ denotes the Banach space of all continuous X-valued functions defined on K. Both endowed with their usual norms. The symbols $L_p(\mu)$ and $C(K)$ stand for the case in which X is the scalar field. To avoid trivial situations the spaces $L_p(\mu)$, $C(K)$ and X will be supposed to be infinite dimensional. The sum in the sense of ℓ_p or c_0 of a sequence of Banach spaces (X_n) is denoted by $(\sum \oplus X_n)_p$ and $(\sum \oplus X_n)_{c_0}$, respectively.

The first answers to the problem were given by Kwapien and Pisier in the seventies, who characterized when $L_p(\mu, X)$ contains copies of c_0 and ℓ_1, respectively. Bourgain provided alternative proofs of these results, and he obtained some extensions which were crucial to achieve other solutions to our problem. E. and P. Saab, Cembranos, Freniche and Drewnowski worked in $C(K, X)$ spaces in the eighties. Around 1990 Bombal, Emmanuele and Mendoza gave the solutions to the problems on $L_p(\mu, X)$, with $1 \leq p < \infty$, not solved by Kwapien and Pisier theorems. Complemented copies of c_0 in $L_\infty(\mu, X)$ were considered by Leung, Räbiger and Díaz in the early nineties. The last solution, about complemented copies of ℓ_1 in $L_\infty(\mu, X)$, is due to Díaz and Kalton, and it was completed last June.

Most of the results show that we have natural solutions to the problem, that is, the fact that an X-valued function space contains a copy (or a complemented copy) of c_0, ℓ_1 or ℓ_∞, very often holds only if the same occurs in X or in the corresponding scalar function space (note that the converse is always trivially true). Thus, since $L_p(\mu)$ never contains c_0 for $1 \leq p < \infty$, and it never contains ℓ_1 for $1 < p < \infty$, the following results given by Kwapien and Pisier, respectively, provide natural answers

For $1 \leq p < \infty$, $L_p(\mu, X)$ contains a copy of c_0 if and only if X does.

For $1 < p < \infty$, $L_p(\mu, X)$ contains a copy of ℓ_1 if and only if X does.

Nevertheless, there are exceptions. The spaces $L_p(\mu, X)$ and $C(K, X)$ contain complemented copies of c_0 more often than expected. For instance, Cembranos-Freniche result assures that

$C(K,X)$ contains a complemented copy of c_0 whatever the space X is.

Thus we can find lots of examples for which $C(K,X)$ contains complemented copies of c_0 but neither $C(K)$ nor X does. One of this examples particularly curious is $C(\beta\mathbb{N},\ell_\infty)$ (where $\beta\mathbb{N}$ denotes the Stone-Cech compactification of the natural numbers), because $C(\beta\mathbb{N},\ell_\infty) \approx C(\beta\mathbb{N}, C(\beta\mathbb{N})) \approx C(\beta\mathbb{N}\times\beta\mathbb{N})$ contains a complemented copy of c_0, while $C(\beta\mathbb{N}) \approx \ell_\infty$ does not.

Another striking example can be obtained taking any space X having a copy of c_0 and no complemented copies of c_0 (for instance $X = \ell_\infty$), because in that case $L_\infty([0,1],X)$ and $\ell_\infty(X)$ are never isomorphic, whereas we know that $L_\infty([0,1])$ and ℓ_∞ are isometric. This is because $L_\infty([0,1],X)$ contains a complemented copy of c_0 but $\ell_\infty(X)$ does not.

c_0-SEQUENCES IN $L_p(\mu,X)$

As we have mentioned, Kwapien (1974) proved that if $L_1(\mu,X)$ contains a copy of c_0 then X must contain a copy of c_0, too. The approach given by Bourgain (1978) of this fact goes further. It says where we can find copies of c_0 in X, as soon as we have a c_0-sequence in $L_1(\mu,X)$. More precisely, if (f_n) is a c_0-sequence in $L_1(\mu,X)$ (that is, a sequence equivalent to the unit vector basis of c_0), then there is a measurable subset A of Ω, with positive measure, such that $(f_n(\omega))$ has a c_0-subsequence for all $\omega\in A$. We would ask if it is possible to have the same subsequence for all $\omega \in A$. The following simple example shows that this is not the case:

Let us consider in $L_1([0,1],c_0)$ the sequence $(f_n(.)) = ((r_n(.)+1)e_n)$, where (r_n) is the sequence of Rademacher functions in $[0,1]$ and (e_n) denotes the canonical basis of c_0. It is easy to see that (f_n) is a c_0-sequence in $L_1([0,1],c_0)$, but for each subsequence (f_{n_k}) of (f_n) and for almost all $t\in[0,1]$ the sequence $(f_{n_k}(t))$ vanishes for infinite natural numbers k, and so it can not be a c_0-sequence.

At this point it is natural to ask if the behavior of c_0-sequences in $L_p(\mu,X)$, for $1<p<\infty$, is the same as those in $L_1(\mu,X)$; or even more, is every c_0-sequence in $L_p(\mu,X)$ a c_0-sequence in $L_1(\mu,X)$? Maybe this is well known but we have found no explicit answer in the literature. Looking for a way to prove this, we found how an argument used by Bourgain in [2] can be applied here obtaining even more than we asked. It uses a technique which we think has interest by itself because it provides a non difficult way to transfer certain problems in $L_p(\mu,X)$ with $1<p<\infty$, to the same problem in $L_1(\mu,X)$. The key is the following "subsequence splitting lemma" for scalar functions, which was noticed by Bourgain [2], although it was contained more or less implicitly in [6]. It is the sharpest "subsequence splitting lemma" we know. It says us that every bounded sequence in $L_1(\mu)$ has a subsequence which can be split in a very good way in two pieces: one piece is disjointly supported and the other one is uniformly integrable.

LEMMA. (Kadec-Pelczynski-Bourgain) *If (f_n) is a bounded sequence in $L_1(\mu)$, then there exist a subsequence (g_n) of (f_n) and a sequence (A_n) of pairwise disjoint measurable sets such that $(\chi_{\Omega\setminus A_n} g_n)$ is uniformly integrable.*

Bourgain shows in [2] how to apply this lemma in the study of vector-valued functions. Following his arguments it is possible to prove the next result

PROPOSITION. *Let $1 < p < \infty$. Then we have:*

- *Every c_0-sequence in $L_p(\mu, X)$ is a c_0-sequence in $L_1(\mu, X)$.*
- *Every ℓ_r-sequence in $L_p(\mu, X)$ is an ℓ_r-sequence in $L_1(\mu, X)$, whenever $1 \leq r < \infty$, $r \neq p$.*
- *If $L_p(\mu, X)$ contains a copy of H and H does not contain complemented copies of ℓ_p, then $L_1(\mu, X)$ contains a copy of H.*

Antecedents of part of this Proposition may be found included in some proofs in the literature (see, for instance, [9], [5] and [1])

THE LAST SOLUTION

The last problem solved among the ones posed at the beginning has been to characterize when $L_\infty(\mu, X)$ contains complemented copies of ℓ_1. Since $L_\infty(\mu)$ never has a complemented copy of ℓ_1 (think, for instance, that it is injective while ℓ_1 is not), if $L_\infty(\mu, X)$ has a complemented copy of ℓ_1, must X contain a complemented copy of ℓ_1?. The answer is "No". This was first noticed by Montgomery-Smith. He realized that an example given by Johnson (in 1972) provided a counterexample. Let us begin with Johnson's example:

The space $\left(\sum \oplus \ell_1^n\right)_\infty$ has a complemented copy of ℓ_1.

In fact, if we denote by H the subspace of $\left(\sum \oplus \ell_1^n\right)_\infty$ of all sequences of the form $((a_1), (a_1, a_2), \dots, (a_1, a_2, \dots, a_n), \dots)$, with (a_k) belonging to ℓ_1, it is possible to prove that H is a 1-complemented subspace of $\left(\sum \oplus \ell_1^n\right)_\infty$ isometrically isomorphic to ℓ_1.

Now, Montgomery-Smith contribution was to notice that if one takes the space $X_0 = (\sum \oplus \ell_1^n)_{c_0}$, then $L_\infty(\mu, X_0)$ contains a complemented copy of $(\sum \oplus \ell_1^n)_\infty$ and therefore, a complemented copy of ℓ_1. Of course, X_0 has not even copies of ℓ_1, because its dual, $(\sum \oplus \ell_\infty^n)_1$, is separable.

This example was very important because it showed what one can not expect. It made difficult to give a reasonable conjecture concerning which Banach spaces X provide $L_\infty(\mu, X)$ with complemented copies of ℓ_1. For this reason, it was surprising when Díaz (1994) realized that the crucial condition satisfied by X_0 is that it contains ℓ_1^n's uniformly complemented. He proved that whenever we have a space X containing ℓ_1^n's uniformly complemented, the space $L_\infty(\mu, X)$ has a complemented copy of ℓ_1. Moreover, he also proved that in many cases the converse is true.

During some time it was not known if this converse was always true. Some people tried to show that it was, and in fact we have read several wrong proofs of this fact. Finally, in last June we have occasion to ask Kalton about this, while he was visiting our University. He showed us the right way to prove the result. The key was to use the notion of locally complemented subspace which can be seen in [7], [3] or [4]. Putting together Díaz and Kalton contributions we have:

THEOREM. (Díaz-Kalton) *$L_\infty(\mu, X)$ contains a complemented copy of ℓ_1 if and only if X contains ℓ_1^n's uniformly complemented*

REFERENCES

[1] BOMBAL, F., HERNANDO, B., On the injection of a Köthe function space into $L_1(\mu)$, *Annal. Soc. Math. Polon. Serie I: Commentationes Math.,* **25** (1995), 49–60.

[2] BOURGAIN, J., The Komlós theorem for vector valued functions, unpublished (1979),

[3] FAKHOURY, H., Sélections linéaires associées au Théorème de Hahn-Banch, *J. Funct. Anal.,* **11** (1972), 436–452.

[4] GODEFROY, G., KALTON, N.J., SAPHAR, P.D., Unconditional ideals in Banach spaces, *Studia Math.,* **104** (1993), 13–59.

[5] HENSGEN, W., Some properties of the vector-valued Banach ideal space $E(X)$ derived from those of E and X, *Collectanea Math.,* **43** (1992), 1–13.

[6] KADEC, M.I., PEŁCZYŃSKI, A., Bases, lacunary sequences and complemented subspaces in the spaces L_p, *Studia Math.,* **21** (1962), 1–13.

[7] KALTON, N.J., Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$, *Math. Nachr.,* **115** (1984), 71–97.

[8] MENDOZA, J., Copies of Classical Sequences Spaces in Vector-Valued Function Banach Spaces, in "Function Spaces the Second Conference", Lecture Notes

in Pure and Applied Mathematics (K. Jarosz, ed.), Vol. 172, Marcel Dekker, 1995, 311 - 320.

[9] RAYNAUD, Y., Sur les sous-espaces de $L_p(L_q)$, in "Seminaire d'Analyse Fonctionelle 1984/85", Publ. Math. Univ. Paris VII, 26, Univ. Paris VII, Paris, 1986, 49 - 71.